Non-relativistic Limit of a Dirac-Maxwell Operator in Relativistic Quantum Electrodynamics

Asao Arai*

Department of Mathematics, Hokkaido University

Sapporo 060-0810, Japan

E-mail: arai@math.sci.hokudai.ac.jp

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Abstract

The non-relativistic (scaling) limit of a particle-field Hamiltonian H, called a Dirac-Maxwell operator, in relativistic quantum electrodynamics is considered. It is proven that the non-relativistic limit of H yields a self-adjoint extension of the Pauli-Fierz Hamiltonian with spin $1/2$ in non-relativistic quantum electrodynamics. This is done by establishing in an abstract framework a general limit theorem on a family of self-adjoint operators partially formed out of strongly anticommuting self-adjoint operators and then by applying it to H.

Keywords: quantum electrodynamics, Dirac operator, Dirac-Maxwell operator, Pauli-Fierz Hamiltonian, non-relativistic limit, scalig limit, Fock space, strongly anticommuting self-adjoint operators

1 Introduction

In a previous paper [3], the author analyzed fundamental properties of a particle-field Hamiltonian H in relativistic quantum electrodynamics (QED), namely, the Hamiltonian of a Dirac particle — a relativistic charged particle with spin $1/2$ — interacting with the quantum radiation field. For convenience in mentioning the particle-field Hamiltonian, we call it a *Dirac-Maxwell operator*. In this paper, we consider the non-relativistic (scaling) limit of H. We prove that the non-relativistic limit of H yields a self-adjoint extension of the Pauli-Fierz Hamiltonian with spin $1/2$ in non-relativistic QED. This establishes a

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mathematically rigorous connection of relativistic QED to non-relativistic QED, which has not been proved so far. The Dirac-Maxwell operator H is of the form $H = H_{\rm D} + H_{\rm rad} + H_I$, where $H_{\rm D}$ is a Dirac opeartor describing the Dirac particle system only, $H_{\rm rad}$ is the free Hamiltonian of the quantum radiation field (a quantum version of the Maxwell Hamiltonian in the Coulomb gauge) and H_I is the interaction term beween the Dirac particle and the quantum radiation field. As for the Dirac operator $H_{\rm D}$, the non-relativistic limit has already been investigated and well understood ([11, Chapter 6] and references therein). We extend the methods used in the case of the Dirac operator $H_{\rm D}$ to the case of H. This can be done in an abstract framework. We remark that the non-relativistic limit theory of $H_{\rm D}$ is included in the theory of scaling limits on strongly anticommuting self-adjoint operators [2]. In view of this structure, we further develop the theory of scaling limits on strongly anticommuting self-adjoint operators in such a way that it can be applied to the non-relativistic limit of H. This is an outline of our method taken in the present paper.

The present paper is organized as follows. In Section 2 we describe the Dirac-Maxwell operator and the Pauli-Fierz Hamiltonian with spin 1/2. In Section 3 we state the main results of the present paper. Section 4 is devoted to an abstract analysis of a family of self-adjoint operators partially formed out of strongly anticommuting self-adjoint operators. We prove a limit theorem and a resolvent formula. These results are generalizations of previously known ones ([2], [11, Chapter 6]). In the last section, applying the general limit theorem established in Section 4, we prove the main results. In Appendix A we present a method to find a self-adjoint extension $\widetilde{S}$ of a Hermitian operator S defined as a finite sum of self-adjoint operators bounded from below. The self-adjoint extension $\widetilde{S}$ may be different from the Friedrichs extension and the one defined as a form sum if S is symmetric, but not essentially self-adjoint. The method here has an advantage in that $\widetilde{S}$ can be approximated by a family $\{S(\kappa)\}_{\kappa>0}$ of self-adjoint operators (as $\kappa \to \infty$) which are defined by "cutting off" S and may be tractable. We apply this abstract method to the construction of a self-adjoint extension of the Pauli-Fierz Hamiltonian without spin (Appendix B) and that with spin 1/2 (Section 3.3).

2 The Dirac-Maxwell Operator and The Pauli-Fierz Hamiltonian

For a linear operator T on a Hilbert space, we denote its domain by $D(T)$, and its adjoint by T^* (provided that T is densely defined). For two objects $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ such that products $a_j b_j$ $(j = 1, 2, 3)$ and their sum can be defined, we set $\boldsymbol{a} \cdot \boldsymbol{b} := \sum_{j=1}^{3} a_j b_j$. We use the physical unit system in which c(the speed of light)$= 1$ and $\hbar = 1$ ($\hbar := h/(2\pi)$; h is the Planck constant).

2.1 The Dirac operator

Let D_j $(j = 1, 2, 3)$ be the generalized partial differential operator in the variable x_j, the j-th component of $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$, and $\nabla := (D_1, D_2, D_3)$. We denote the mass and the charge of the Dirac particle by $m > 0$ and $q \in \mathbf{R} \setminus \{0\}$ respectively. We consider the situation where the Dirac particle is in a potential V which is a *Hermitian-matrix-valued Borel measurable function* on $\mathbf{R}^3$. Then the Hamiltonian of the Dirac particle is given by the Dirac operator

$$H_{\rm D} := \boldsymbol{\alpha} \cdot (-i\nabla) + m\beta + V \tag{2.1}$$

acting in the Hilbert space

$$\mathcal{H}_{\rm D} := \oplus^4 L^2(\mathbf{R}^3) \tag{2.2}$$

with domain $D(H_{\rm D}) := [\oplus^4 H^1(\mathbf{R}^3)] \cap D(V)$ ($H^1(\mathbf{R}^3)$ is the Sobolev space of order 1), where α_j ($j = 1, 2, 3$) and β are 4×4 Hermitian matrices satisfying the anticommutation relations

$$\{\alpha_j, \, \alpha_k\} = 2\delta_{jk}, \quad j, k = 1, 2, 3, \tag{2.3}$$

$$\{\alpha_j, \, \beta\} = 0, \quad \beta^2 = 1, \quad j = 1, 2, 3, \tag{2.4}$$

$\{A, B\} := AB + BA$ and δ_{jk} is the Kronecker delta. We assume the following:

Hypothesis (A)

Each matrix element of V is almost everywhere (a.e.) finite with respect to the three-dimensional Lebesgue measure $d\boldsymbol{x}$ and the subspace $\cap_{j=1}^3 [D(D_j) \cap D(V)]$ is dense in $\mathcal{H}_{\rm D}$.

Under this hypothesis, $H_{\rm D}$ is a symmetric operator. Detailed analysis of the Dirac operator is given in [11].

Example 2.1 A typical example for V is

$$V_{\rm em} := \phi - q\boldsymbol{\alpha} \cdot \boldsymbol{A}^{\rm ex}$$

with $\phi : \mathbf{R}^3 \to \mathbf{R}$ an external scalar potential and $\boldsymbol{A}^{\rm ex} := (A_1^{\rm ex}, A_2^{\rm ex}, A_3^{\rm ex}) : \mathbf{R}^3 \to \mathbf{R}^3$ an external vector potential, where $A_j^{\rm ex}$ and ϕ are in the set

$$L^2_{\rm loc}(\mathbf{R}^3) := \left\{ f : \mathbf{R}^3 \to \mathbf{C}; \text{Borel measurable} \middle| \int_{|\boldsymbol{x}| \le R} |f(\boldsymbol{x})|^2 d\boldsymbol{x} < \infty, \forall R > 0 \right\}.$$

Then $D(V_{\rm em}) \supset \oplus^4 C_0^\infty(\mathbf{R}^3)$, where $C_0^\infty(\mathbf{R}^3)$ is the set of C^∞-functions on $\mathbf{R}^3$ with compact support. Hence $\cap_{j=1}^3 [D(D_j) \cap D(V_{\rm em})]$ is dense. Thus $V_{\rm em}$ obeys Hypothesis (A).

2.2 The quantum radiation field

The Hilbert space of one-photon states in momentum representation is given by

$$\mathcal{H}_{\rm ph} := L^2(\mathbf{R}^3) \oplus L^2(\mathbf{R}^3), \tag{2.5}$$

where $\mathbf{R}^3 := \{\boldsymbol{k} = (k_1, k_2, k_3) | k_j \in \mathbf{R}, \ j = 1, 2, 3\}$ physically means the momentum space of photons. Then a Hilbert space for the quantum radiation field in the Coulomb gauge is given by

$$\mathcal{F}_{\rm rad} := \oplus_{n=0}^\infty \otimes_{\rm s}^n \mathcal{H}_{\rm ph}, \tag{2.6}$$

the Boson Fock space over $\mathcal{H}_{\rm ph}$, where $\otimes_{\rm s}^n \mathcal{H}_{\rm ph}$ denotes the n-fold symmetric tensor product of $\mathcal{H}_{\rm ph}$ and $\otimes_{\rm s}^0 \mathcal{H}_{\rm ph} := \mathbf{C}$. For basic facts on the theory of the Boson Fock space, we refer the reader to [9, §X.7].

We denote by $a(F)$ $(F \in \mathcal{H}_{\rm ph})$ the annihilation operator with test vector F on $\mathcal{F}_{\rm rad}$; its adjoint is given by

$$(a(F)^*\Psi)^{(n)} = \sqrt{n} S_n(F \otimes \Psi^{(n-1)}), \quad n \geq 0, \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in D(a(F)^*),$$

where S_n is the symmetrization operator on $\otimes^n \mathcal{H}_{\rm ph}$ and $\Psi^{-1} := 0$. For each $f \in L^2(\mathbf{R}^3)$, we define

$$a^{(1)}(f) := a(f,0), \quad a^{(2)}(f) := a(0,f). \tag{2.7}$$

The mapping : $f \to a^{(r)}(f^*)$ restricted to $\mathcal{S}(\mathbf{R}^3)$ (the Schwartz space of rapidly decreasing C^∞-functions on $\mathbf{R}^3$) defines an operator-valued distribution (f^* denotes the complex conjugate of f). We denote its symbolical kernel by $a^{(r)}(\boldsymbol{k})$: $a^{(r)}(f) = \int a^{(r)}(\boldsymbol{k}) f(\boldsymbol{k})^* d\boldsymbol{k}$.

We take a nonnegative Borel measurable function ω on $\mathbf{R}^3$ to denote the one free photon energy. We assume that, for a.e. $\boldsymbol{k} \in \mathbf{R}^3$ with respect to the Lebesgue measure on $\mathbf{R}^3$, $0 < \omega(\boldsymbol{k}) < \infty$. Then the function ω defines uniquely a multiplication operator on $\mathcal{H}_{\rm ph}$ which is nonnegative, self-adjoint and injective. We denote it by the same symbol ω. The free Hamiltonian of the quantum radiation field is then defined by

$$H_{\rm rad} := d\Gamma(\omega), \tag{2.8}$$

the second quantization of ω [8, p.302, Example 2] and [9, §X.7]. The operator $H_{\rm rad}$ is a nonnegative self-adjoint operator. The symbolical expression of $H_{\rm rad}$ is $H_{\rm rad} = \sum_{r=1}^2 \int \omega(\boldsymbol{k}) a^{(r)}(\boldsymbol{k})^* a^{(r)}(\boldsymbol{k}) d\boldsymbol{k}$.

Remark 2.1 Usually ω is taken to be of the form $\omega_{\rm phys}(\boldsymbol{k}) := |\boldsymbol{k}|, \quad \boldsymbol{k} \in \mathbf{R}^3$, but, in this paper, for mathematical generality, we do not restrict ourselves to this case.

There exist $\mathbf{R}^3$-valued Borel measurable functions $\mathbf{e}^{(r)}$ $(r = 1,2)$ on $\mathbf{R}^3$ such that, for a.e. $\boldsymbol{k}$

$$\mathbf{e}^{(r)}(\boldsymbol{k}) \cdot \mathbf{e}^{(s)}(\boldsymbol{k}) = \delta_{rs}, \quad \mathbf{e}^{(r)}(\boldsymbol{k}) \cdot \boldsymbol{k} = 0, \quad r, s = 1,2. \tag{2.9}$$

These vector-valued functions $\mathbf{e}^{(r)}$ are called the polarization vectors of a photon.

The time-zero quantum radiation field is given by $\boldsymbol{A}(\boldsymbol{x}) := (A_1(\boldsymbol{x}), A_2(\boldsymbol{x}), A_3(\boldsymbol{x}))$ with

$$A_j(\boldsymbol{x}) := \sum_{r=1}^2 \int d\boldsymbol{k} \frac{e_j^{(r)}(\boldsymbol{k})}{\sqrt{2(2\pi)^3 \omega(\boldsymbol{k})}} \left\{ a^{(r)}(\boldsymbol{k})^* e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} + a^{(r)}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \right\}, \quad j = 1,2,3, \tag{2.10}$$

in the sense of operator-valued distribution. Let ϱ be a real tempered distribution on $\mathbf{R}^3$ such that

$$\frac{\hat{\varrho}}{\sqrt{\omega}}, \quad \frac{\hat{\varrho}}{\omega} \in L^2(\mathbf{R}^3), \tag{2.11}$$

where $\hat{\varrho}$ denotes the Fourier transform of ϱ. The quantum radiation field

$$\boldsymbol{A}^\varrho := (A_1^\varrho, A_2^\varrho, A_3^\varrho) \tag{2.12}$$

with momentum cutoff $\hat{\varrho}$ is defined by

$$A_j^{\varrho}(\boldsymbol{x}) := \sum_{r=1}^{2} \int d\boldsymbol{k} \frac{e_j^{(r)}(\boldsymbol{k})}{\sqrt{2\omega(\boldsymbol{k})}} \left\{ a^{(r)}(\boldsymbol{k})^* e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\varrho}(\boldsymbol{k})^* + a^{(r)}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\varrho}(\boldsymbol{k}) \right\}. \tag{2.13}$$

Symbolically $A_j^{\varrho}(\boldsymbol{x}) = \int A_j(\boldsymbol{x}-\boldsymbol{y})\varrho(\boldsymbol{y})d\boldsymbol{y}$.

2.3 The Dirac-Maxwell operator

The Hilbert space of state vectors for the coupled system of the Dirac particle and the quantum radiation field is taken to be

$$\mathcal{F} := \mathcal{H}_{\mathrm{D}} \otimes \mathcal{F}_{\mathrm{rad}}. \tag{2.14}$$

This Hilbert space can be identified as

$$\mathcal{F} = L^2(\mathbf{R}^3; \oplus^4 \mathcal{F}_{\mathrm{rad}}) = \int_{\mathbf{R}^3}^{\oplus} \oplus^4 \mathcal{F}_{\mathrm{rad}} d\boldsymbol{x} \tag{2.15}$$

the Hilbert space of $\oplus^4\mathcal{F}_{\mathrm{rad}}$-valued Lebesgue square integrable functions on $\mathbf{R}^3$ (the constant fibre direct integral with base space $(\mathbf{R}^3, d\boldsymbol{x})$ and fibre $\oplus^4\mathcal{F}_{\mathrm{rad}}$ [10, §XIII.6]). We freely use this identification. The total Hamiltonian of the coupled system — a particle-field Hamiltonian — is defined by

$$H := H_{\mathrm{D}} + H_{\mathrm{rad}} - q\boldsymbol{\alpha}\cdot\boldsymbol{A}^{\varrho} = \boldsymbol{\alpha}\cdot(-i\nabla - q\boldsymbol{A}^{\varrho}) + m\beta + V + H_{\mathrm{rad}}. \tag{2.16}$$

We call H a *Dirac-Maxwell operator*. The (essential) self-adjointness of H is discussed in [3].

2.4 The Pauli-Fierz Hamiltonian with spin $1/2$

A Hamiltonian which describes a quantum system of non-relativistic charged particles interacting with the quantum radiation filed is called a Pauli-Fierz Hamiltonian [7]. Here we consider a non-relativistic charged particle with mass m, charge q and spin $1/2$. Suppose that the particle is in an external electromagnetic vector potential $A^{\mathrm{ex}} = (\boldsymbol{A}^{\mathrm{ex}}, \phi)$, where $\boldsymbol{A}^{\mathrm{ex}} := (A_1^{\mathrm{ex}}, A_2^{\mathrm{ex}}, A_3^{\mathrm{ex}}) : \mathbf{R}^3 \to \mathbf{R}^3$ and $\phi : \mathbf{R}^3 \to \mathbf{R}$ are Borel measurable and a.e. finite with respect to $d\boldsymbol{x}$. Let

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.17}$$

the Pauli spin matrices, and set

$$\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3). \tag{2.18}$$

Then the Pauli-Fierz Hamiltonian of this quantum system is defined by

$$H_{\mathrm{PF}} := \frac{\{\boldsymbol{\sigma}\cdot(-i\nabla - q\boldsymbol{A}^{\varrho} - q\boldsymbol{A}^{\mathrm{ex}})\}^2}{2m} + \phi + H_{\mathrm{rad}} \tag{2.19}$$

acting in the Hilbert space

$$\mathcal{F}_{\mathrm{PF}} := L^2(\mathbf{R}^3; \mathbf{C}^2) \otimes \mathcal{F}_{\mathrm{rad}} = L^2(\mathbf{R}^3; \oplus^2\mathcal{F}_{\mathrm{rad}}) = \int_{\mathbf{R}^3}^{\oplus} \oplus^2 \mathcal{F}_{\mathrm{rad}} d\boldsymbol{x}. \tag{2.20}$$

For the Pauli-Fierz Hamiltonian without spin, see Appendix B.

3 Main Results

3.1 A Dirac operator coupled to the quantum radiation field

We use the following representation of α_j and β [11, p.3]:

$$\alpha_j := \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{3.1}$$

where I_2 is the 2×2 identity matrix. Hence the eigenspaces $\mathcal{H}_{\mathrm{D}}^{\pm}$ of β with eigenvalue ± 1 take the forms respectively

$$\mathcal{H}_D^+ = \left\{ \begin{pmatrix} f \\ g \\ 0 \\ 0 \end{pmatrix} \middle| f, g \in L^2(\mathbf{R}^3) \right\}, \quad \mathcal{H}_D^- = \left\{ \begin{pmatrix} 0 \\ 0 \\ f \\ g \end{pmatrix} \middle| f, g \in L^2(\mathbf{R}^3) \right\}. \tag{3.2}$$

and we have

$$\mathcal{H}_{\mathrm{D}} = \mathcal{H}_{\mathrm{D}}^+ \oplus \mathcal{H}_{\mathrm{D}}^-. \tag{3.3}$$

Let $P_\pm$ be the orthogonal projections onto $\mathcal{H}_{\mathrm{D}}^{\pm}$. Then we have

$$V = V_0 + V_1 \tag{3.4}$$

with

$$V_0 = P_+VP_+ + P_-VP_-, \quad V_1 = P_+VP_- + P_-VP_+. \tag{3.5}$$

Note that

$$[V_0, \beta] = 0, \quad \{V_1, \beta\} = 0,$$

where $[A, B] := AB - BA$. In operator-matrix form relative to the orthogonal decomposition (3.3), we have

$$V_0 = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0 & W^* \\ W & 0 \end{pmatrix}, \tag{3.6}$$

where $U_\pm$ are 2×2 Hermitian matrix-valued functions on $\mathbf{R}^3$ and W is a 2×2 complex matrix-valued function on $\mathbf{R}^3$.

Let

$$\not{D}(V_1) := \boldsymbol{\alpha} \cdot (-i\nabla - q\boldsymbol{A}^\varrho) + V_1 \tag{3.7}$$

Then, recalling that A_j^ϱ is $H_{\mathrm{rad}}^{1/2}$-bounded [3], we see that $\not{D}(V_1)$ is densely defined and symmetric with $D(\not{D}(V_1)) \supset \left(\cap_{j=1}^3 [D(D_j) \cap D(V)]\right) \otimes_{\mathrm{alg}} D(H_{\mathrm{rad}}^{1/2})$, where $\otimes_{\mathrm{alg}}$ means algebraic tensor product.

By (3.3), we have the following orthogonal decomposition of $\mathcal{F}$:

$$\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-, \tag{3.8}$$

where

$$\mathcal{F}_\pm := \mathcal{H}_{\mathrm{D}}^{\pm} \otimes \mathcal{F}_{\mathrm{rad}} \cong \mathcal{F}_{\mathrm{PF}}. \tag{3.9}$$

Relative to this orthogonal decomposition, we can write

$$\not{D}(V_1) = \begin{pmatrix} 0 & D_{W^*} \\ D_W & 0 \end{pmatrix}, \tag{3.10}$$

where

$$D_W := \boldsymbol{\sigma}\cdot(-i\nabla - q\boldsymbol{A}^{\varrho}) + W, \tag{3.11}$$
$$D_{W^*} := \boldsymbol{\sigma}\cdot(-i\nabla - q\boldsymbol{A}^{\varrho}) + W^* \tag{3.12}$$

acting in $\mathcal{F}_{\mathrm{PF}}$.

For a closable linear operator T on a Hilbert space, we denote its closure by $\bar{T}$ unless otherwise stated. Note that D_W is densely defined as an operator on $\mathcal{F}_{\mathrm{PF}}$ and $(D_W)^* \supset D_{W^*}$. Hence $(D_W)^*$ is densely defined. Thus D_W is closable. Based on this fact, we can define

$$\widetilde{\not{D}}(V_1) := \begin{pmatrix} 0 & \left(\overline{D}_W\right)^* \\ \overline{D}_W & 0 \end{pmatrix}. \tag{3.13}$$

Lemma 3.1 *Under Hypothesis (A), $\widetilde{\not{D}}(V_1)$ is a self-adjoint extension of $\not{D}(V_1)$.*

Proof: The self-adjointness of $\not{D}(V_1)$ follows from a general theorem (e.g., [11, p.142, Lemma 5.3]). It is obvious that $\widetilde{\not{D}}(V_1)|D(D_W)\oplus D(D_{W^*}) = \not{D}(V_1)$, where, for a linear operator T and a subspace $D \subset D(T)$, $T|D$ denotes the restriction of T to D. Hence $\widetilde{\not{D}}(V_1)$ is a self-adjoint extension of $\not{D}(V_1)$. ∎

Remark 3.1 The operator

$$\widehat{\not{D}}(V_1) := \begin{pmatrix} 0 & \overline{D}_{W^*} \\ (\overline{D}_{W^*})^* & 0 \end{pmatrix} \tag{3.14}$$

is also a self-adjoint extension of $\not{D}(V_1)$. But, for simplicity, we consider here only $\widetilde{\not{D}}(V_1)$. Discussions on $\widetilde{\not{D}}(V_1)$ presented below apply also to $\widehat{\not{D}}(V_1)$ with suitable modifications.

3.2 A scaled Dirac-Maxwell operator

For a self-adjoint operator A, we denote the spectrum and the spectral measure of A by $\sigma(A)$ and $E_A(\,\cdot\,)$ respectively. In the case where A is bounded from below, we set

$$\mathcal{E}_0(A) := \inf\sigma(A), \quad A' := A - \mathcal{E}_0(A) \geq 0.$$

Let $\Lambda : (0,\infty) \to (0,\infty)$ be a nondecreasing function such that $\Lambda(\kappa) \to \infty$ as $\kappa \to \infty$ and A be a self-adjoint operator on a Hilbert space. Then, for each $\kappa > 0$, we define $A^{(\kappa)}$ by

$$A^{(\kappa)} := \begin{cases} E_{A'}([0,\Lambda(\kappa)])A'E_{A'}([0,\Lambda(\kappa)]) + \mathcal{E}_0(A) & \text{if } A \text{ is bounded from below} \\ & \text{and } \mathcal{E}_0(A) < 0 \\ E_{|A|}([0,\Lambda(\kappa)])AE_{|A|}([0,\Lambda(\kappa)]) & \text{if } A \text{ is nonnegative} \\ & \text{or } A \text{ is not bounded from below} \end{cases} \tag{3.15}$$

Then $A^{(\kappa)}$ is a bounded self-adjoint operator with

$$\|A^{(\kappa)}\| \leq \Lambda(\kappa). \tag{3.16}$$

Proposition 3.2 *The following hold:*

(i) *For all* $\psi \in D(A)$, $\mathrm{s}\text{ - }\lim_{\kappa\to\infty} A^{(\kappa)}\psi = A\psi$, *where* s - lim *means strong limit.*

(ii) *For all* $z \in \mathbf{C}\setminus\mathbf{R}$, $\mathrm{s}\text{ - }\lim_{\kappa\to\infty}(A^{(\kappa)} - z)^{-1} = (A - z)^{-1}$.

Proof: Part (i) follows from the functional calculus of A. Part (ii) follows from (i) and a general convergence theorem [8, p.292, Theorem VIII.25(a)]. ■

With this preliminary, we define for $\kappa > 0$ a scaled Dirac-Maxwell operator

$$H(\kappa) := \kappa\widetilde{\not{D}}(V_1) + \kappa^2 m\beta - \kappa^2 m + V_{0,\kappa} + H_{\mathrm{rad}}^{(\kappa)}, \tag{3.17}$$

where

$$V_{0,\kappa} := \begin{pmatrix} U_+^{(\kappa)} & 0 \\ 0 & U_-^{(\kappa)} \end{pmatrix}. \tag{3.18}$$

Some remarks may be in order on this definition. The parameter κ in $H(\kappa)$ means the speed of light *concerning the Dirac particle only*. The speed of light related to the external potential $V = V_0 + V_1$ and the quantum radiation field $\boldsymbol{A}^{\varrho}$ is absorbed in them respectively. The third term $-\kappa^2 m$ on the right hand side of (3.17) is a subtraction of the rest energy of the Dirac particle. Hence taking the scaling limit $\kappa \to \infty$ in $H(\kappa)$ in a suitable sense corresponds in fact to a *partial* non-relativistic limit of the quantum system under consideration.

If one considers the non-relativistic limit in a way similar to the usual Dirac operator H_{D}, then one may define

$$\widehat{H}(\kappa) := \kappa\widetilde{\not{D}}(V_1) + \kappa^2 m\beta - \kappa^2 m + V_0 + H_{\mathrm{rad}} \tag{3.19}$$

as a scaled Dirac-Maxwell operator, where no cutoffs on V_0 and H_{rad} are made. In this form, however, we find that, besides the (essential) self-adjointness problem of $\widehat{H}(\kappa)$, the methods used in the usual Dirac type operators ([11, Chapter 6] or those in [2]) seem not to work. This is because of the existence of the operator H_{rad} in $\widehat{H}(\kappa)$ which is singular as a perturbation of $H_0(\kappa) := \kappa\widetilde{\not{D}}(V_1) + \kappa^2 m\beta - \kappa^2 m + V_0$ (if one would try to apply the methods on sacaling limits discussed in the cited literatures, then one would have to treat H_{rad} as a perturbation of $H_0(\kappa)$). To avoid this difficulty, we replace H_{rad} in $\widehat{H}(\kappa)$ by a bounded self-adjoint operator which is obtained by cutting off H_{rad}. This is one of the basic ideas of the present paper. We apply the same idea to V_0 which also may be singular as a perturbation of $\kappa\widetilde{\not{D}}(V_1) + \kappa^2 m\beta - \kappa^2 m$. In this way we arrive at Definition (3.17) of a scaled Dirac-Maxwell operator.

Lemma 3.3 *Under Hypothesis (A),* $H(\kappa)$ *is self-adjoint with* $D(H(\kappa)) = D(\widetilde{\not{D}}(V_1))$.

Proof: The operator $\kappa^2 m\beta - \kappa^2 m + V_{0,\kappa} + H_{\mathrm{rad}}^{(\kappa)}$ is a bounded self-adjoint operator. Hence, by the Kato-Rellich theorem, the assertion follows. ■

3.3 Self-adjoint extension of the Pauli-Fierz Hamiltonian

Essential self-adjointness of the the Pauli-Fierz Hamiltonian H_{PF} given by (2.19) and its generalizations is discussed in [4, 5]. These papers show that, under additional conditions on $\hat{\varrho}, \omega, \boldsymbol{A}^{\mathrm{ex}}$ and ϕ, the Pauli-Fierz Hamiltonians are essentially self-adjoint. In the present paper, we do not intend to discuss essential self-adjointness problem of the Pauli-Fierz type Hamiltonians. Instead, we define a self-adjoint extension of H_{PF}, which may not be known before.

We define

$$H_{\mathrm{PF}}(\kappa; W, U_+) := \frac{\left(\overline{D}_W\right)^* \overline{D}_W}{2m} + U_+^{(\kappa)} + H_{\mathrm{rad}}^{(\kappa)}, \quad \kappa > 0 \tag{3.20}$$

acting in $\mathcal{F}_{\mathrm{PF}}$.

Lemma 3.4 *Under Hypotheses (A), $H_{\mathrm{PF}}(\kappa; W, U_+)$ is self-adjoint and bounded from below.*

Proof: By von Neumann's theorem (e.g., [9, p.180, Theorem X.25]), the operator $(2m)^{-1}\left(\overline{D}_W\right)^* \overline{D}_W$ is self-adjoint and nonnegative. The operator $U_+^{(\kappa)} + H_{\mathrm{rad}}^{(\kappa)}$ is bounded and self-adjoint. Hence, by the Kato-Rellich theorem, $H_{\mathrm{PF}}(\kappa; W, U_+)$ is self-adjoint and bounded from below. ■

A generalization of the Pauli-Fierz Hamiltonian H_{PF} is defined by

$$H_{\mathrm{PF}}(W, U_+) := \frac{D_{W^*} D_W}{2m} + U_+ + H_{\mathrm{rad}} \tag{3.21}$$

acting in $\mathcal{F}_{\mathrm{PF}}$.

We formulate additional conditions:

Hypothesis (B)

The function U_+ is bounded from below. In this case we set

$$u_0 := \mathcal{E}_0(U_+).$$

Remark 3.2 Under Hypothesis (A), $D(H_{\mathrm{PF}}(W, U_+))$ is not necessarily dense in $\mathcal{F}_{\mathrm{PF}}$, but, $D(\bar{D}_W) \cap D(U_+) \cap D(H_{\mathrm{rad}})$ is dense in $\mathcal{F}_{\mathrm{PF}}$. Hence $D(\bar{D}_W) \cap D(|U_+|^{1/2}) \cap D(H_{\mathrm{rad}}^{1/2})$ also is dense in $\mathcal{F}_{\mathrm{PF}}$. Therefore we can define a densely defined symmetric form s_{PF} as follows:

$$D(\mathsf{s}_{\mathrm{PF}}) := D(\bar{D}_W) \cap D(|U_+|^{1/2}) \cap D(H_{\mathrm{rad}}^{1/2}) \text{ (form domain)}, \tag{3.22}$$

$$\mathsf{s}_{\mathrm{PF}}(\Psi, \Phi) := \frac{1}{2m}(\bar{D}_W \Psi, \bar{D}_W \Phi) + (\Psi, U_+ \Phi) + (H_{\mathrm{rad}}^{1/2} \Psi, H_{\mathrm{rad}}^{1/2} \Phi), \tag{3.23}$$

$$\Psi, \Phi \in D(\mathsf{s}_{\mathrm{PF}}). \tag{3.24}$$

Assume Hypothesis (B) in addition to Hypothesis (A). Then it is easy to see that s_{PF} is closed. Let $H_{\mathrm{PF}}^{(\mathrm{f})}$ be the self-adjoint operator associated with s_{PF}. Then $H_{\mathrm{PF}}^{(\mathrm{f})} \geq u_0$ and $H_{\mathrm{PF}}^{(\mathrm{f})}$ is a self-adjoint extension of $H_{\mathrm{PF}}(W, U_+)$.

Theorem 3.5 *Under Hypotheses (A) and (B), there exists a self-adjoint extension of* $\widetilde{H}_{\mathrm{PF}}(W,U_+)$ *of* $H_{\mathrm{PF}}(W,U_+)$ *which have the following properties:*

(i) $\widetilde{H}_{\mathrm{PF}}(W,U_+) \geq u_0$.

(ii) $D(|\widetilde{H}_{\mathrm{PF}}(W,U_+)|^{1/2}) \subset D(\overline{D}_W) \cap D(|U_+|^{1/2}) \cap D(H_{\mathrm{rad}}^{1/2})$

(iii) *For all* $z \in (\mathbf{C} \setminus \mathbf{R}) \cup \{\xi \in \mathbf{R} | \xi < u_0\}$,

$$\mathrm{s}\text{ - }\lim_{\kappa\to\infty}(H_{\mathrm{PF}}(\kappa;W,U_+) - z)^{-1} = (\widetilde{H}_{\mathrm{PF}}(W,U_+) - z)^{-1},$$

where s - lim *means strong limit.*

(iv) *For all* $\xi < u_0$ *and* $\Psi \in D(|\widetilde{H}_{\mathrm{PF}}(W,U_+)|^{1/2})$,

$$\mathrm{s}\text{ - }\lim_{\kappa\to\infty}(H_{\mathrm{PF}}(\kappa;W,U_+) - \xi)^{1/2}\Psi = (\widetilde{H}_{\mathrm{PF}}(W,U_+) - \xi)^{1/2}\Psi.$$

Proof: We need only to apply Theorem A.1 in Appendix A to the following case:

$$\mathcal{H} = \mathcal{F}_{\mathrm{PF}},\ N = 2,\ A = \frac{\left(\overline{D}_W\right)^* \overline{D}_W}{2m},\quad B_1 = U_+,\ B_2 = H_{\mathrm{rad}},\ L = \Lambda.$$

∎

Remark 3.3 As for conditions for $\hat{\rho}$ and ω for Theorem 3.5 to hold, we only need condition (2.11); no additional conditions is necessary.

Remark 3.4 In the same manner as in Theorem 3.5, we can define a self-adjoint extension of the Pauli-Fierz Hamiltonian without spin (see Appendix B).

Remark 3.5 Under Hypotheses (A), (B) and that $D(H_{\mathrm{PF}}(W,U_+))$ is dense, $H_{\mathrm{PF}}(W,U_+)$ is a symmetric operator bounded from below. Hence it has the Friedrichs extension $\widehat{H}_{\mathrm{PF}}(W,U_+)$. But it is not clear that, in the case where $H_{\mathrm{PF}}(W,U_+)$ is not essentially self-adjoint, $\widetilde{H}_{\mathrm{PF}}(W,U_+) = \widehat{H}_{\mathrm{PF}}(W,U_+)$ or $\widetilde{H}_{\mathrm{PF}}(W,U_+) = H_{\mathrm{PF}}^{(\mathrm{f})}$ (Remark 3.2) or both of them do not hold.

3.4 Main theorems

We now state main results on the non-relativistic limit of $H(\kappa)$.

Theorem 3.6 *Let Hypotheses (A) and (B) be satisfied. Suppose that*

$$\lim_{\kappa\to\infty}\frac{\Lambda(\kappa)^2}{\kappa} = 0. \tag{3.25}$$

Then, all $z \in \mathbf{C} \setminus \mathbf{R}$,

$$\mathrm{s}\text{ - }\lim_{\kappa\to\infty}(H(\kappa) - z)^{-1} = \begin{pmatrix} \left(\widetilde{H}_{\mathrm{PF}}(W,U_+) - z\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.26}$$

In the case where U_+ is not necessarily bounded from below, we have the following.

Theorem 3.7 *Let Hypothesis (A) and (3.25) be satisfied. Suppose that $H_{\rm PF}(W, U_+)$ is essentially self-adjoint. Then, all $z \in \mathbf{C} \setminus \mathbf{R}$,*

$$\text{s - }\lim_{\kappa\to\infty}(H(\kappa) - z)^{-1} = \begin{pmatrix} \left(\overline{H_{\rm PF}(W, U_+)} - z\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.27}$$

Remark 3.6 Under additional conditions on ϱ, ω, W and U_+, one can prove that $H_{\rm PF}(W, U_+)$ is essentially self-adjoint for all values of the coupling constant q [4, 5].

We now apply Theorems 3.6 and 3.7 to the case where $V = V_{\rm em} = \phi - q\boldsymbol{\alpha} \cdot \boldsymbol{A}^{\rm ex}$ (Example 2.1), i.e., the case where $W = -q\boldsymbol{\sigma} \cdot \boldsymbol{A}^{\rm ex}$ and $U_\pm = \phi I_2$. We assume the following.

Hypothesis (C)

(C.1) The subspace $\cap_{j=1}^3[D(D_j) \cap D(A_j^{\rm ex}) \cap D(\phi)]$ is dense in $L^2(\mathbf{R}^3)$.

(C.2) ϕ is bounded from below. In this case we set $\phi_0 := \inf \sigma(\phi)$.

Under Hypothesis (C), we have a self-adjoint opeartor

$$\widetilde{H}_{\rm PF} := \widetilde{H}_{\rm PF}(-q\boldsymbol{\sigma} \cdot \boldsymbol{A}^{\rm ex}, \phi), \tag{3.28}$$

which is a self-adjoint extension of the original Pauli-Fierz Hamiltonian $H_{\rm PF}$ given by (2.19).

Let

$$H_{\rm DM}(\kappa) := \kappa \not{D}(-q\boldsymbol{\alpha} \cdot \boldsymbol{A}^{\rm ex}) + \kappa^2 m\beta - \kappa^2 m + \phi^{(\kappa)} + H_{\rm rad}^{(\kappa)}, \tag{3.29}$$

Then $H_{\rm DM}(\kappa)$ is the Dirac-Maxwell operator $H(\kappa)$ with $V_1 = -q\boldsymbol{\alpha} \cdot \boldsymbol{A}^{\rm ex}$ and $V_0 = \phi$. Theorems 3.6 and 3.7 immediately yield the following results on the non-relativistic limit of $H_{\rm DM}(\kappa)$.

Corollary 3.8 *Let Hypothesis (C) and (3.25) be satisfied. Then, for all $z \in \mathbf{C} \setminus \mathbf{R}$,*

$$\text{s - }\lim_{\kappa\to\infty}(H_{\rm DM}(\kappa) - z)^{-1} = \begin{pmatrix} \left(\widetilde{H}_{\rm PF} - z\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.30}$$

Corollary 3.9 *Assume (C.1) and (3.25). Suppose that $H_{\rm PF}$ is essentially self-adjoint. Then, all $z \in \mathbf{C} \setminus \mathbf{R}$,*

$$\text{s - }\lim_{\kappa\to\infty}(H_{\rm DM}(\kappa) - z)^{-1} = \begin{pmatrix} \left(\overline{H}_{\rm PF} - z\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.31}$$

Thus a mathematically rigorous connection of relativistic QED to non-relativistic QED is established.

4 Limit Theorem on Strongly Anticommuting Self-adjoint Operators

In this section we prove a limit theorem concerning strongly anticommuting self-adjoint operators. For a review of the fundamental abstract theory of strongly anticommuting self-adjoint operators, see [1].

Definition 4.1 Let A and B be self-adjoint operators on a Hilbert space $\mathcal{H}$.

(i) We say that A and B *strongly commute* if their spectral measures E_A and E_B commute (i.e., for all Borel sets $J, K \subset \mathbf{R}$, $E_A(J)E_B(K) = E_B(K)E_A(J)$).

(ii) We say that A and B *strongly anticommute* if, for all $\psi \in D(A)$ and $t \in \mathbf{R}$, $e^{-itB}\psi \in D(A)$ and $Ae^{-itB}\psi = e^{itB}A\psi$ (i.e. $e^{itB}A \subset Ae^{-itB}$).

Let $A \neq 0$ and B be strongly anticommuting self-adjoint operators on a Hilbert space $\mathcal{H}$. *We assume that B is injective.* For each $\kappa > 0$, we define

$$T_0(\kappa) := \kappa A + \kappa^2(B - |B|). \tag{4.1}$$

The operator $\kappa A + \kappa^2 B$ is an abstract form of Dirac-type operators and $-\kappa^2|B|$ is a "renormalization" term. It is shown that $T_0(\kappa)$ is essentially self-adjoint (Lemma 3.1 in [2]). We consider a perturbation of $T_0(\kappa)$. Let $C(\kappa)$ ($\kappa > 0$) be a symmetric operator on $\mathcal{H}$ and

$$T(\kappa) := \overline{T_0(\kappa)} + C(\kappa). \tag{4.2}$$

The main purpose of this section is to consider the limit $\kappa \to \infty$ of $T(\kappa)$ in the strong resolvent sense under a general condition for $C(\kappa)$. A basic assumption for $C(\kappa)$ is as follows:

Hypothesis (I)

$D(\overline{T_0(\kappa)}) \subset D(C(\kappa))$ and $T(\kappa)$ is self-adjoint with $D(T(\kappa)) = D(\overline{T_0(\kappa)})$.

To state the main result we need some preliminaries. Let $B = U_B|B|$ be the polar decomposition. Then U_B is self-adjoint and unitary and $\sigma(U_B) = \{\pm 1\}$, where, for a linear operator T, $\sigma(T)$ denotes the spectrum of T (see p.141 in [2]). The operators

$$P_\pm^B := \frac{1}{2}(I \pm U_B), \tag{4.3}$$

are respectively the orthogonal projections onto the eigenspaces

$$\mathcal{H}_\pm := \ker(U_B \mp I) \tag{4.4}$$

of U_B with eigenvalues ± 1 and we have the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-. \tag{4.5}$$

It is known that A and $|B|$ strongly commute (Lemma 2.2(v) in [2]). Hence the product spectral measure $E := E_A \otimes E_{|B|}$ of A and $|B|$ can be defined with spectral representations

$$A = \int_{\mathbf{R}^2} \lambda dE(\lambda, \mu), \quad |B| = \int_{\mathbf{R}^2} \mu dE(\lambda, \mu).$$

With the spectral measure E, we can define a nonnegative self-adjoint operator

$$K_0 := \frac{1}{2} \int_{\mathbf{R}^2} \frac{\lambda^2}{\mu} dE(\lambda, \mu) \geq 0. \tag{4.6}$$

Note that

$$K_0 = \frac{A^2 |B|^{-1}}{2} \quad \text{on } D(A^2|B|^{-1}) \cap D(|B|^{-1}A^2). \tag{4.7}$$

It is shown that K_0 is reduced by $\mathcal{H}_\pm$ (see Lemma 2.4 in [2]). We denote $K_{0,\pm}$ the reduced part of K_0 to $\mathcal{H}_\pm$ respectively. Thus we have

$$K_0 = \begin{pmatrix} K_{0,+} & 0 \\ 0 & K_{0,-} \end{pmatrix}, \tag{4.8}$$

where the operator-matrix representation is relative to the orthogonal decomposition (4.5):

$$P_+^B = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_-^B = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{4.9}$$

We define

$$K(\kappa) := K_0 + P_+^B C(\kappa) P_+^B. \tag{4.10}$$

Hypothesis (II)

Let $\kappa_0 > 0$ be a constant.

(II.1) For all $\kappa \geq \kappa_0$, $C(\kappa)$ is reduced by $\mathcal{H}_\pm$ so that it has the operator-matrix representation

$$C(\kappa) = \begin{pmatrix} C_+(\kappa) & 0 \\ 0 & C_-(\kappa) \end{pmatrix}, \tag{4.11}$$

where $C_\pm(\kappa)$ are the reduced parts of $C(\kappa)$ to $\mathcal{H}_\pm$ respectively.

(II.2) For all $\kappa \geq \kappa_0$, $D(K_0^{1/2}) \subset D(C(\kappa))$ and there exist nonnegative constants $a(\kappa)$ and $b(\kappa)$ such that

$$\|C(\kappa)f\| \leq a(\kappa)\|K_0^{1/2}f\| + b(\kappa)\|f\|, \quad f \in D(K_0^{1/2}). \tag{4.12}$$

Lemma 4.2 *Let Hypothesis (II) be satisfied and let*

$$K_+(\kappa) := K_{0,+} + C_+(\kappa). \tag{4.13}$$

Then, for all $\kappa \geq \kappa_0$, $K(\kappa)$ is self-adjoint with $D(K(\kappa)) = D(K_0)$ and bounded from below. Moreover, $K(\kappa)$ is reduced by $\mathcal{H}_\pm$ with

$$K(\kappa) = K_+(\kappa) \oplus K_{0,-} = \begin{pmatrix} K_+(\kappa) & 0 \\ 0 & K_{0,-} \end{pmatrix}. \tag{4.14}$$

Proof: By (II.2), $D(K_0) \subset D(C(\kappa)) \subset D(P_+^B C(\kappa) P_+^B)$. Hence $D(K(\kappa)) = D(K_0)$. Let $f \in D(K_0)$. Then we have for all $\varepsilon > 0$

$$\|K_0^{1/2} f\|^2 \leq \|f\| \|K_0 f\| \leq \varepsilon^2 \|K_0 f\|^2 + \frac{\|f\|^2}{4\varepsilon^2}.$$

Hence

$$\|K_0^{1/2} f\| \leq \varepsilon \|K_0 f\| + \frac{\|f\|}{2\varepsilon}. \tag{4.15}$$

This estimate and (4.12) imply

$$\|C(\kappa) f\| \leq a(\kappa)\varepsilon \|K_0 f\| + \left(\frac{a(\kappa)}{2\varepsilon} + b(\kappa) \right) \|f\|. \tag{4.16}$$

By the reducibility of $C(\kappa)$ by $\mathcal{H}_\pm$, we have $\|P_+^B C(\kappa) P_+^B f\| \leq \|C(\kappa) f\|$. Since $\varepsilon > 0$ is arbitrary, it follows from the Kato-Rellich theorem that $K(\kappa)$ is self-adjoint and bounded from below. The last assertion is easy to prove. ■

Hypothesis (III)

Under Hypothesis (II) (so that, by Lemma 4.2, for all $\kappa \geq \kappa_0$, $K_+(\kappa)$ is self-adjoint), there exists a self-adjoint operator K_+ on $\mathcal{H}_+$ such that, for all $z \in \mathbf{C} \setminus \mathbf{R}$,

$$\text{s - } \lim_{\kappa \to \infty} (K_+(\kappa) - z)^{-1} = (K_+ - z)^{-1}. \tag{4.17}$$

The main result of this section is the following:

Theorem 4.3 *Assume Hypotheses (I)–(III). Suppose that*

$$\lim_{\kappa \to \infty} \frac{a(\kappa)^3}{\kappa} = 0, \quad \lim_{\kappa \to \infty} \frac{b(\kappa)^2}{\kappa} = 0, \quad \lim_{\kappa \to \infty} \frac{a(\kappa)^2 b(\kappa)}{\kappa} = 0 \tag{4.18}$$

and

$$M := \inf \sigma(|B|) > 0. \tag{4.19}$$

Then, for all $z \in \mathbf{C} \setminus \mathbf{R}$,

$$\text{s - } \lim_{\kappa \to \infty} (T(\kappa) - z)^{-1} = \begin{pmatrix} (K_+ - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.20}$$

We prove Theorem 4.3 by a series of lemmas. *In what follows, we assume (4.19).* Then $|B|^{-1}$ is bounded with

$$\||B|^{-1}\| \leq \frac{1}{M}. \tag{4.21}$$

For $z \in \mathbf{C} \setminus \mathbf{R}$, we define

$$K(\kappa, z) := K(\kappa) - z - \frac{z^2}{2\kappa^2} |B|^{-1}. \tag{4.22}$$

and set

$$d(\kappa, z) := \frac{|z|^2}{2\kappa^2 M |\text{Im}\, z|} > 0. \tag{4.23}$$

Lemma 4.4 *Assume Hypothesis (II) and (4.19). Let* $z \in \mathbf{C} \setminus \mathbf{R}$, $\kappa \geq \kappa_0$ *and*

$$L(\kappa, z) := 1 - \frac{z^2}{2\kappa^2}|B|^{-1}(K(\kappa) - z)^{-1}. \tag{4.24}$$

Let

$$d(\kappa, z) < 1. \tag{4.25}$$

Then the following hold:

(i) $L(\kappa, z)$ *is bijective with*

$$L(\kappa, z)^{-1} = \sum_{n=0}^{\infty} \left(\frac{z^2}{2\kappa^2}\right)^n \left(|B|^{-1}(K(\kappa) - z)^{-1}\right)^n \tag{4.26}$$

in operator norm topology and

$$\|L(\kappa, z)^{-1}\| \leq \frac{1}{1 - d(\kappa, z)}. \tag{4.27}$$

(ii) $K(\kappa, z)$ *is bijective and*

$$\begin{aligned} K(\kappa, z)^{-1} &= (K(\kappa) - z)^{-1} L(\kappa, z)^{-1} && (4.28) \\ &= \sum_{n=0}^{\infty} \left(\frac{z^2}{2\kappa^2}\right)^n (K(\kappa) - z)^{-1} \left(|B|^{-1}(K(\kappa) - z)^{-1}\right)^n && (4.29) \end{aligned}$$

in operator norm topology with

$$\|K(\kappa, z)^{-1}\| \leq r(\kappa, z), \tag{4.30}$$

where

$$r(\kappa, z) := \frac{1}{|\mathrm{Im}\, z|(1 - d(\kappa, z))}. \tag{4.31}$$

Proof: (i) We have by (4.21)

$$\left\| \frac{z^2}{2\kappa^2}|B|^{-1}(K(\kappa) - z)^{-1} \right\| \leq d(\kappa, z) < 1.$$

Hence, by C. Neumann's theorem, the bijectivity of $L(\kappa, z)$ follows with Neumann expansion (4.26). Inequality (4.27) follows from the general fact that, for all bounded linear operators T with $\|T\| < 1$, $\|(1 - T)^{-1}\| \leq (1 - \|T\|)^{-1}$. (ii) We have $K(\kappa, z) = L(\kappa, z)(K(\kappa) - z)$, which implies that $K(\kappa, z)$ is bijective with (4.28). Expansion (4.29) follows from (4.28) and (4.26). Using (4.27) and (4.28), we obtain (4.30). ∎

The following fact is an important key to the analysis here.

Theorem 4.5 *Assume Hypotheses (I), (II) and (4.19). Let $z \in \mathbf{C} \setminus \mathbf{R}$ and $d(\kappa, z) < 1$ with $\kappa \geq \kappa_0$. Then the operator $1 + \dfrac{C(\kappa)}{2\kappa^2}(\kappa A + z)|B|^{-1}K(\kappa, z)^{-1}$ is bijective and*

$$
\begin{aligned}
(T(\kappa) - z)^{-1} &= \left(P_+^B + \frac{1}{2\kappa^2}(\kappa A + z)|B|^{-1}\right) K(\kappa, z)^{-1} \\
&\quad \times \left(1 + \frac{C(\kappa)}{2\kappa^2}(\kappa A + z)|B|^{-1}K(\kappa, z)^{-1}\right)^{-1}. \qquad (4.32)
\end{aligned}
$$

Proof: Informal (heuristic) manupilations to obtain (4.32) are similar to the case of an abstract Dirac operator [11, p.180, Theorem 6.4] or to a case previously discussed by the present author [2, p.155, Theorem 4.3]. But, for completeness (since the assumption here is slightly different from those in [2, 11]), we give an outline of proof. Introducing an operator

$$W(\kappa, z) := 1 + C(\kappa)(\overline{T_0(\kappa)} - z)^{-1},$$

which is well-defined by Hypothesis (I), we have

$$T(\kappa) - z = W(\kappa, z)(\overline{T_0(\kappa)} - z).$$

This implies that $W(\kappa, z)$ is bijective and

$$(T(\kappa) - z)^{-1} = (\overline{T_0(\kappa)} - z)^{-1}W(\kappa, z)^{-1}.$$

On the other hand, we have

$$(\overline{T_0(\kappa)} - z)^{-1} = \frac{1}{2\kappa^2}(S_0(\kappa) + z)|B|^{-1}K_0(\kappa, z)^{-1}, \qquad (4.33)$$

where

$$
\begin{aligned}
S_0(\kappa) &:= \kappa A + \kappa^2(B + |B|), \\
K_0(\kappa, z) &:= K_0 - z - \frac{z^2}{2\kappa^2}|B|^{-1} = K(\kappa, z) - P_+^B C(\kappa) P_+^B,
\end{aligned}
$$

see p.147, (3.17) and (3.18) in [2]. Hence

$$(T(\kappa) - z)^{-1} = \frac{1}{2\kappa^2}(S_0(\kappa) + z)|B|^{-1}K_0(\kappa, z)^{-1}W(\kappa, z)^{-1}. \qquad (4.34)$$

Let

$$X(\kappa, z) := 1 + P_+^B C(\kappa) P_+^B K_0(\kappa, z)^{-1}.$$

Using (4.33), we have

$$W(\kappa, z) = X(\kappa, z) + \frac{C(\kappa)}{2\kappa^2}(\kappa A + z)|B|^{-1}K_0(\kappa, z)^{-1},$$

where we have used that $B + |B| = 2P_+^B|B|$ and $C(\kappa)P_+^B = P_+^B C(\kappa) P_+^B$. Note that

$$K(\kappa, z) = X(\kappa, z)K_0(\kappa, z).$$

This implies that $X(\kappa, z)$ is bijective with

$$X(\kappa, z)^{-1} = K_0(\kappa, z)K(\kappa, z)^{-1}.$$

Hence we obtain

$$\begin{aligned} W(\kappa, z) &= \left(1 + \frac{C(\kappa)}{2\kappa^2}(\kappa A + z)|B|^{-1}K_0(\kappa, z)^{-1}X(\kappa, z)^{-1}\right) X(\kappa, z) \\ &= \left(1 + \frac{C(\kappa)}{2\kappa^2}(\kappa A + z)|B|^{-1}K(\kappa, z)^{-1}\right) X(\kappa, z), \end{aligned}$$

which implies that

$$Y(\kappa, z) := 1 + \frac{C(\kappa)}{2\kappa^2}(\kappa A + z)|B|^{-1}K(\kappa, z)^{-1}$$

is also bijective with

$$W(\kappa, z)^{-1} = X(\kappa, z)^{-1}Y(\kappa, z)^{-1} = K_0(\kappa, z)K(\kappa, z)^{-1}Y(\kappa, z)^{-1}.$$

Putting this equation into (4.34), we obtain (4.32). ■

Lemma 4.6 *Assume Hypothesis (II) and (4.19). Let $\varepsilon > 0$. Then, for all $f \in D(K_0)$,*

$$\|C(\kappa)|B|^{-1}f\| \le \frac{\varepsilon a(\kappa)}{M}\|K_0 f\| + \frac{1}{M}\left(\frac{a(\kappa)}{2\varepsilon} + b(\kappa)\right)\|f\|. \tag{4.35}$$

Proof: We see by functional calculus that, for all $f \in D(K_0)$, $|B|^{-1}f \in D(K_0)$ and $K_0|B|^{-1}f = |B|^{-1}K_0 f$. Using this fact, (4.16) and (4.21), we obtain (4.35). ■

Lemma 4.7 *Assume (4.19). Then $D(K_0) \subset D(A|B|^{-1})$ and*

$$\|A|B|^{-1}f\| \le \varepsilon\|K_0 f\| + \frac{1}{\varepsilon M}\|f\|, \quad f \in D(K_0), \tag{4.36}$$

where $\varepsilon > 0$ is arbitrary.

Proof: Let $g \in D := D(A^2|B|^{-1}) \cap D(|B|^{-1}A^2)$, we have

$$\begin{aligned} \|A|B|^{-1}g\|^2 &= 2(|B|^{-1}g, K_0 g) \le \frac{2\|g\|}{M}\|K_0 g\| \\ &\le \varepsilon^2\|K_0 g\|^2 + \frac{1}{\varepsilon^2 M^2}\|g\|^2, \end{aligned}$$

where $\varepsilon > 0$ is arbitrary. Hence

$$\|A|B|^{-1}g\| \le \varepsilon\|K_0 g\| + \frac{1}{\varepsilon M}\|g\|.$$

Since D is a core of K_0 (p.143, Lemma 2.4 in [2]) and $|B|^{-1}$ is bounded, the assertion follows from a limiting argument. ■

Lemma 4.8 *Assume Hypothesis (II) and (4.19). Then* $D(K_0) \subset D(C(\kappa)A|B|^{-1})$ *and*

$$\|C(\kappa)A|B|^{-1}f\| \le \left(\frac{\sqrt{2}a(\kappa)}{\sqrt{M}} + \varepsilon b(\kappa)\right)\|K_0 f\| + \frac{b(\kappa)}{\varepsilon M}\|f\|, \quad f \in D(K_0), \tag{4.37}$$

where $\varepsilon > 0$ *is arbitrary.*

Proof: Let $f \in D(K_0)$. Then it follows from the functional calculus on the product spectral measure E and (4.12) that $f \in D(K_0^{1/2}A|B|^{-1}) \subset D(C(\kappa)A|B|^{-1})$ and

$$\begin{aligned} \|C(\kappa)A|B|^{-1}f\| &\le a(\kappa)\|K_0^{1/2}A|B|^{-1}f\| + b(\kappa)\|A|B|^{-1}f\| \\ &= a(\kappa)\|\sqrt{2}|B|^{-1/2}K_0 f\| + b(\kappa)\|A|B|^{-1}f\| \\ &\le \frac{\sqrt{2}a(\kappa)}{\sqrt{M}}\|K_0 f\| + b(\kappa)\|A|B|^{-1}f\|. \end{aligned}$$

This estimate and (4.36) give (4.37). ■

Lemma 4.9 *Assume Hypothesis (II) and (4.19). Let* $\delta > 0$ *be a constant such that* $a(\kappa)\delta < 1$*. Then, for all* $f \in D(K_0)$ *and* $\kappa \ge \kappa_0$*,*

$$\begin{aligned} \|K_0 f\| \le\ & \frac{1}{1-a(\kappa)\delta}\|K(\kappa, z)f\| \\ & + \frac{1}{1-a(\kappa)\delta}\left(|z| + \frac{|z|^2}{2\kappa^2 M} + \frac{a(\kappa)}{2\delta} + b(\kappa)\right)\|f\|. \end{aligned} \tag{4.38}$$

Proof: Using (4.16), we have

$$\begin{aligned} \|K_0 f\| &\le \|K(\kappa)f\| + \|C(\kappa)P_+^B f\| \\ &\le \|K(\kappa)f\| + a(\kappa)\delta\|K_0 f\| + \left(\frac{a(\kappa)}{2\delta} + b(\kappa)\right)\|f\|, \end{aligned}$$

where $\delta > 0$ is arbitrary. Taking $\delta > 0$ such that $a(\kappa)\delta < 1$, we obtain

$$\|K_0 f\| \le \frac{1}{1-a(\kappa)\delta}\|K(\kappa)f\| + \frac{1}{1-a(\kappa)\delta}\left(\frac{a(\kappa)}{2\delta} + b(\kappa)\right)\|f\|. \tag{4.39}$$

On the other hand, we have

$$\|K(\kappa)f\| \le \|K(\kappa, z)f\| + \left(|z| + \frac{|z|^2}{2\kappa^2 M}\right)\|f\|.$$

Thus (4.38) follows. ■

Lemma 4.10 *Assume Hypothesis (II), (4.19) and (4.25). Let* $\delta > 0$ *be a constant such that* $a(\kappa)\delta < 1$ *and* $\varepsilon > 0$. *Let*

$$\begin{aligned} G_1(\kappa, z, \varepsilon, \delta) \quad &:= \quad \frac{\varepsilon a(\kappa)}{M(1 - a(\kappa)\delta)} \left\{ 1 + r(\kappa, z) \left(|z| + \frac{|z|^2}{2\kappa^2 M} + \frac{a(\kappa)}{2\delta} + b(\kappa) \right) \right\} \\ & \quad + r(\kappa, z) \left(\frac{a(\kappa)}{2\varepsilon M} + \frac{b(\kappa)}{M} \right). \end{aligned} \tag{4.40}$$

Then $C(\kappa)|B|^{-1}K(\kappa, z)^{-1}$ *is bounded with*

$$\|C(\kappa)|B|^{-1}K(\kappa, z)^{-1}\| \leq G_1(\kappa, z, \varepsilon, \delta). \tag{4.41}$$

Proof: This follows from Lemma 4.6 and Lemma 4.9. ■

Lemma 4.11 *Assume Hypothesis (II), (4.19) and (4.25). Let* $\delta > 0$ *be a constant such that* $a(\kappa)\delta < 1$ *and* $\varepsilon > 0$. *Let*

$$\begin{aligned} G_2(\kappa, z, \varepsilon, \delta) \quad &:= \quad \frac{1}{1 - a(\kappa)\delta} \\ & \quad \times \left(\frac{\sqrt{2}a(\kappa)}{\sqrt{M}} + \varepsilon b(\kappa) \right) \left\{ 1 + r(\kappa, z) \left(|z| + \frac{|z|^2}{2\kappa^2 M} + \frac{a(\kappa)}{2\delta} + b(\kappa) \right) \right\} \\ & \quad + \frac{r(\kappa, z) b(\kappa)}{\varepsilon M}. \end{aligned} \tag{4.42}$$

Then $C(\kappa)A|B|^{-1}K(\kappa, z)^{-1}$ *is bounded with*

$$\|C(\kappa)A|B|^{-1}K(\kappa, z)^{-1}\| \leq G_2(\kappa, z, \varepsilon, \delta). \tag{4.43}$$

Proof: This follows from Lemma 4.8 and Lemma 4.9. ■

Lemma 4.12 *Assume Hypotheses (II), (III) and (4.19). Then*

$$\text{s - }\lim_{\kappa \to \infty} P_+^B K(\kappa, z)^{-1} = \begin{pmatrix} (K_+ - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.44}$$

Proof: Let

$$K := K_+ \oplus K_{0,+}.$$

By Lemma 4.4, we have

$$K(\kappa, z)^{-1} = (K(\kappa) - z)^{-1} + (K(\kappa) - z)^{-1}V(\kappa)$$

with $V(\kappa) := \sum_{n=1}^{\infty} \left(\frac{z^2}{2\kappa^2} \right)^n (|B|^{-1}(K(\kappa) - z)^{-1})^n$. Hence

$$K(\kappa, z)^{-1} - (K - z)^{-1} = (K(\kappa) - z)^{-1} - (K - z)^{-1} + (K(\kappa) - z)^{-1}V(\kappa).$$

It is easy to see that $\|V(\kappa)\| \to 0$ as $\kappa \to \infty$. By Hypothesis (III), we have

$$\text{s - }\lim_{\kappa\to\infty}(K(\kappa) - z)^{-1} = (K - z)^{-1}.$$

Hence

$$\text{s - }\lim_{\kappa\to\infty} K(\kappa, z)^{-1} = (K - z)^{-1},$$

which implies that

$$\text{s - }\lim_{\kappa\to\infty} P_+^B K(\kappa, z)^{-1} = P_+^B (K - z)^{-1} = \begin{pmatrix} (K_+ - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus (4.44) holds. ■

Proof of Theorem 4.3

By Lemmas 4.10 and 4.11, we have

$$\left\| \frac{C(\kappa)}{2\kappa^2}(\kappa A + z)|B|^{-1}K(\kappa, z)^{-1} \right\| \le \frac{G_2(\kappa, z, \varepsilon, \delta)}{2\kappa} + \frac{|z|}{2\kappa^2} G_1(\kappa, z, \varepsilon, \delta).$$

Let $0 < \alpha < 1$ be fixed and set $\delta = \alpha/a(\kappa)$ so that $a(\kappa)\delta = \alpha < 1$. Let $\kappa_1 > 0$ be a constant such that $d(\kappa_1, z) < 1$ and $\kappa_1 \ge \max\{\kappa_0, 1\}$. Let $\kappa \ge \kappa_1$. Then

$$\begin{aligned} G_1(\kappa, z, \varepsilon, \delta) &\le C_1[a(\kappa) + a(\kappa)^3 + a(\kappa)b(\kappa) + b(\kappa)], \\ G_2(\kappa, z, \varepsilon, \delta) &\le C_2[a(\kappa) + a(\kappa)^3 + a(\kappa)b(\kappa) + b(\kappa) + b(\kappa)a(\kappa)^2 + b(\kappa)^2], \end{aligned}$$

where C_1 and C_2 are constants independent of $\kappa \ge \kappa_1$. Hence, under condition (4.18), we have

$$\lim_{\kappa\to\infty} \frac{G_1(\kappa, z, \varepsilon, \delta)}{\kappa^2} = 0, \quad \frac{G_2(\kappa, z, \varepsilon, \delta)}{\kappa} = 0.$$

Hence

$$\lim_{\kappa\to\infty} \left\| \frac{C(\kappa)}{2\kappa^2}(\kappa A + z)|B|^{-1}K(\kappa, z)^{-1} \right\| = 0,$$

which implies that

$$\lim_{\kappa\to\infty} \left(1 + \frac{C(\kappa)}{2\kappa^2}(\kappa A + z)|B|^{-1}K(\kappa, z)^{-1} \right)^{-1} = 1 \tag{4.45}$$

in operator-norm topology. By Lemmas 4.7 and 4.9, we have

$$\|A|B|^{-1}K(\kappa, z)^{-1}\| \le \frac{\varepsilon}{1 - a(\kappa)\delta} + \frac{r(\kappa, z)\varepsilon}{1 - a(\kappa)\delta}\left(|z| + \frac{|z|^2}{2\kappa^2 M} + \frac{a(\kappa)}{2\delta} + b(\kappa) \right) + \frac{r(\kappa, z)}{\varepsilon M}.$$

Hence, in the same way as above, we can show that

$$\lim_{\kappa\to\infty} \frac{1}{2\kappa^2}(\kappa A + z)|B|^{-1}K(\kappa, z)^{-1} = 0$$

in operator-norm topology. These facts together with Theorem 4.5 and Lemma 4.12 imply (4.20). ■

Remark 4.1 Higher order corrections to the limiting formula (4.20) can be computed by using Theorem 4.5 and (4.29).

5 Proof of The Main Theorems

5.1 Proof of Theorem 3.6

We apply Theorem 4.3. For this purpose, we first prove the following lemma.

Lemma 5.1 *The self-adjoint operator $\widetilde{\not{D}}(V_1)$ strongly anticommutes with $m\beta$.*

Proof: We have for all $t \in \mathbf{R}$

$$e^{-itm\beta} = \begin{pmatrix} e^{-itm} I_2 & 0 \\ 0 & e^{itm} I_2 \end{pmatrix}.$$

This implies that, for all $\Psi \in D\left(\widetilde{\not{D}}(V_1)\right) = D\left(\overline{D}_W\right) \oplus D\left(\left(\overline{D}_W\right)^*\right)$, $e^{-itm\beta}\Psi \in D(\widetilde{\not{D}}(V_1))$ and $\widetilde{\not{D}}(V_1)e^{-itm\beta}\Psi = e^{itm\beta}\widetilde{\not{D}}(V_1)\Psi$. Hence $\widetilde{\not{D}}(V_1)$ strongly anticommutes with $m\beta$. ■Let

$$A = \widetilde{\not{D}}(V_1), \quad B = m\beta, \quad C(\kappa) = V_{0,\kappa} + H_{\mathrm{rad}}^{(\kappa)}.$$

Then $|B| = m$ and we can write

$$H(\kappa) = \kappa A + \kappa^2(B - |B|) + C(\kappa).$$

By Lemma 5.1, A and B strongly anticommute. Hence $H(\kappa)$ is of the form $T(\kappa)$ in Section 4. We need only to check that $T(\kappa) = H(\kappa)$ satisfies the assumption of Theorem 4.3. Since $C(\kappa)$ is bounded, Hypothesis (I) holds. In the present case we have $P_\pm^B = P_\pm$ and $C(\kappa)$ is reduced by $\mathcal{F}_\pm$ with

$$C(\kappa) = \begin{pmatrix} U_+^{(\kappa)} + H_{\mathrm{rad}}^{(\kappa)} & 0 \\ 0 & U_-^{(\kappa)} + H_{\mathrm{rad}}^{(\kappa)} \end{pmatrix}. \tag{5.1}$$

Hence Hypothesis (II.1) holds.

In the present case we have

$$K_0 = \frac{\widetilde{\not{D}}(V_1)^2}{2m} = \begin{pmatrix} \dfrac{(\overline{D}_W)^*\overline{D}_W}{2m} & 0 \\ 0 & \dfrac{\overline{D}_W(\overline{D}_W)^*}{2m} \end{pmatrix}. \tag{5.2}$$

By (3.16), $\|C(\kappa)\Psi\| \le 2\Lambda(\kappa)\|\Psi\|$ for all $\Psi \in \mathcal{F}$. Hence Hypothesis (II.2) holds with

$$a(\kappa) = 0, \quad b(\kappa) = 2\Lambda(\kappa). \tag{5.3}$$

By (5.1) and (5.2), we have

$$K_+(\kappa) = H_{\mathrm{PF}}(\kappa; W, U_+).$$

By Theorem 3.5, Hypothesis (III) holds with $K_+ = \widetilde{H}_{\mathrm{PF}}(W, U_+)$. By (5.3) and (3.25), (4.18) holds. Thus the assumption of Theorem 4.3 is satisfied. Hence we can apply Theorem 4.3 to obtain (3.26).

5.2 Proof of Theorem 3.6

Hypotheses (I) and (II) hold in this case too. But it is not immediately obvious if Hypothesis (III) holds, since, in this case, we can not use Theorem 3.5. We note that

$$\lim_{\kappa\to\infty} H_{\rm PF}(\kappa; W, U_+)\Psi = H_{\rm PF}(W, U_+)\Psi, \quad \Psi \in D(H_{\rm PF}(W, U_+)).$$

By the assumption on the essential self-adjointness of $H_{\rm PF}(W, U_+)$, we can apply a general convergence theorem [8, p.292, Theorem VIII.25(a)] to conclude that, for all $z \in \mathbf{C} \setminus \mathbf{R}$,

$$\text{s - }\lim_{\kappa\to\infty} (H_{\rm PF}(\kappa; W, U_+) - z)^{-1} = \left(\overline{H_{\rm PF}(W, U_+)} - z\right)^{-1}.$$

Hence Hypothesis (III) holds with $K_+ = \overline{H_{\rm PF}(W, U_+)}$. Then, in the same way as in the proof of Theorem 3.5, we obtain Theorem 3.6.

Appendix

A A Class of Self-adjoint Extensions of Hermitian Operators

We say that a linear operator S on a Hilbert space $\mathcal{H}$ is *Hermitian* if $(\psi, S\phi) = (S\psi, \phi)$ for all $\psi, \phi \in D(S)$. In this definition, we do not assume the denseness of $D(S)$. A densely defined Hermitian operator is called a symmetric operator.

In this appendix we present a class of self-adjoint extensions of Hermitian operators. To the author's best knowledge, this class is new. Let $\mathcal{H}$ be a complex Hilbert space. Let A be a nonnegative self-adjoint operator on $\mathcal{H}$ and B_j ($j = 1, 2, \cdots, N,\ N \in \mathbf{N}$) be self-adjoint operators bounded from below with $B_j \geq b_j$ ($b_j \in \mathbf{R}$ is a constant) such that $\cap_{j=1}^N \left[D\left(A^{1/2}\right) \cap D\left(|B_j|^{1/2}\right)\right]$ is dense in $\mathcal{H}$. Let

$$c_0 := \sum_{j=1}^N b_j.$$

Then the operator

$$S := A + \sum_{j=1}^N B_j$$

is Hermitian and bounded from below with $S \geq c_0$.

Remark A.1 If S is densely defined (i.e., $D(S) = \cap_{j=1}^N [D(A) \cap D(B_j)]$ is dense), then S is a symmetric operator bounded from below and hence S has a self-adjoint extension $S_{\rm F}$, called the Friedrichs extension (e.g., [9, p.177, Theorem X.23]).

Remark A.2 The operator S has another type of self-fadjoint extension $S_{\rm f}$ which is given by the form sum $S_{\rm f} := A\dot{+}B_1\dot{+}\cdots\dot{+}B_N$, i.e., the self-adjoint operator associated with the densely defined symmetric closed form s_0 given by

$$\begin{aligned} D(\mathsf{s}_0) &:= \cap_{j=1}^N \left[D\left(A^{1/2}\right)\cap D\left(|B_j|^{1/2}\right)\right] \text{ (form domain)}, \\ \mathsf{s}_0(\psi,\phi) &:= (A^{1/2}\psi, A^{1/2}\phi) + \sum_{j=1}^N (\widehat{B}^{1/2}\psi, \widehat{B}^{1/2}\phi) + c_0(\psi,\phi), \quad \psi,\phi\in D(\mathsf{s}_0), \end{aligned}$$

where

$$\widehat{B}_j := B_j - b_j$$

and $(\,\cdot\,,\,\cdot\,)$ denotes the inner product of $\mathcal{H}$.

Here we want to construct a self-adjoint extension of S which may be different from $S_{\rm F}$ and $S_{\rm f}$ if S is symmetric, but not essentially self-adjoint. For this purpose we first introduce an approximate or a "cutoff" version of S.

Remark A.3 If each B_j is bounded, then, by the Kato-Rellich theorem, S is self-adjoint. Thus the arguments below are nontrivial only if A and at least one of B_j $(j = 1,\cdots,N)$ are unbounded.

Let $L : (0,\infty) \to (0,\infty)$ be a nondecreasing function such that $L(\kappa)\to\infty$ as $\kappa\to\infty$ and

$$\widehat{B}_j(\kappa) := E_{\widehat{B}_j}([0,L(\kappa)])\widehat{B}_j E_{\widehat{B}_j}([0,L(\kappa)]), \quad \kappa > 0,$$

where $E_{\widehat{B}_j}$ is the spectral measure of $\widehat{B}_j$. It is easy to see that each $\widehat{B}_j(\kappa)$ is a nonnegative bounded self-adjoint operator with $\|\widehat{B}_j(\kappa)\| \le L(\kappa)$. Let

$$S(\kappa) := A + \sum_{j=1}^N \widehat{B}_j(\kappa) + c_0.$$

Then, by the Kato-Rellich theorem, $S(\kappa)$ is self-adjoint with $S(\kappa)\ge c_0$. Moreover, for all $\psi\in\cap_{j=1}^N[D(A)\cap D(B_j)]$, we have

$$\text{s - }\lim_{\kappa\to\infty} S(\kappa)\psi = S\psi.$$

In this sense $S(\kappa)$ may be regarded as an approximate version of S.

Theorem A.1 *Let A, B_j, S and $S(\kappa)$ be as above. Then there exists a unique self-adjoint extension $\widetilde{S}$ of S such that the following properties hold:*

(i) $\widetilde{S}\ge c_0$.

(ii) $D(|\widetilde{S}|^{1/2}) \subset \cap_{j=1}^N \left[D(A^{1/2})\cap D(\widehat{B}_j^{1/2})\right]$.

(iii) *For all* $z \in (\mathbf{C} \setminus \mathbf{R}) \cup \{\xi \in \mathbf{R} | \xi < c_0\}$,

$$\text{s - } \lim_{\kappa \to \infty} (S(\kappa) - z)^{-1} = (\widetilde{S} - z)^{-1}.$$

(iv) *For all* $\xi < c_0$ *and* $\psi \in D(|\widetilde{S}|^{1/2})$,

$$\text{s - } \lim_{\kappa \to \infty} (S(\kappa) - \xi)^{1/2} \psi = (\widetilde{S} - \xi)^{1/2} \psi.$$

Proof: For each $\kappa > 0$, we define a symmetric form s_κ with form domain $D(\mathsf{s}) = D(A^{1/2})$ by

$$\mathsf{s}_\kappa(\psi, \phi) := (A^{1/2}\psi, A^{1/2}\phi) + \sum_{j=1}^{N} (\psi, \widehat{B}_j(\kappa)\phi) + c_0(\psi, \phi), \quad \psi, \phi \in D(A^{1/2}).$$

This is the densely defined closed symmetric form associated with the self-adjoint operator $S(\kappa)$. Since $(\psi, \widehat{B}_j(\kappa)\psi)$ is nondecreasing in κ for all $\psi \in \mathcal{H}$ with

$$0 \leq (\phi, \widehat{B}_j(\kappa)\phi) \leq (\widehat{B}_j^{1/2}\phi, \widehat{B}_j^{1/2}\phi), \quad \phi \in D(\widehat{B}_j^{1/2}),$$

it follows that, for all $\kappa, \kappa' > 0$ with $\kappa < \kappa'$,

$$c_0 \leq \mathsf{s}_\kappa \leq \mathsf{s}_{\kappa'} \leq \mathsf{s}_0.$$

Hence we can apply a general convergence theorem on nondecreasing symmetric forms ([6, p.461, Theorem 3.13]) to conclude that there exists a self-adjoint operator $\widetilde{S}$ on $\mathcal{H}$ such that (i), (iii) and (iv) hold with $\mathsf{s}_\kappa \leq \mathsf{s}$, where s is the symmetric form associated with $\widetilde{S}$, so that $D(|\widetilde{S}|^{1/2}) \subset D(A^{1/2})$. To show that $\widetilde{S}$ is a self-adjoint extension of S, let $\psi \in D(S) = \cap_{j=1}^{N} [D(A) \cap D(B_j)]$ and $\phi \in D(\widetilde{S}) = D(\widetilde{S} - c_0 + 1)$. Then

$$(\psi, (\widetilde{S} - c_0 + 1)\phi) = ((S(\kappa) - c_0 + 1)\psi, (S(\kappa) - c_0 + 1)^{-1}(\widetilde{S} - c_0 + 1)\phi).$$

Note that s - $\lim_{\kappa\to\infty}(S(\kappa) - c_0 + 1)\psi = (S - c_0 + 1)\psi$ and, by property (iii),

$$\text{s - } \lim_{\kappa \to \infty} (S(\kappa) - c_0 + 1)^{-1} = (\widetilde{S} - c_0 + 1)^{-1}.$$

Hence

$$(\psi, (\widetilde{S} - c_0 + 1)\phi) = ((S - c_0 + 1)\psi, (\widetilde{S} - c_0 + 1)^{-1}(\widetilde{S} - c_0 + 1)\phi) = ((S - c_0 + 1)\psi, \phi),$$

which implies that $\psi \in D(\widetilde{S} - c_0 + 1) = D(\widetilde{S})$ and $(\widetilde{S} - c_0 + 1)\psi = (S - c_0 + 1)\psi$, i.e., $\widetilde{S}\psi = S\psi$. Thus $\widetilde{S}$ is a self-adjoint extension of S. We next prove (ii). It follows from the inequality $\mathsf{s}_\kappa \leq \mathsf{s}$ as shown above and the nondecreasingness of s_κ in κ that $D(\mathsf{s}) \subset D(\mathsf{s}_\kappa) = D(A^{1/2})$ and that, for all $\psi \in D(\mathsf{s}) = D(|\widetilde{S}|^{1/2})$, $\lim_{\kappa\to\infty} \mathsf{s}_\kappa(\psi, \psi)$ exists. This implies that $\lim_{\kappa\to\infty}$ $(\widehat{B}_j(\kappa)^{1/2}\psi, \widehat{B}_j(\kappa)^{1/2}\psi)$ exists $(j = 1, \cdots, N)$. By using the spectral representation for $(\widehat{B}_j(\kappa)^{1/2}\psi, \widehat{B}_j(\kappa)^{1/2}\psi)$ and the monotone convergence theorem, we see that $\psi \in D(\widehat{B}_j^{1/2})$, $j = 1, \cdots, N$. Thus part (ii) follows. The uniqueness of $\widetilde{S}$ follows from property (iii). ■

Remark A.4 The self-adjoint extension $\widetilde{S}$ may depend on the choice of the function L. Unfortunately we have been unable to make clear whether $S_\# = \widetilde{S}$ or not $(\# = \mathrm{F}, \mathrm{f})$ in the case where S is symmetric, but not essentially self-adjoint.

B Self-adjoint Extension of the Pauli-Fierz Hamiltonian Without Spin

Let $\boldsymbol{A}^{\mathrm{ex}}$ and ϕ be as in Example 2.1 in Section 2 and

$$P_j := -iD_j - qA_j^{\varrho} - qA_j^{\mathrm{ex}}.$$

We set $\boldsymbol{P} = (P_1, P_2, P_3)$. Then the Pauli-Fierz Hamiltonian without spin is given by

$$h_{\mathrm{PF}} := \frac{\boldsymbol{P}^2}{2m} + \phi + H_{\mathrm{rad}}$$

acting in the Hilbert space $L^2(\mathbf{R}^3) \otimes \mathcal{F}_{\mathrm{rad}} = L^2(\mathbf{R}^3; \mathcal{F}_{\mathrm{rad}}) = \int_{\mathbf{R}^3}^{\oplus} \mathcal{F}_{\mathrm{rad}} d\boldsymbol{x}$. It is easy to see that h_{PF} is Hermitian. We assume Hypothesis (C) in Section 3. Then each P_j is symmetric. Hence we can define a nonnegative self-adjoint operator $K_{\mathrm{PF}}^{(\mathrm{f})}$ as the form sum

$$K_{\mathrm{PF}}^{(\mathrm{f})} := \frac{1}{2m}\left\{ \left(\bar{P}_1\right)^* \bar{P}_1 \dot{+} \left(\bar{P}_2\right)^* \bar{P}_2 \dot{+} \left(\bar{P}_3\right)^* \bar{P}_3 \right\},$$

which is a self-adjoint extension of $K_{\mathrm{PF},0} := (2m)^{-1}\boldsymbol{P}^2$. Hence $K_{\mathrm{PF},0}$ has a self-adjoint extension which is nonnegative. Let K_{PF} be any self-adjoint extension of $K_{\mathrm{PF},0}$ such that $K_{\mathrm{PF}} \geq 0$ and $D(K_{\mathrm{PF}}^{1/2}) \cap D(|\phi|^{1/2}) \cap D(H_{\mathrm{rad}}^{1/2})$ is dense. Then we define

$$h_{\mathrm{PF}}(\kappa) := K_{\mathrm{PF}} + H_{\mathrm{rad}}^{(\kappa)} + \phi^{(\kappa)},$$

where

$$\begin{aligned} H_{\mathrm{rad}}^{(\kappa)} &:= E_{H_{\mathrm{rad}}}([0, L(\kappa)]) H_{\mathrm{rad}} E_{H_{\mathrm{rad}}}([0, L(\kappa)]), \\ \phi^{(\kappa)} &:= (\phi - \phi_0)\chi_{[0,L(\kappa)]}(\phi - \phi_0) + \phi_0, \end{aligned}$$

where $\chi_{[0,L(\kappa)]}$ is the characteristic function of the interval $[0, L(\kappa)]$. Since $H_{\mathrm{rad}}^{(\kappa)} + \phi^{(\kappa)}$ is bounded and symmetric, $h_{\mathrm{PF}}(\kappa)$ is self-adjoint and bounded from below with $h_{\mathrm{PF}}(\kappa) \geq \phi_0$.

Theorem B.1 *Assume Hypothesis (C) in Section 3. Then there exists a unique self-adjoint extension $\widetilde{h}_{\mathrm{PF}}$ of h_{PF} such that the following properties hold:*

(i) $\widetilde{h}_{\mathrm{PF}} \geq \phi_0$.

(ii) $D(|\widetilde{h}_{\mathrm{PF}}|^{1/2}) \subset D(K_{\mathrm{PF}}^{1/2}) \cap D(|\phi|^{1/2}) \cap D(H_{\mathrm{rad}}^{1/2})$.

(iii) *For all $z \in (\mathbf{C} \setminus \mathbf{R}) \cup \{\xi \in \mathbf{R} | \xi < \phi_0\}$,*

$$\operatorname*{s-lim}_{\kappa\to\infty} (h_{\mathrm{PF}}(\kappa) - z)^{-1} = (\widetilde{h}_{\mathrm{PF}} - z)^{-1}.$$

(iv) *For all $\xi < \phi_0$ and $\Psi \in D(|\widetilde{h}_{\mathrm{PF}}|^{1/2})$,*

$$\operatorname*{s-lim}_{\kappa\to\infty} (h_{\mathrm{PF}}(\kappa) - \xi)^{1/2}\Psi = (\widetilde{h}_{\mathrm{PF}} - \xi)^{1/2}\Psi.$$

Proof: We only need to apply Theorem A.1 to the following case:

$$\mathcal{H} = L^2(\mathbf{R}^3; \mathcal{F}_{\mathrm{rad}}),\ A = K_{\mathrm{PF}},\ N = 2,\ B_1 = \phi,\ B_2 = H_{\mathrm{rad}}.$$

■

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