A lower bound on the spectral gap of the 3-dimensional stochastic Ising models

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1 Introduction

Let us consider the Glauber dynamics at low temperature (large $\beta > 0$) which evolves on a cube $\Lambda_d(L) = (-L, L]^d \cap \mathbb{Z}^d$ ($L \in \mathbb{N}$) whose side-length is $2L$ with a boundary condition ω. By $\mathrm{gap}(\Lambda_d(L), \omega)$, we will denote the spectral gap corresponding to a boundary condition ω. Especially, By $\mathrm{gap}(\Lambda_d(L), \phi)$ and $\mathrm{gap}(\Lambda_d(L), +)$, we will mean spectral gaps corresponding to free and + boundary conditions, respectively. L. E. Thomas proved in [Tho,89] that

$$\mathrm{gap}(\Lambda_d(L), \phi) \leq B \exp(-\beta C L^{d-1}) \quad \text{for any } L \in \mathbb{N} \tag{1.1}$$

for any $d \geq 2$ and sufficiently large $\beta > 0$, where $B = B(\beta, d) > 0$ and $C = C(\beta, d) > 0$. For $d = 2$ and any $\beta > \beta_c(2)$, it is known that the speed at which $\mathrm{gap}(\Lambda_2(L), +)$ shrinks to zero as $L \nearrow \infty$ is different from the one at which $\mathrm{gap}(\Lambda_2(L), \phi)$ does (see [Ma,94], [Ma,99] and [CGMS,96]). In this paper, we confirm that it is also true for $d \geq 3$ and sufficiently large $\beta > 0$. In fact, we prove that for sufficiently large $\beta > 0$, some $B > 0$ and some $C > 0$,

$$\mathrm{gap}(\Lambda_d(L), +) \geq B \exp(-\beta C L^{d-2} (\log L)^2) \quad \text{for any } L \in \mathbb{N}. \tag{1.2}$$

For each $\delta \in [0, 1]$, we will consider the boundary condition η_δ which is defined by

$$\eta_\delta(x) = \begin{cases} +1 & \text{if } x^d = -L \text{ and } -\delta L < x^i \leq \delta L \ (i \neq d), \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

For $d = 3$, we also prove that for sufficiently large $\beta > 0$, some $B > 0$ and some $C > 0$,

$$\mathrm{gap}(\Lambda_3(L), \eta_1) \geq B \exp(-\beta C L^{\frac{5}{3}} (\log L)^2) \quad \text{for any } L \in \mathbb{N}, \tag{1.4}$$

which implies at least that the speed at which $\mathrm{gap}(\Lambda_3(L), \eta_1)$ shrinks to zero as $L \nearrow \infty$ is different from the one at which $\mathrm{gap}(\Lambda_3(L), \phi)$ does, as was expected from the result for

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$d = 2$ (see [Ma,94]). The proof of (1.2) and (1.4) goes along the line of [Ma,94], but the dimensionality comes in, and we have to introduce some new geometrical lemmas besides estimates given in [D,72].

Organization of the paper. In Section 1, we will introduce our results and key ingredients of cluster expansion and the notion of standard walls. In Section 2, we will introduce Propositions 2.1 and 2.2 and sketch the proof of our results along the line of [Ma,94]. In Section 3, we will introduce a lemma about cluster expansion and give the proof of Proposition 2.1. In Section 4, we will give the proof of Proposition 2.2. Because of the boundary condition η_1, the standard wall which includes much more boundary faces than interior faces has less energy than we expect from the size of it. For this reason, we decompose such a standard wall into pieces which do not belong the boundary of $Q(\Lambda_3(L))$ (see (4.38)-(4.46)). We analyze the energy-entropy competition coming from these pieces. In Appendix, we will prove geometrical lemmas used in Section 4.

Basic Definitions. For $x = (x^i)_{i=1}^d \in \mathbb{Z}^d$, we will use the l_1-norm $\|x\|_1 = \sum_{i=1}^d |x^i|$ and l_∞-norm $\|x\|_\infty = \max\{|x^1|, \cdots, |x^d|\}$. Let $p = 1$ or $p = \infty$. A set $\Lambda \subset \mathbb{Z}^d$ is said to be l_p-connected if for each distinct $x, y \in \Lambda$, we can find some $\{z_0, \cdots, z_m\} \subset \Lambda$ with $z_0 = x$, $z_m = y$ and $\|z_i - z_{i-1}\|_p = 1$ for all $i \leq m$. The interior and exterior boundaries of a set $\Lambda \subset \mathbb{Z}^d$ will be denoted respectively by

$$\partial_{in}\Lambda = \{x \in \Lambda; \|x - y\|_1 = 1 \text{ for some } y \notin \Lambda\}$$

and

$$\partial_{ex}\Lambda = \{y \notin \Lambda; \|x - y\|_1 = 1 \text{ for some } x \in \Lambda\}.$$

We will use the notation $\Lambda \subset\subset \mathbb{Z}^d$ to indicate that Λ is a non-empty finite subset of $\mathbb{Z}^d$. The number of points contained in a set $\Lambda \subset\subset \mathbb{Z}^d$ will be denoted by $|\Lambda|$.

The boundary conditions and the Gibbs states. In addition to the usual spin configuration spaces

$$\Omega_\Lambda = \{\sigma = (\sigma(x))_{x\in\Lambda}; \sigma(x) = +1 \text{ or } -1\} \quad \text{for any } \Lambda \subset \mathbb{Z}^d,$$

we will introduce a configuration space $\Omega_{\text{b.c.}}$ for boundary conditions

$$\Omega_{\text{b.c.}} = \{\omega = (\omega(x))_{x\in\mathbb{Z}^d}; \omega(x) = +1,\ 0 \text{ or } -1\}.$$

We define $\phi \in \Omega_{\text{b.c.}}$ and $+ \in \Omega_{\text{b.c.}}$ by

$$\phi(x) = 0 \quad \text{for all } x \in \mathbb{Z}^d \quad \text{and} \quad +(x) = +1 \quad \text{for all } x \in \mathbb{Z}^d, \quad \text{respectively.}$$

The set of all real functions on Ω_Λ will be denoted by $\mathcal{C}_\Lambda$. For each $\Lambda \subset\subset \mathbb{Z}^d$ and each $\omega \in \Omega_{\text{b.c.}}$, the *Hamiltonian* $H_\Lambda^\omega \in \mathcal{C}_\Lambda$ is defined by

$$H_\Lambda^\omega(\sigma) = -\frac{1}{2} \sum_{\substack{x,y\in\Lambda \\ \|x-y\|_1=1}} \sigma(x)\sigma(y) - \sum_{\substack{x\in\Lambda,\, y\notin\Lambda \\ \|x-y\|_1=1}} \sigma(x)\omega(y).$$

A *Gibbs state* in $\Lambda \subset\subset \mathbb{Z}^d$ with a boundary condition $\omega \in \Omega_{\text{b.c.}}$ and inverse temperature $\beta > 0$ is the probability distribution μ_Λ^ω such that the probability of each configuration $\sigma \in \Omega_\Lambda$ is given by

$$\mu_\Lambda^\omega(\{\sigma\}) = \frac{1}{Z_\Lambda^\omega} \exp[-\beta H_\Lambda^\omega(\sigma)],$$

where Z_Λ^ω is the normalization constant.

Stochastic Ising models. Let $\Lambda \subset\subset \mathbb{Z}^d$ and let $\omega \in \Omega_{\text{b.c.}}$. The generator of a stochastic Ising model is the linear operator $A_\Lambda^\omega : \mathcal{C}_\Lambda \to \mathcal{C}_\Lambda$ given by

$$A_\Lambda^\omega f(\sigma) = \sum_{x \in \Lambda} c_x^\omega(\sigma)[f(\sigma^x) - f(\sigma)],$$

where $c_x^\omega \in \mathcal{C}_\Lambda$ is the *transition rate* and σ^x is the configuration obtained from σ by replacing $\sigma(x)$ with $-\sigma(x)$. We will assume the following conditions on the transition rate c_x^ω:

(H1) *Detailed balance condition.* It holds that

$$c_x^\omega(\sigma) \exp[-\beta H_{\{x\}}^{\sigma_\Lambda \omega}(\sigma(x))] = c_x^\omega(\sigma^x) \exp[-\beta H_{\{x\}}^{\sigma_\Lambda \omega}(-\sigma(x))], \tag{1.5}$$

where $\sigma_\Lambda \omega$ is the configuration such that $\sigma_\Lambda \omega = \sigma$ on Λ and $\sigma_\Lambda \omega = \omega$ on Λ^c.

(H2) *Positivity and boundedness.* There exist $c_m = c_m(\beta, d) \in (0, \infty)$ and $c_M = c_M(\beta, d) \in (0, \infty)$ such that for any $\Lambda \subset\subset \mathbb{Z}^d$

$$\begin{aligned} c_m &\le \inf\{c_x^\omega(\sigma); x \in \Lambda,\ \omega \in \Omega_{\text{b.c.}} \text{ and } \sigma \in \Omega_\Lambda\} \\ &\le \sup\{c_x^\omega(\sigma); x \in \Lambda,\ \omega \in \Omega_{\text{b.c.}} \text{ and } \sigma \in \Omega_\Lambda\} \le c_M. \end{aligned} \tag{1.6}$$

(H3) *Nearest neighbor interaction.* If $\sigma(y) = \sigma'(y)$ for all y with $\|y - x\|_1 = 1$, then it holds that $c_x^\omega(\sigma) = c_x^\omega(\sigma')$.

(H4) *Attractivity.* If $\sigma \le \sigma'$ and $\sigma(x) = \sigma'(x)$, then it holds that

$$\sigma(x) c_x^\omega(\sigma) \ge \sigma'(x) c_x^\omega(\sigma'). \tag{1.7}$$

An example of functions c_x^ω is given by

$$\begin{aligned} c_x^\omega(\sigma) &= \exp\Big[-\frac{\beta}{2}\Big(H_{\{x\}}^{\sigma_\Lambda \omega}(\sigma^x) - H_{\{x\}}^{\sigma_\Lambda \omega}(\sigma)\Big)\Big] \\ &= \exp\Big[-\beta \sigma(x)\Big(\sum_{y \in \Lambda; \|x-y\|_1 = 1} \sigma(y) + \sum_{y \notin \Lambda; \|x-y\|_1 = 1} \omega(y)\Big)\Big]. \end{aligned}$$

It can be seen by (1.5) that for any $f, g \in \mathcal{C}_\Lambda$

$$\begin{aligned} -\mu_\Lambda^\omega[f A_\Lambda^\omega g] &= -\mu_\Lambda^\omega[g A_\Lambda^\omega f] \\ &= \frac{1}{2} \sum_{x \in \Lambda} \sum_{\sigma \in \Omega_\Lambda} \mu_\Lambda^\omega(\sigma) c_x^\omega(\sigma)[f(\sigma^x) - f(\sigma)][g(\sigma^x) - g(\sigma)]. \end{aligned}$$

Finally, we define

$$\mathrm{gap}(\Lambda,\omega)=\inf\Big\{\frac{-\mu_\Lambda^\omega[fA_\Lambda^\omega f]}{\mu_\Lambda^\omega[\,|f-\mu_\Lambda^\omega[f]|^2\,]};f\in\mathcal{C}_\Lambda\Big\},\tag{1.8}$$

which is the smallest positive eigenvalue of $-A_\Lambda^\omega$, and hence it is called the *spectral gap*.

Main Results. Let $\Lambda_d(L)=(-L,L]^d\cap\mathbb{Z}^d$ for each $L\in\mathbb{N}$. For each $\omega\in\Omega_{\mathrm{b.c.}}$, we define

$$F_L^+(\omega)=\{y\in\partial_{ex}\Lambda_d(L);\omega(y)=+1\}.$$

Theorem 1.1. *Let $d\geq 3$. Consider a stochastic Ising model on the square $\Lambda_d(L)$. Then, there exists $\beta_0=\beta_0(d)>0$ such that for any $\beta\geq\beta_0$ and any $L\in\mathbb{N}$*

$$\mathrm{gap}(\Lambda_d(L),+)\geq B\exp(-\beta CL^{d-2}(\log L)^2)\tag{1.9}$$

holds, where $B=B(c_m,c_M,d)>0$ and $C=C(\beta,d)>0$.

Theorem 1.2. *Let $d=3$. Consider a stochastic Ising model on the square $\Lambda_3(L)$. Suppose that a boundary condition $\omega\in\Omega_{\mathrm{b.c.}}$ satisfies that $\omega(x)\geq 0$ for all $x\in\mathbb{Z}^d$ and*

$$F_L^+(\omega)\supset\{y\in\partial_{ex}\Lambda_3(L);y^3=-L\}.\tag{1.10}$$

Then, there exists $\beta_0'>0$ such that for any $\beta\geq\beta_0'$ and any $L\in\mathbb{N}$

$$\mathrm{gap}(\Lambda_3(L),\omega)\geq B'\exp(-\beta C'L^{\frac{5}{3}}(\log L)^2)\tag{1.11}$$

holds, where $B'=B'(c_m,c_M)>0$ and $C'=C'(\beta)>0$.

Hereafter, we will introduce key ingredients for the proof of Theorems 1.1 and 1.2. We will also introduce some basic lemmas which we will use here.

Block Dynamics (See [Ma,94] and [Ma,99]). From now on, we will use the following modified Hamiltonian for convenience: For each $\Lambda\subset\subset\mathbb{Z}^d$, each $\omega\in\Omega_{\mathrm{b.c.}}$ and each $\mathbb{J}=(\mathrm{J}_{x,y})\in[0,1]^{\mathbb{Z}^d\times\mathbb{Z}^d}$, we define

$$H_\Lambda^{\omega,\mathbb{J}}(\sigma)=-\frac{1}{2}\sum_{\substack{x,y\in\Lambda\\ \|x-y\|_1=1}}(\sigma(x)\sigma(y)-1)-\sum_{\substack{x\in\Lambda,\,y\notin\Lambda\\ \|x-y\|_1=1}}\mathrm{J}_{x,y}(\sigma(x)\omega(y)-1).\tag{1.12}$$

By $\mu_\Lambda^{\omega,\mathbb{J}}$ and $\mathrm{Z}(\Lambda,\omega,\mathbb{J})$, we will denote the Gibbs state in Λ with having $H_\Lambda^{\omega,\mathbb{J}}$ as its Hamiltonian and the normalization constant, respectively. By $\mathrm{gap}(\Lambda,\omega,\mathbb{J})$, we will also denote the spectral gap of the generator

$$(A_\Lambda^{\omega,\mathbb{J}}f)(\sigma)=\sum_{x\in\Lambda}c_x^{\omega,\mathbb{J}}(\sigma)[f(\sigma^x)-f(\sigma)],$$

where $c_x^{\omega,\mathbb{J}}$ satisfy (H1) for $H_\Lambda^{\omega,\mathbb{J}}$, (H3), (H4) and that for any $\Lambda \subset\subset \mathbb{Z}^d$

$$\begin{aligned} c_m &\le \inf\{c_x^{\omega,\mathbb{J}}(\sigma); x \in \Lambda,\ \omega \in \Omega_{\text{b.c.}},\ \sigma \in \Omega_\Lambda \text{ and } \mathbb{J} \in [0,1]^{\mathbb{Z}^d \times \mathbb{Z}^d}\} \\ &\le \sup\{c_x^{\omega,\mathbb{J}}(\sigma); x \in \Lambda,\ \omega \in \Omega_{\text{b.c.}},\ \sigma \in \Omega_\Lambda \text{ and } \mathbb{J} \in [0,1]^{\mathbb{Z}^d \times \mathbb{Z}^d}\} \le c_M. \end{aligned}$$

For a finite family $\{Q_i\}$ with $Q_i \subset \Lambda$ and $\cup_i Q_i = \Lambda$, we define

$$(A_\Lambda^{\{Q_i\},\tau,\mathbb{J}} f)(\sigma) = \sum_i \sum_{\eta \in \Omega_{Q_i}} \mu_{Q_i}^{\sigma_\Lambda \tau,\mathbb{J}}(\eta)[f(\sigma^\eta) - f(\sigma)], \tag{1.13}$$

where σ^η is the configuration such that $\sigma^\eta = \eta$ on Q_i and $\sigma^\eta = \sigma$ on Q_i^c. The dynamics having $A_\Lambda^{\{Q_i\},\tau,\mathbb{J}}$ as its generator is called the *block dynamics*. By $\text{gap}(\Lambda, \{Q_i\}, \omega, \mathbb{J})$, we will denote the spectral gap of the generator $A_\Lambda^{\{Q_i\},\tau,\mathbb{J}}$, and we have that

$$\text{gap}(\Lambda, \{Q_i\}, \omega, \mathbb{J}) = \inf\Big\{ \frac{-\mu_\Lambda^{\omega,\mathbb{J}}[f A_\Lambda^{\{Q_i\},\tau,\mathbb{J}} f]}{\mu_\Lambda^{\omega,\mathbb{J}}[\,|f - \mu_\Lambda^{\omega,\mathbb{J}}[f]|^2]}; f \in \mathcal{C}_\Lambda \Big\}.$$

For each $l \le 2L$, set

$$Q_i = \{x \in \Lambda_d(L); -L + (i-1)l < x^d \le -L + (i+3)l\}.$$

Then, we have (see Section 2 in [Ma,94] or the proof of Theorem 5 in [Sch,94]) that

$$\begin{aligned} &\text{gap}(\Lambda_d(L), \omega, \mathbb{J}) \\ \ge\ & \frac{c_m}{4c_M|Q_i|} \exp\Big[-4\beta\Big(4l \sum_{k=2}^d (2L)^{k-2} + 1\Big)\Big] \,\text{gap}(\Lambda_d(L), \{Q_i\}, \omega, \mathbb{J}). \end{aligned} \tag{1.14}$$

Contours and The Cluster Expansion (See [KP,86]). Let $Q(x) = \prod_{i=1}^d [x^i - (1/2), x^i + (1/2)] \subset \mathbb{R}^d$ and let $Q(V) = \cup_{x \in V} Q(x) \subset \mathbb{R}^d$ for each $V \subset \mathbb{R}^d$. By $\partial Q(V)$, we will denote the boundary of $Q(V)$ in $\mathbb{R}^d$. Let us fix $L \in \mathbb{N}$. For each $L_1 \in \mathbb{Z} \cup \{-\infty, \infty\}$ and each $L_2 \in \mathbb{Z} \cup \{-\infty, \infty\}$, we set

$$\begin{aligned} \Lambda(L_1, L_2) &= \Lambda_d(L_1, L_2) \\ &= \{x \in \mathbb{Z}^d; L_1 < x^d < L_2 + 1,\ -L < x^1, \cdots, x^{d-1} \le L\}. \end{aligned}$$

We will call $\gamma \subset \mathbb{R}^d$ a *contour* (in $\Lambda(L_1, L_2)$) if $\gamma = \partial Q(\Theta)$ for a finite l_∞-connected set $\Theta \subset \Lambda(L_1, L_2)$ which satisfies that Θ^c is l_∞-connected. By $\mathcal{C}(L_1, L_2)$, we will denote the collection of contours in $\Lambda(L_1, L_2)$. For each $n \in \mathbb{Z}$, we define a boundary condition $\omega_n \in \Omega_{\text{b.c.}}$ by

$$\omega_n(x) = \begin{cases} -1 & \text{if } x^d \ge n, \\ +1 & \text{otherwise.} \end{cases} \tag{1.15}$$

We will write $\omega = \omega_0$. Let us fix a negative integer L_1 and $L_2 \in \mathbb{N}$. For each $\sigma \in \Omega_{\Lambda(L_1,L_2)}$, let

$$\Lambda(L_1, L_2, \sigma, \omega) = \left\{ (x,y);\ \begin{array}{l} x, y \in \Lambda(L_1, L_2) \cup \partial_{ex}\Lambda(L_1, L_2) \\ \text{such that } \|x-y\|_1 = 1 \text{ and} \\ \sigma_{\Lambda(L_1,L_2)}\omega(x) \neq \sigma_{\Lambda(L_1,L_2)}\omega(y) \end{array} \right\}.$$

For a given configuration $\sigma \in \Omega_{\Lambda(L_1,L_2)}$, we decompose

$$\cup_{(x,y)\in\Lambda(L_1,L_2,\sigma,\omega)}(Q(x) \cap Q(y)) \tag{1.16}$$

into the connected components. Then, there exists a unique component which does not belong to $\mathcal{C}(L_1, L_2)$. We will call such a component an *open contour* in σ. By $\Gamma^{\omega}_{\Lambda(L_1,L_2)}(\sigma)$, we will denote the open contour in σ. By $\mathcal{O}(L_1, L_2)$, we will denote the collection of open contours for some $\sigma \in \Omega_{\Lambda(L_1,L_2)}$. We define $\mathcal{C}(-\infty, \infty) = \cup_{N\in\mathbb{N}}\mathcal{C}(-N, N)$ and $\mathcal{O}(-\infty, \infty) = \cup_{N\in\mathbb{N}}\mathcal{O}(-N, N)$. We define the maps $\mathrm{h}^+ : \mathcal{O}(-\infty, \infty) \to \mathbb{Z}$ and $\mathrm{h}^- : \mathcal{O}(-\infty, \infty) \to \mathbb{Z}$, respectively, by

$$\mathrm{h}^+(\Gamma) = \max\{x^d + (1/2); x \in \Gamma\} \quad \text{and} \quad \mathrm{h}^-(\Gamma) = \min\{x^d + (1/2); x \in \Gamma\}. \tag{1.17}$$

Let us fix $\Gamma \in \mathcal{O}(L_1, L_2)$ and $\gamma \in \mathcal{C}(L_1, L_2)$ such that $\Gamma \cap \gamma = \emptyset$. Then, there exists the unique configuration $\sigma_\Gamma \in \Omega_{\Lambda(L_1,L_2)}$ which satisfies that Γ is the open contour in σ_Γ, and that there are no contours in σ_Γ. We can also see that there exists the unique configuration $\sigma_{\Gamma,\gamma} \in \Omega_{\Lambda(L_1,L_2)}$ in which Γ is the open contour and γ is also the unique contour. We define

$$\Phi_{\mathbb{J}}(\Gamma) = \exp[-\beta H^{\omega,\mathbb{J}}_{\Lambda(L_1,L_2)}(\sigma_\Gamma)]$$

and

$$\Phi_{\mathbb{J}}(\gamma) = \exp[-\beta(H^{\omega,\mathbb{J}}_{\Lambda(L_1,L_2)}(\sigma_{\Gamma,\gamma}) - H^{\omega,\mathbb{J}}_{\Lambda(L_1,L_2)}(\sigma_\Gamma))].$$

For each pair $\{\gamma_1, \gamma_2\} \subset \mathcal{C}(L_1, L_2) \cup \mathcal{O}(L_1, L_2)$, we will mean by $\gamma_1\iota\gamma_2$ that $\gamma_1 \cap \gamma_2 \neq \emptyset$. We will call a nonempty set $C \subset \mathcal{C}(L_1, L_2)$ a *cluster* if it is not decomposable into two nonempty sets, $C = C_1 \cup C_2$, such that there are no pairs $(\gamma_1, \gamma_2) \in C_1 \times C_2$ satisfying that $\gamma_1\iota\gamma_2$. For each $\Gamma \in \mathcal{O}(L_1, L_2)$ and each cluster $C \subset \mathcal{C}(L_1, L_2)$, $\Gamma\iota C$ will indicate that $\Gamma \cap C \neq \emptyset$.

Let $X \subset \mathcal{C}(L_1, L_2)$. By $\mathcal{D}(X)$, we will denote the family of subsets $\underline{\gamma} \subset X$ such that there are no pairs $\{\gamma_1, \gamma_2\} \subset \underline{\gamma}$ with $\gamma_1\iota\gamma_2$. We define

$$\mathrm{Z}(X; \Phi_{\mathbb{J}}) = \sum_{\underline{\gamma}\in\mathcal{D}(X)} \prod_{\gamma\in\underline{\gamma}} \Phi_{\mathbb{J}}(\gamma)$$

if X is non-empty, and define $\mathrm{Z}(\emptyset; \Phi_{\mathbb{J}}) = 1$. Let $\mathcal{P}_{L_1,L_2}$ be the family of all subsets of $\mathcal{C}(L_1, L_2)$.

Lemma 1.3. *Let functions $a : \mathcal{C}(L_1, L_2) \to [0, \infty)$ and $d : \mathcal{C}(L_1, L_2) \to [0, \infty)$ be such that*

$$\sum_{\gamma' \in \mathcal{C}(L_1, L_2); \gamma' \iota \gamma} \exp[a(\gamma') + d(\gamma')] |\Phi_{\mathbb{J}}(\gamma')| \leq a(\gamma) \tag{1.18}$$

for each $\gamma \in \mathcal{C}(L_1, L_2)$. Then, $\mathrm{Z}(X; \Phi_{\mathbb{J}}) \neq 0$ and there exists a unique function $\Phi_{\mathbb{J}}^{\mathrm{T}} : \mathcal{P}_{L_1, L_2} \to \mathbb{R}$ such that

$$\log \mathrm{Z}(X; \Phi_{\mathbb{J}}) = \sum_{C; C \subset X} \Phi_{\mathbb{J}}^{\mathrm{T}}(C) \tag{1.19}$$

for every $X \subset \mathcal{C}(L_1, L_2)$. Moreover, the function $\Phi_{\mathbb{J}}^{\mathrm{T}}$ is given by the formula

$$\Phi_{\mathbb{J}}^{\mathrm{T}}(C) = \sum_{B; B \subset C} (-1)^{|C| - |B|} \log \mathrm{Z}(B; \Phi_{\mathbb{J}}), \tag{1.20}$$

the estimate

$$\sum_{C \subset X; C \iota \gamma} |\Phi_{\mathbb{J}}^{\mathrm{T}}(C)| \exp[\sum_{\gamma' \in C} d(\gamma')] \leq a(\gamma) \tag{1.21}$$

holds true for every $\gamma \in \mathcal{C}(L_1, L_2)$, and

$$\Phi_{\mathbb{J}}^{\mathrm{T}}(C) = 0 \text{ whenever } C \text{ is not a cluster.} \tag{1.22}$$

Standard Walls (See [D,72]). Let $d \geq 3$. We will introduce the notion of standard walls to express contour models and to use it in the proof of key propositions (Propositions 2.1 and 2.2). We define $\mathrm{H}(z) = \{x \in \mathbb{R}^d; x^d = z\}$ for each $z \in \mathbb{R}$. Let π be the orthogonal projection: $\mathbb{R}^d \to \mathrm{H}(-\frac{1}{2})$ and let $\mathbb{M}^n$ be the n-dimensional standard Hausdorff measure in $\mathbb{R}^d$ for each $n \leq d-1$. We will call w a *face* if

$$w \in \{Q(x) \cap Q(y); x, y \in \mathbb{Z}^d \text{ such that } \|x - y\|_1 = 1\}.$$

Let us fix $\Lambda(L_1, L_2)$ and $\Gamma \in \mathcal{O}(L_1, L_2)$. For a face $w \subset \Gamma$, we will call w a *ceiling face* (of Γ) if $\mathbb{M}^{d-1}(\pi(w)) = 1$ and there is no other face $w' \subset \Gamma$ such that $\pi(w) = \pi(w')$. The other faces (in Γ) will be called *wall faces* (of Γ). By *walls* (of Γ), we will mean connected components of the set

$$\{v \in \mathbb{R}^d; \text{for some wall face } w \text{ of } \Gamma, v \in w\}.$$

Let W' be a wall. Then, there exists a configuration $\sigma \in \Omega_{\Lambda(L_1, L_2)}$ such that the family of walls corresponding to $\Gamma^{\omega}_{\Lambda(L_1, L_2)}(\sigma)$ consists of the only wall W which is obtained by the vertical shift of W'. We will call W a *standard wall*. For the family of walls $\{W_i'\}$ (of Γ), we have a unique family of standard walls $\{W_i\}$ (corresponding to Γ). By $\mathcal{SW}(L_1, L_2)$, we will denote the collection of families of standard walls which correspond to $\Gamma^{\omega}_{\Lambda(L_1, L_2)}(\sigma)$ for some $\sigma \in \Omega_{\Lambda(L_1,L_2)}$. We define $\mathcal{SW}(-\infty,\infty) = \cup_{N\in\mathbb{N}}\mathcal{SW}(-N,N)$. Note that there is one-to-one correspondence between $\mathcal{O}(-\infty,\infty)$ and $\mathcal{SW}(-\infty,\infty)$. By $\mathbb{W}(\Gamma)$ and $\Gamma(\mathbb{W})$, we will mean the family of standard walls corresponding to the open contour $\Gamma \in \mathcal{O}(-\infty,\infty)$ and the open contour corresponding to the family of standard walls $\mathbb{W} \in \mathcal{SW}(-\infty,\infty)$, respectively. For each $\mathbb{W} \in \mathcal{SW}(-\infty,\infty)$, we define

$$\mathrm{h}^+(\mathbb{W}) = \mathrm{h}^+(\Gamma(\mathbb{W})) \qquad \text{and} \qquad \mathrm{h}^-(\mathbb{W}) = \mathrm{h}^-(\Gamma(\mathbb{W})). \tag{1.23}$$

We will introduce a partial order $\prec$ for standard walls of each $\mathbb{W} \in \mathcal{SW}(-\infty,\infty)$. We define the map $\hat{\pi} : \mathbb{R}^d \supset V \mapsto \hat{\pi}(V) \subset \mathrm{H}(-\frac{1}{2})$ by

$$\hat{\pi}(V) = \{x \in \mathrm{H}(-\tfrac{1}{2}); \text{any path } \gamma : x \to \infty \text{ in } \mathrm{H}(-\tfrac{1}{2}) \text{ intersects } \pi(V)\}. \tag{1.24}$$

We define the partial order $\prec$ naturally induced by $(\{\hat{\pi}(W); W \in \mathbb{W}\}, \subset)$ for each $\mathbb{W} \in \mathcal{SW}(-\infty,\infty)$. Let $\mathbb{W} \in \mathcal{SW}(-\infty,\infty)$ and let $W \in \mathbb{W}$. We define

$$\mathbb{W}(W) = \{W' \in \mathbb{W}\backslash\{W\}; W' \succ W\}. \tag{1.25}$$

Note that $\mathbb{W}(W) \in \mathcal{SW}(-\infty,\infty)$. We can see that $x^d = y^d$ for any $x, y \in \Gamma(\mathbb{W}(W))$ such that $\pi(x) \in \hat{\pi}(W)$ and $\pi(y) \in \hat{\pi}(W)$. Then,

$$\mathrm{b}(W,\mathbb{W}) = x^d + (1/2) \quad \text{for some } x \in \Gamma(\mathbb{W}(W)) \text{ such that } \pi(x) \in \hat{\pi}(W) \tag{1.26}$$

is well-defined. For each $\Gamma \in \mathcal{O}(-\infty,\infty)$ and each $W \in \mathbb{W}(\Gamma)$, we also define

$$\mathrm{b}(W,\Gamma) = \mathrm{b}(W,\mathbb{W}(\Gamma)). \tag{1.27}$$

Lemma 1.4. *Let $\mathbb{W} \in \mathcal{SW}(-\infty,\infty)$. Suppose that $\{W_i\}_{i=1}^n \subset \mathbb{W}$ and $W_1 \prec \cdots \prec W_n$. Then, it holds that $\mathbb{M}^{d-1}(W_n) \geq n$.*

Lemma 1.5. *Let $\mathbb{W} \in \mathcal{SW}(-\infty,\infty)$. Suppose that $\{W_1, W_2\} \subset \mathbb{W}$ and $W_1 \prec W_2$. Then, it holds that*

$$\mathrm{h}^+(\mathbb{W}(W_2)) \leq \mathrm{h}^+(\mathbb{W}(W_1))$$

and

$$\mathrm{h}^-(\mathbb{W}(W_2)) \geq \mathrm{h}^-(\mathbb{W}(W_1)).$$

Lemma 1.6. *Let $\mathbb{W} \in \mathcal{SW}(-\infty,\infty)$. Then, it holds that*

$$\mathrm{h}^+(\mathbb{W}) = \max\{\mathrm{h}^+(\mathbb{W}(W) \cup \{W\}); W \textit{ is a minimal standard wall in } \mathbb{W}\}$$

and

$$\mathrm{h}^-(\mathbb{W}) = \min\{\mathrm{h}^-(\mathbb{W}(W) \cup \{W\}); W \textit{ is a minimal standard wall in } \mathbb{W}\}.$$

We omit the proof of these lemmas.

Lemma 1.7. *Suppose that* $\{\mathbb{W}, \mathbb{W}'\} \subset \mathcal{SW}(-\infty, \infty)$ *such that* $\mathbb{W} \cap \mathbb{W}' = \emptyset$ *and* $\mathbb{W} \cup \mathbb{W}' \in \mathcal{SW}(-\infty, \infty)$. *Then, we have that*

$$\mathrm{h}^+(\mathbb{W} \cup \mathbb{W}') \leq \mathrm{h}^+(\mathbb{W}) + \mathrm{h}^+(\mathbb{W}')$$

and

$$\mathrm{h}^-(\mathbb{W} \cup \mathbb{W}') \geq \mathrm{h}^-(\mathbb{W}) + \mathrm{h}^-(\mathbb{W}').$$

Proof. Without loss of generality, we can assume by Lemma 1.6 that there exists a unique minimal standard wall $W \in \mathbb{W} \cup \mathbb{W}'$. Moreover, we can assume that $W \in \mathbb{W}$. Note that there exist unique minimal standard walls in $\mathbb{W}$ and $\mathbb{W}'$, respectively, and that there exist unique maximal standard walls in $\mathbb{W} \cup \mathbb{W}'$, $\mathbb{W}$ and $\mathbb{W}'$, respectively. Rearrange $\mathbb{W} \cup \mathbb{W}'$ in order of $\succ$. The family of standard walls $\mathbb{W} \cup \mathbb{W}'$ is divided into blocks of standard walls each of which is a subset of either $\mathbb{W}$ or $\mathbb{W}'$. We will show Lemma 1.7 only for h^+ by induction on the number of these blocks. To be more precise, we will show by induction that for any $k \in \mathbb{N}$,

$$\mathrm{h}^+(\mathbb{W} \cup \mathbb{W}') \leq \mathrm{h}^+(\mathbb{W}) + \mathrm{h}^+(\mathbb{W}')$$

if

$$\mathbb{W} \cup \mathbb{W}' = \{W^{k+1}_{n_{k+1}}, \cdots, W^{k+1}_1, W^k_{n_k}, \cdots, W^k_1, \cdots, W^1_{n_1}, \cdots, W^1_1\},$$
$$\{W^j_i\}_{i=1}^{n_j} \subset \mathbb{W} \quad \text{for odd } j, \qquad \{W^j_i\}_{i=1}^{n_j} \subset \mathbb{W}' \quad \text{for even } j$$

and

$$\{W^{k+1}_{n_{k+1}} \succ \cdots \succ W^{k+1}_1 \succ W^k_{n_k} \succ \cdots \succ W^k_1 \succ \cdots \succ W^1_{n_1} \succ \cdots \succ W^1_1\}. \tag{1.28}$$

The inequality for h^- can be obtained in a similar way.

We will first consider the case where $k = 1$. In this case, we can see that $W^2_1 \succ W^1_{n_1}$ for the maximal standard wall $W^1_{n_1} \in \mathbb{W}$ and the minimal standard wall $W^2_1 \in \mathbb{W}'$. Then, we can see that

$$(\mathbb{W} \cup \mathbb{W}')(W^1_{n_1}) = \mathbb{W}' \qquad \text{and} \qquad \mathrm{b}(W^1_{n_1}, \mathbb{W} \cup \mathbb{W}') \leq \mathrm{h}^+(\mathbb{W}').$$

Therefore, we have that

$$\begin{aligned} & \mathrm{h}^+(\mathbb{W} \cup \mathbb{W}') \\ = \ & \max\{\mathrm{h}^+((\mathbb{W} \cup \mathbb{W}')(W^1_{n_1})),\ \mathrm{b}(W^1_{n_1}, \mathbb{W} \cup \mathbb{W}') + \mathrm{h}^+(\mathbb{W})\} \\ \leq \ & \mathrm{h}^+(\mathbb{W}') + \mathrm{h}^+(\mathbb{W}), \end{aligned} \tag{1.29}$$

which implies (1.28) for $k = 1$.

We will next consider the case where $k = m$ assuming that (1.28) is true when $k = m - 1$. Suppose that the maximal standard wall in $\mathbb{W} \cup \mathbb{W}'$ is the maximal standard wall in $\mathbb{W}$ (m is even). Note that

$$\mathrm{b}(W^m_{n_m}, \mathbb{W} \cup \mathbb{W}') = \mathrm{b}(W^{m-1}_{n_{m-1}}, \mathbb{W}). \tag{1.30}$$

Then, we have by induction hypothesis, (1.30) and Lemmas 1.5 and 1.6 that

$$\begin{aligned}
& \mathrm{h}^+(\mathbb{W}\cup\mathbb{W}') \\
= \;& \max\{\mathrm{h}^+((\mathbb{W}\cup\mathbb{W}')(W^m_{n_m})), \\
& \qquad\qquad \mathrm{b}(W^m_{n_m}, \mathbb{W}\cup\mathbb{W}') + \mathrm{h}^+(\{W^m_{n_m}, \cdots, W^1_1\})\} \\
\leq \;& \max\{\mathrm{h}^+(\mathbb{W}),\ \mathrm{b}(W^{m-1}_{n_{m-1}}, \mathbb{W}) + \mathrm{h}^+(\mathbb{W}\backslash\{W^{m+1}_i\}_{i=1}^{n_{m+1}}) + \mathrm{h}^+(\mathbb{W}')\} \\
\leq \;& \mathrm{h}^+(\mathbb{W}) + \mathrm{h}^+(\mathbb{W}'),
\end{aligned} \tag{1.31}$$

which implies (1.28) for $k = m$. Similarly, we can obtain (1.31) in the case where the maximal standard wall in $\mathbb{W}\cup\mathbb{W}'$ is the maximal standard wall in $\mathbb{W}'$ (m is odd).

2 Key Propositions for The Proof of Theorems 1.1 and 1.2

We have only to modify Propositions 3.1 and 4.1 in [Ma,94] to prove Theorems 1.1 and 1.2, respectively. Throughout this section, we assume that $\beta > 0$ is sufficiently large. For $r > 0$, $\lceil r\rceil$ stands for the smallest integer larger or equal to r. We will introduce a modified proposition for the proof of Theorem 1.1. We consider the block dynamics generated by $A^{\{Q_i\},+,\mathbb{J}\equiv 1}_{\Lambda_d(L)}$.

Proposition 2.1. *Suppose that $\beta > 0$ is sufficiently large. Let $l = \lceil K_1(\log L)^2\rceil$ and $M \in \mathbb{N}$ with $4l \leq M \leq 2L - l$. Then, for any $\varepsilon > 0$ there exists $K_1 = K_1(\beta,\varepsilon) > 0$ such that for sufficiently large $L \in \mathbb{N}$ and any $x \in \Lambda(-L, -L+M-3l)$*

$$\mu^{+,\mathbb{J}\equiv 1}_{\Lambda(-L,-L+M)}(\sigma(x) = +1) - \mu^{\eta,\mathbb{J}\equiv 1}_{\Lambda(-L,-L+M)}(\sigma(x) = +1) \leq \varepsilon L^{-d}, \tag{2.32}$$

where we write $\eta = \omega_{-L+M}$ (see (1.15)).

We can obtain from (2.32) (see Section 3 in [Ma,94]) that for sufficiently large $L \in \mathbb{N}$,

$$\mathrm{gap}(\Lambda_d(L), \{Q_i\}, +, \mathbb{J}\equiv 1) \geq \exp[-L]. \tag{2.33}$$

From (1.14) and (2.33), we have that for some $B = B(c_m, c_M, d) > 0$, some $C = C(K_1, \beta, d) > 0$ and for any $L \in \mathbb{N}$

$$\mathrm{gap}(\Lambda_d(L), +, \mathbb{J}\equiv 1) \geq B\exp[-\beta C L^{d-2}(\log L)^2],$$

which implies (1.9).

We will introduce a modified proposition for the proof of Theorem 1.2. Let $\hat{\mathbb{J}}_\delta$ be given by

$$\bar{\mathrm{J}}_{x,y} = \begin{cases} 1 & \text{if } x \in \Lambda_3(L),\ x^3 = -L+1,\ y \in \partial_{ex}\Lambda_3(L) \text{ and } y^3 = -L, \\ \delta & \text{otherwise.} \end{cases}$$

From the direct calculation, we have that

$$\mathrm{gap}(\Lambda_3(L), +, \hat{\mathbb{J}}_0) \geq \frac{c_m}{c_M} \exp[-160\beta\delta L^2]\,\mathrm{gap}(\Lambda_3(L), +, \hat{\mathbb{J}}_\delta). \tag{2.34}$$

We consider the block dynamics generated by $A^{\{Q_i\},+,\hat{\mathbb{J}}_\delta}_{\Lambda_3(L)}$.

Proposition 2.2. *Suppose that $d = 3$ and $\beta > 0$ is sufficiently large. Let $l = \lceil K_2 L^{\frac{2}{3}} (\log L)^2 \rceil$ and $M \in \mathbb{N}$ with $4l \leq M \leq 2L - l$. Let $\delta = L^{-\frac{2}{3}} \log L$. Then, for any $\varepsilon > 0$ there exists $K_2 = K_2(\beta, \varepsilon) > 0$ such that for sufficiently large $L \in \mathbb{N}$ and any $x \in \Lambda(-L, -L + M - 3l)$*

$$\mu^{+,\hat{\mathbb{J}}_\delta}_{\Lambda(-L,-L+M)}(\sigma(x) = +1) - \mu^{\eta,\hat{\mathbb{J}}_\delta}_{\Lambda(-L,-L+M)}(\sigma(x) = +1) \leq \varepsilon L^{-3}, \tag{2.35}$$

where we write $\eta = \omega_{-L+M}$ (see (1.15)).

We can obtain from (2.35) that for sufficiently large $L \in \mathbb{N}$,

$$\mathrm{gap}(\Lambda_3(L), \{Q_i\}, +, \hat{\mathbb{J}}_\delta) \geq \exp[-L]. \tag{2.36}$$

From (1.14), (2.34) and (2.36), we have that for some $B' = B'(c_m, c_M) > 0$, some $C' = C'(K_2, \beta) > 0$ and for any $L \in \mathbb{N}$,

$$\mathrm{gap}(\Lambda_3(L), +, \hat{\mathbb{J}}_0) \geq B' \exp[-\beta C' L^{\frac{5}{3}} (\log L)^2],$$

which implies (1.11).

3 Proof of Proposition 2.1

For simplicity we will prove Proposition 2.1 for $d = 3$. Let us fix $M \in \mathbb{N}$ with $4l \leq M \leq 2L - l$. We will omit the notation $\mathbb{J} \equiv 1$. We will write $\eta = \omega_{-L+M}$ and $\zeta = \omega_{-L+M+2l}$ (see (1.15)) in this section. We assume that $\beta > 0$ is sufficiently large throughout this section. We define open contours for each $\sigma \in \Omega_{\Lambda(-L,-L+M)}$ under the boundary conditions η and ζ (see (1.16)). By $\Gamma^\eta_{\Lambda(-L,-L+M)}(\sigma)$ and $\Gamma^\zeta_{\Lambda(-L,-L+M)}(\sigma)$, we will denote the open contours for each $\sigma \in \Omega_{\Lambda(-L,-L+M)}$ under the boundary conditions η and ζ, respectively. We consider the events

$$\mathcal{A}^\eta_{\Lambda(-L,-L+M)} = \{\sigma \in \Omega_{\Lambda(-L,-L+M)}; \mathrm{h}^-(\Gamma^\eta_{\Lambda(-L,-L+M)}(\sigma)) > -L + M - 3l\}$$

and

$$\mathcal{A}^\zeta_{\Lambda(-L,-L+M)} = \{\sigma \in \Omega_{\Lambda(-L,-L+M)}; \mathrm{h}^-(\Gamma^\zeta_{\Lambda(-L,-L+M)}(\sigma)) > -L + M - 3l\}.$$

Then, we have by FKG inequality that for any $x \in \Lambda(-L,-L+M-3l)$,

$$\begin{aligned}
& \mu^{+}_{\Lambda(-L,-L+M)}(\sigma(x)=+1)-\mu^{\eta}_{\Lambda(-L,-L+M)}(\sigma(x)=+1) \\
\leq\ & \mu^{+}_{\Lambda(-L,-L+M)}(\sigma(x)=+1) \\
& \quad -\mu^{\eta}_{\Lambda(-L,-L+M)}(\sigma(x)=+1 \mid \mathcal{A}^{\eta}_{\Lambda(-L,-L+M)})\mu^{\eta}_{\Lambda(-L,-L+M)}(\mathcal{A}^{\eta}_{\Lambda(-L,-L+M)}) \\
\leq\ & \mu^{\eta}_{\Lambda(-L,-L+M)}((\mathcal{A}^{\eta}_{\Lambda(-L,-L+M)})^{c}) \\
\leq\ & \mu^{\zeta}_{\Lambda(-L,-L+M)}((\mathcal{A}^{\zeta}_{\Lambda(-L,-L+M)})^{c}).
\end{aligned} \tag{3.1}$$

We define the event

$$\mathcal{A}^{\omega}_{\Lambda(-2L+2l,2l)}=\{\sigma\in\Omega_{\Lambda(-2L+2l,2l)};\mathrm{h}^{-}(\Gamma^{\omega}_{\Lambda(-2L+2l,2l)}(\sigma))>-l\},$$

where we write $\omega=\omega_0$ (see (1.15)). We also have by FKG inequality that

$$\mu^{\zeta}_{\Lambda(-L,-L+M)}((\mathcal{A}^{\zeta}_{\Lambda(-L,-L+M)})^{c})\leq\mu^{\omega}_{\Lambda(-2L+2l,2l)}((\mathcal{A}^{\omega}_{\Lambda(-2L+2l,2l)})^{c}). \tag{3.2}$$

Therefore, we can obtain Proposition 2.1 from (3.1), (3.2) and the following lemma.

Lemma 3.1. *Suppose that $\beta>0$ is sufficiently large. Let $l=\lceil K_1(\log L)^2\rceil$. Then, for any $\varepsilon>0$ there exists $K_1=K_1(\beta,\varepsilon)>0$ such that for sufficiently large $L\in\mathbb{N}$*

$$\mu^{\omega}_{\Lambda(-2L+2l,2l)}((\mathcal{A}^{\omega}_{\Lambda(-2L+2l,2l)})^{c})\leq\varepsilon L^{-3}. \tag{3.3}$$

We will use cluster expansion to prove Lemma 3.1. Recall that by $\mathbb{W}(\Gamma)$ and $\Gamma(\mathbb{W})$, we will mean the family of standard walls corresponding to the open contour Γ and the open contour corresponding to the family of standard walls $\mathbb{W}$, respectively, and that $\mathcal{SW}(-\infty,\infty)=\cup_{N\in\mathbb{N}}\mathcal{SW}(-N,N)$. We define $\mathcal{SW}(-L_1,\infty)=\cup_{N\in\mathbb{N}}\mathcal{SW}(-L_1,N)$ and $\mathcal{SW}(-\infty,L_1)=\cup_{N\in\mathbb{N}}\mathcal{SW}(-N,L_1)$ for each $L_1\in\mathbb{N}$. For each $\mathbb{W}\in\mathcal{SW}(-\infty,\infty)$, we define

$$\begin{aligned}
\Phi(\mathbb{W}) &= \Phi(\Gamma(\mathbb{W}))/e^{-8\beta L^2} \\
&= \exp[-\beta H^{\omega}_{\Lambda(-N,N)}(\sigma_{\Gamma(\mathbb{W})})]/e^{-8\beta L^2}
\end{aligned} \tag{3.4}$$

for some $N\in\mathbb{N}$ such that $\Gamma(\mathbb{W})\in\mathcal{O}(-N,N)$. We can see that it is well-defined. We define

$$\mathbf{C}=\cup_{N\in\mathbb{N}}\{C\subset\mathcal{C}(-N,N);C\text{ is a cluster}\}.$$

We also define for each $n\in\mathbb{N}$

$$\mathbf{C}(n)=\{C\in\mathbf{C};\mathrm{diam}(C)\geq n\}, \tag{3.5}$$

where $\mathrm{diam}(C)=\sup\{\mathrm{dist}_1(x,y);x,y\in C\}$ and dist_1 is the metric induced by l_1-norm. For each $\Gamma\in\mathcal{O}(-\infty,\infty)$, let $\sum_{C\iota\Gamma}$ stand for the summation over all elements $C\in\mathbf{C}$ such that $C\iota\Gamma$. For each $\sigma\in\Omega_{\Lambda(-\infty,\infty)}$, let

$$\Lambda(\sigma,\omega)=\left\{(x,y);\ \begin{array}{l} x,y\in\Lambda(-\infty,\infty)\cup\partial_{ex}\Lambda(-\infty,\infty)\text{ such that} \\ \|x-y\|_1=1\text{ and }\sigma_{\Lambda(-\infty,\infty)}\omega(x)\neq\sigma_{\Lambda(-\infty,\infty)}\omega(y)\end{array}\right\}.$$

For a given configuration $\sigma \in \Omega_{\Lambda(-\infty,\infty)}$, we decompose

$$\cup_{(x,y)\in\Lambda(\sigma,\omega)}(Q(x)\cap Q(y))$$

into the connected components. Then, there exists a unique connected component which includes a point $(L+1,0,-1/2)$. We will denote such a component by $\Gamma^{\omega}_{\Lambda(-\infty,\infty)}(\sigma)$. Note that $\Gamma^{\omega}_{\Lambda(-\infty,\infty)}(\sigma)$ includes

$$\cup_{(x,y)\in\Lambda_{ex}(\omega)}(Q(x)\cap Q(y)),$$

where

$$\Lambda_{ex}(\omega) = \left\{ (x,y);\ \begin{array}{l} x,y\in\partial_{ex}\Lambda(-\infty,\infty) \text{ such that} \\ \|x-y\|_1 = 1 \text{ and } \omega(x)\neq\omega(y) \end{array} \right\}.$$

Let $L_1 \in \mathbb{N}$. We similarly define $\Gamma^{\omega}_{\Lambda(-L_1,\infty)}(\sigma)$ and $\Gamma^{\omega}_{\Lambda(-\infty,L_1)}(\sigma)$ for each $\sigma \in \Omega_{\Lambda(-L_1,\infty)}$ and each $\sigma \in \Omega_{\Lambda(-\infty,L_1)}$, respectively.

Lemma 3.2. *Suppose that $\beta > 0$ is sufficiently large, $L_1 \in \mathbb{N}$ and $L_2 \in \mathbb{N}$. Let $\mathbf{1}_{L_1,L_2} : \mathbf{C} \to \{0,1\}$ be the indicator function of the event that $C \subset \mathcal{C}(-L_1,L_2)$. Then, there exists a function Φ^{T} such that for each $\Gamma \in \mathcal{O}(-L_1,L_2)$*

$$\begin{aligned}\mu^{\omega}_{\Lambda(-L_1,L_2)}&(\Gamma^{\omega}_{\Lambda(-L_1,L_2)}(\sigma)=\Gamma) \\ &= \mathrm{Z}(\mathcal{SW}(-L_1,L_2))^{-1}\Phi(\mathbb{W}(\Gamma))\exp[-\sum_{C\iota\Gamma}\mathbf{1}_{L_1,L_2}(C)\Phi^{\mathrm{T}}(C)],\end{aligned} \tag{3.6}$$

where

$$\mathrm{Z}(\mathcal{SW}(-L_1,L_2)) = \sum_{\mathbb{W}\in\mathcal{SW}(-L_1,L_2)}\Phi(\mathbb{W})\exp[-\sum_{C\iota\Gamma(\mathbb{W})}\mathbf{1}_{L_1,L_2}(C)\Phi^{\mathrm{T}}(C)].$$

Moreover, we have that

$$\mathrm{Z}(\mathcal{SW}(-\infty,\infty)) = \sum_{\mathbb{W}\in\mathcal{SW}(-\infty,\infty)}\Phi(\mathbb{W})\exp[-\sum_{C\iota\Gamma(\mathbb{W})}\Phi^{\mathrm{T}}(C)] < \infty, \tag{3.7}$$

and that for each $\Gamma \in \mathcal{O}(-\infty,\infty)$

$$\begin{aligned}\mu^{\omega}_{\Lambda(-\infty,\infty)}&(\Gamma^{\omega}_{\Lambda(-\infty,\infty)}(\sigma)=\Gamma) \\ &= \mathrm{Z}(\mathcal{SW}(-\infty,\infty))^{-1}\Phi(\mathbb{W}(\Gamma))\exp[-\sum_{C\iota\Gamma}\Phi^{\mathrm{T}}(C)].\end{aligned} \tag{3.8}$$

We also have that for each $\Gamma \in \mathcal{O}(-L_1,\infty)$

$$\begin{aligned}\mu^{\omega}_{\Lambda(-L_1,\infty)}&(\Gamma^{\omega}_{\Lambda(-L_1,\infty)}(\sigma)=\Gamma) \\ &= \mathrm{Z}(\mathcal{SW}(-L_1,\infty))^{-1}\Phi(\mathbb{W}(\Gamma))\exp[-\sum_{C\iota\Gamma}\mathbf{1}_{L_1,\infty}(C)\Phi^{\mathrm{T}}(C)],\end{aligned} \tag{3.9}$$

where

$$\mathrm{Z}(\mathcal{SW}(-L_1,\infty)) = \sum_{\mathbb{W}\in\mathcal{SW}(-L_1,\infty)} \Phi(\mathbb{W})\exp[-\sum_{C\iota\Gamma(\mathbb{W})} \mathbf{1}_{L_1,\infty}(C)\Phi^{\mathrm{T}}(C)], \tag{3.10}$$

and that for each $\Gamma \in \mathcal{O}(-\infty, L_1)$

$$\begin{aligned} &\mu^{\omega}_{\Lambda(-\infty,L_1)}(\Gamma^{\omega}_{\Lambda(-\infty,L_1)}(\sigma) = \Gamma) \\ &\quad = \mathrm{Z}(\mathcal{SW}(-\infty,L_1))^{-1}\Phi(\mathbb{W}(\Gamma))\exp[-\sum_{C\iota\Gamma}\mathbf{1}_{\infty,L_1}(C)\Phi^{\mathrm{T}}(C)], \end{aligned} \tag{3.11}$$

where

$$\mathrm{Z}(\mathcal{SW}(-\infty,L_1)) = \sum_{\mathbb{W}\in\mathcal{SW}(-\infty,L_1)} \Phi(\mathbb{W})\exp[-\sum_{C\iota\Gamma(\mathbb{W})} \mathbf{1}_{\infty,L_1}(C)\Phi^{\mathrm{T}}(C)]. \tag{3.12}$$

Proof. Let us fix $L_1 \in \mathbb{N}$ and $L_2 \in \mathbb{N}$. Set $\Lambda = \Lambda(-L_1, L_2)$ and $\Gamma(\cdot) = \Gamma^{\omega}_{\Lambda(-L_1,L_2)}(\cdot)$. Let $\Delta(\Gamma) = \{x \in \mathbb{Z}^3; \mathrm{dist}_\infty(x,\Gamma) = 1/2\}$ for each $\Gamma \in \mathcal{O}(-L_1, L_2)$, where dist_∞ is the metric induced by l_∞-norm. Let us fix $\Gamma \in \mathcal{O}(-L_1, L_2)$. We decompose $\Lambda\backslash\Delta(\Gamma)$ into two sets R^+_Γ and R^-_Γ which lie, in a natural way, below and above Γ, respectively. We have by Lemma 1.3 that

$$\begin{aligned} \sum_{\sigma\in\Omega_\Lambda;\Gamma(\sigma)=\Gamma} \exp[-\beta H^{\omega}_{\Lambda}(\sigma)] &= \Phi(\Gamma)\mathrm{Z}(R^+_\Gamma,+)\mathrm{Z}(R^-_\Gamma,-) \\ &= \mathrm{Z}(\Lambda,+)\Phi(\Gamma)\exp[-\sum_{C\iota\Gamma}\mathbf{1}_{L_1,L_2}(C)\Phi^{\mathrm{T}}(C)]. \end{aligned} \tag{3.13}$$

Set in Lemma 1.3, $a(\cdot) = 6\mathbb{M}^2(\cdot)$ and $d(\cdot) = (\beta - \hat{\beta})\mathbb{M}^2(\cdot)$ for sufficiently large $\hat{\beta} > 0$. We have that for any $\gamma \in \mathcal{C}(-L_1, L_2)$

$$\Phi(\gamma) = \exp[-2\beta\mathbb{M}^2(\gamma)]. \tag{3.14}$$

Therefore, we can see by Lemma 1.3 that there exist $c_1 > 0$ and $c_2 > 0$ such that for each $x \in \Lambda(-\infty,\infty)$ and any $n \in \mathbb{N}$ (see (3.5)),

$$\sum_{C\in\mathbf{C}(n);C\iota\partial Q(x)} |\Phi^{\mathrm{T}}(C)| \le c_1\exp[-(\beta-\hat{\beta})c_2 n] \tag{3.15}$$

Note that $\hat{\beta}$, c_1 and c_2 do not depend on $\beta > 0$, $L_1 \in \mathbb{N}$, $L_2 \in \mathbb{N}$ and $L \in \mathbb{N}$.

For each $\Gamma \in \mathcal{O}(-L_1, L_2)$, we can see from (3.4) and (3.13) that

$$\begin{aligned} &\sum_{\sigma\in\Omega_\Lambda;\Gamma(\sigma)=\Gamma} \exp[-\beta H^{\omega}_{\Lambda}(\sigma)] \\ &\quad = e^{-8\beta L^2}\mathrm{Z}(\Lambda,+)\Phi(\mathbb{W}(\Gamma))\exp[-\sum_{C\iota\Gamma}\mathbf{1}_{L_1,L_2}(C)\Phi^{\mathrm{T}}(C)]. \end{aligned} \tag{3.16}$$

Then, we have that

$$\mu^\omega_\Lambda(\Gamma(\sigma)=\Gamma) = \mathrm{Z}(\mathcal{SW}(-L_1,L_2))^{-1}\Phi(\mathbb{W}(\Gamma))\exp[-\sum_{C\iota\Gamma}\mathbf{1}_{L_1,L_2}(C)\Phi^{\mathrm{T}}(C)], \tag{3.17}$$

where

$$\mathrm{Z}(\mathcal{SW}(-L_1,L_2)) = \sum_{\mathbb{W}\in\mathcal{SW}(-L_1,L_2)}\Phi(\mathbb{W})\exp[-\sum_{C\iota\Gamma(\mathbb{W})}\mathbf{1}_{L_1,L_2}(C)\Phi^{\mathrm{T}}(C)].$$

We obtained (3.6). We can also obtain from (3.4) and (3.15) that

$$\sum_{\mathbb{W}\in\mathcal{SW}(-\infty,\infty)}\Phi(\mathbb{W})\exp[\sum_{C\iota\Gamma(\mathbb{W})}|\Phi^{\mathrm{T}}(C)|] < \infty,$$

which implies (3.7). We have by the definition of $\mathrm{Z}(\mathcal{SW}(-\infty,\infty))$ that

$$\lim_{N\to\infty}\mathrm{Z}(\mathcal{SW}(-N,N)) = \mathrm{Z}(\mathcal{SW}(-\infty,\infty)),$$

which implies (3.8). Similarly, we can obtain (3.9) and (3.11).

Let $L_1\in\mathbb{N}\cup\{\infty\}$ and $L_2\in\mathbb{N}\cup\{\infty\}$. For each $\mathbb{W}\in\mathcal{SW}(-L_1,L_2)$, we define

$$P_{\Lambda(-L_1,L_2)}(\mathbb{W}) = \mathrm{Z}(\mathcal{SW}(-L_1,L_2))^{-1}\Phi(\mathbb{W})\exp[-\sum_{C\iota\Gamma(\mathbb{W})}\mathbf{1}_{L_1,L_2}(C)\Phi^{\mathrm{T}}(C)]. \tag{3.18}$$

Proof of Lemma 3.1. Let $\Lambda=\Lambda(-2L+2l,2l)$ and $L'=2L-2l$. We have by FKG inequality that

$$\mu^\omega_\Lambda((\mathcal{A}^\omega_\Lambda)^c) \le \mu^\omega_{\Lambda(-\infty,2l)}(\mathcal{A}'_1), \tag{3.19}$$

where

$$\mathcal{A}'_1 = \{\sigma\in\Omega_{\Lambda(-\infty,2l)};\mathrm{h}^-(\Gamma^\omega_{\Lambda(-\infty,2l)}(\sigma))\le -l\}.$$

We also have by FKG inequality that

$$\mu^\omega_{\Lambda(-\infty,2l)}(\mathcal{A}'_2) \le \mu^\omega_{\Lambda(-\infty,\infty)}(\mathcal{A}''_2), \tag{3.20}$$

where

$$\mathcal{A}'_2 = \{\sigma\in\Omega_{\Lambda(-\infty,2l)};\mathrm{h}^+(\Gamma^\omega_{\Lambda(-\infty,2l)}(\sigma))\ge l\}$$

and

$$\mathcal{A}''_2 = \{\sigma\in\Omega_{\Lambda(-\infty,\infty)};\mathrm{h}^+(\Gamma^\omega_{\Lambda(-\infty,\infty)}(\sigma))\ge l\}.$$

Let

$$\widehat{\mathcal{SW}}^- = \{\mathbb{W}\in\mathcal{SW}(-\infty,2l);\mathrm{h}^-(\mathbb{W})\le -l,\ \mathrm{h}^+(\mathbb{W})< l\}$$

and let

$$\widehat{\mathcal{SW}}^{+} = \{\mathbb{W} \in \mathcal{SW}(-\infty,\infty); \mathrm{h}^{+}(\mathbb{W}) \geq l\}.$$

Then, we have from (3.8), (3.11), (3.18), (3.19) and (3.20) that

$$\begin{aligned}\mu^{\omega}_{\Lambda}((\mathcal{A}^{\omega}_{\Lambda})^{c}) &\leq \mu^{\omega}_{\Lambda(-\infty,2l)}(\mathcal{A}'_{1}\backslash\mathcal{A}'_{2}) + \mu^{\omega}_{\Lambda(-\infty,\infty)}(\mathcal{A}''_{2}) \\ &= \sum_{\mathbb{W}\in\widehat{\mathcal{SW}}^{-}} P_{\Lambda(-\infty,2l)}(\mathbb{W}) + \sum_{\mathbb{W}\in\widehat{\mathcal{SW}}^{+}} P_{\Lambda(-\infty,\infty)}(\mathbb{W}). \end{aligned} \tag{3.21}$$

We will estimate the first term in RHS of (3.21) from above. Note that Φ^{T} is invariant under the vertical shift. Therefore, we can show from (3.15) that there exists $c_3 = c_3(\beta, c_1, c_2) > 0$ such that for any $\mathbb{W} \in \mathcal{SW}(-\infty,\infty)$,

$$\Big|\sum_{C\iota\Gamma(\mathbb{W}\backslash\{W\})} \Phi^{\mathrm{T}}(C) - \sum_{C\iota\Gamma(\mathbb{W})} \Phi^{\mathrm{T}}(C)\Big| \leq c_3\mathbb{M}^2(W) \quad \text{for all } W \in \mathbb{W}. \tag{3.22}$$

Note that c_3 is independent of $L \in \mathbb{N}$. We can also see from (3.15) that

$$2\sum_{x\in\Lambda; x^3=2l} \Big|\sum_{C\in\mathbf{C}(n); C\iota\partial Q(x)} \Phi^{\mathrm{T}}(C)\Big| \leq 8L^2 c_1 \exp[-(\beta-\hat{\beta})c_2K_1(\log L)^2/2]. \tag{3.23}$$

Let $\{c_i\}_{i\geq 4}$ be some positive constants which may depend on $\hat{\beta} > 0$ and $\beta > 0$. Note that by the definition of standard walls

$$\mathbb{M}^2(\pi(W)) \leq \frac{1}{3}\mathbb{M}^2(W) \tag{3.24}$$

for any $\{W\} \in \mathcal{SW}(-\infty,\infty)$. For any pair $\{\mathbb{W},\mathbb{W}'\} \subset \mathcal{SW}(-\infty,2l)$ such that $\mathbb{W}' = \mathbb{W}\backslash\{W\}$ for some $W \in \mathbb{W}$, $\mathrm{h}^{+}(\mathbb{W}) < l$ and $\mathrm{h}^{+}(\mathbb{W}') \leq 3l/2$, we have from (3.22), (3.23) and (3.24) that for sufficiently large $L \in \mathbb{N}$,

$$\begin{aligned} & P_{\Lambda(-\infty,2l)}(\mathbb{W})\Big/P_{\Lambda(-\infty,2l)}(\mathbb{W}') \\ =\; & \Big(\Phi(\mathbb{W})\Big/\Phi(\mathbb{W}')\Big) \times \\ & \qquad \Big(\exp[-\sum_{C\iota\Gamma(\mathbb{W})}\mathbf{1}_{\infty,2l}(C)\Phi^{\mathrm{T}}(C)]\Big/\exp[-\sum_{C\iota\Gamma(\mathbb{W}')}\mathbf{1}_{\infty,2l}(C)\Phi^{\mathrm{T}}(C)]\Big) \\ \leq\; & \Phi(\{W\})\exp[c_3\mathbb{M}^2(W) + c_4e^{-c_5(\beta-\hat{\beta})K_1(\log L)^2}] \\ \leq\; & c_6\exp[-2\beta\big(\mathbb{M}^2(W) - \mathbb{M}^2(\pi(W))\big) + c_3\mathbb{M}^2(W)] \\ \leq\; & c_6\exp[-(\beta-c_3)\mathbb{M}^2(W)]. \end{aligned} \tag{3.25}$$

Let $K_3 > 0$ be a large constant to be specified later. Suppose that $L \in \mathbb{N}$ is sufficiently large and $K_1 > 2(K_3)^2$. Then, for any $\mathbb{W} \in \widehat{\mathcal{SW}}^{-}$ there exists $W \in \mathbb{W}$ such that $\mathbb{M}^2(W) \geq$

$K_3 \log L$, $\mathbb{W}\backslash\{W\} \in \mathcal{SW}(-\infty, 2l)$ and $\mathrm{h}^+(\mathbb{W}\backslash\{W\}) \leq 3l/2$. To see this, we will first show that the set

$$\mathbb{W}_{big} = \{W' \in \mathbb{W}; \mathbb{M}^2(W') \geq K_3 \log L\} \tag{3.26}$$

is not empty. Assuming that $\mathbb{W}_{big}$ is empty, we can see by Lemma 1.4 that

$$\text{the number of elements of } \mathbb{W}(W) < \lfloor K_3 \log L \rfloor$$

for all $W \in \mathbb{W}$, where $\lfloor r \rfloor$ denotes the integer part of r for each $r \in \mathbb{R}$. By this, the hypothesis that $\mathbb{M}^2(W) < K_3 \log L$ for all $W \in \mathbb{W}$, Lemmas 1.5 and 1.7, we have that

$$\mathrm{h}^-(\mathbb{W}) > -(K_3 \log L)^2 > -l, \tag{3.27}$$

which contradicts that $\mathbb{W} \in \widehat{\mathcal{SW}}^-$. Since $\mathbb{W}_{big}$ is not empty, we can take a minimal element W from $\mathbb{W}_{big}$. We will next show that $\mathrm{h}^+(\mathbb{W}\backslash\{W\}) \leq 3l/2$. We can see by Lemmas 1.5, 1.6 and 1.7 that

$$\begin{aligned}
\mathrm{h}^+(\mathbb{W}\backslash\{W\}) &= \max\{\mathrm{h}^+(\mathbb{W}),\ \mathrm{h}^+(\mathbb{W}(W) \cup \{W' \in \mathbb{W}\backslash\{W\}; W' \prec W\})\} \\
&\leq \max\{\mathrm{h}^+(\mathbb{W}),\ \mathrm{h}^+(\mathbb{W}(W)) + \mathrm{h}^+(\{W' \in \mathbb{W}\backslash\{W\}; W' \prec W\})\} \\
&\leq \mathrm{h}^+(\mathbb{W}) + \mathrm{h}^+(\{W' \in \mathbb{W}\backslash\{W\}; W' \prec W\}).
\end{aligned} \tag{3.28}$$

We can also see by Lemmas 1.4, 1.7 and the definition of W that

$$\mathrm{h}^+(\{W' \in \mathbb{W}\backslash\{W\}; W' \prec W\}) < (K_3 \log L)^2 < l/2, \tag{3.29}$$

which together with $\mathbb{W} \in \widehat{\mathcal{SW}}^-$ and (3.28) implies that $\mathrm{h}^+(\mathbb{W}\backslash\{W\}) \leq 3l/2$. Note that W is the union of all faces belonging to W. Therefore, we have from (3.25) that

$$\begin{aligned}
&\sum_{\mathbb{W} \in \widehat{\mathcal{SW}}^-} P_{\Lambda(-\infty,2l)}(\mathbb{W}) \\
&\leq \Big(\sum_{\mathbb{W}' \in \mathcal{SW}(-\infty,2l)} P_{\Lambda(-\infty,2l)}(\mathbb{W}') \Big) \times \Big(4L^2 \sum_{n \geq K_3 \log L} \kappa^n c_6 e^{-(\beta - c_3)n} \Big) \\
&\leq c_7 \exp[-(\beta - c_8) K_3 \log L],
\end{aligned} \tag{3.30}$$

where $\kappa > 0$ is the connectivity constant.

Similarly, we can estimate the second term in RHS of (3.21) from above by

$$c_9 \exp[-(\beta - c_{10}) K_3 \log L]. \tag{3.31}$$

From (3.21), (3.30) and (3.31), we have that

$$\mu_\Lambda^\omega((\mathcal{A}_\Lambda^\omega)^c) \leq 2c_{11} \exp[-(\beta - c_{12}) K_3 \log L]. \tag{3.32}$$

Let us fix $\varepsilon > 0$. From (3.32), we can obtain (3.3) for some $K_3 > 0$ and $K_1 > 2(K_3)^2$ large enough and for sufficiently large $L \in \mathbb{N}$, and we finished the proof of Proposition 2.1.

4 Proof of Proposition 2.2

Let $L' = 2L - 2l$ and $L'' = L' - l$. We define $\mathbb{J}_\delta$ by

$$\mathrm{J}_{x,y} = \begin{cases} \delta & \text{if } x \in \Lambda(-L', 2l),\ y \in \partial_{ex}\Lambda(-L', 2l) \text{ and } -L'' \le x^3,\ y^3 < 0, \\ 1 & \text{otherwise.} \end{cases}$$

In the same way as in the proof of Proposition 2.1, we can obtain Proposition 2.2 from the following lemma. Throughout this section, we assume that $\beta > 0$ is sufficiently large. Let $\{c_i\}_{i \ge 13}$ be some positive constants which may depend on $\bar{\beta} > 0$ (see (4.4)) and $\beta > 0$.

Lemma 4.1. *Suppose that $\beta > 0$ is sufficiently large. Let $l = \lceil K_2 L^{\frac{2}{3}} (\log L)^2 \rceil$ and $\delta = L^{-\frac{2}{3}} \log L$. Then, for any $\varepsilon > 0$ there exists $K_2 = K_2(\beta, \varepsilon) > 0$ such that for sufficiently large $L \in \mathbb{N}$*

$$\mu^{\omega, \mathbb{J}_\delta}_{\Lambda(-L',2l)}((\mathcal{A}^\omega_{\Lambda(-L',2l)})^c) \le \varepsilon L^{-3}. \tag{4.1}$$

Let $\Lambda = \Lambda(-L', 2l)$. For each $\delta \in [0,1]$ and each $\mathbb{W} \in \mathcal{SW}(-L', 2l)$, we set

$$\begin{aligned} \Phi_\delta(\mathbb{W}) &= \Phi_\delta(\Gamma(\mathbb{W}))/e^{-8\beta L^2} \\ &= \exp[-\beta H^{\omega, \mathbb{J}_\delta}_\Lambda(\sigma_{\Gamma(\mathbb{W})})]/e^{-8\beta L^2}. \end{aligned} \tag{4.2}$$

Lemma 4.2. *Suppose that $\beta > 0$ is sufficiently large and $\delta \in [0,1]$. Then, there exists a function Φ^{T}_δ such that for each $\Gamma \in \mathcal{O}(-L', 2l)$*

$$\begin{aligned} & \mu^{\omega, \mathbb{J}_\delta}_\Lambda(\Gamma^\omega_\Lambda(\sigma) = \Gamma) \\ = \ & \mathrm{Z}(\mathcal{SW}(-L', 2l), \mathbb{J}_\delta)^{-1} \Phi_\delta(\mathbb{W}(\Gamma)) \exp[-\sum_{C \iota \Gamma} \mathbf{1}_{L',2l}(C) \Phi^{\mathrm{T}}_\delta(C)], \end{aligned} \tag{4.3}$$

where

$$\mathrm{Z}(\mathcal{SW}(-L', 2l), \mathbb{J}_\delta) = \sum_{\mathbb{W} \in \mathcal{SW}(-L',2l)} \Phi_\delta(\mathbb{W}) \exp[-\sum_{C \iota \Gamma(\mathbb{W})} \mathbf{1}_{L',2l}(C) \Phi^{\mathrm{T}}_\delta(C)].$$

Moreover, there exist $\bar{\beta} > 0$, $c'_1 > 0$ and $c'_2 > 0$ such that for each $x \in \Lambda$ and any $n \in \mathbb{N}$ (see (3.5)),

$$\sum_{C \in \mathbf{C}(n); C \iota \partial Q(x)} |\Phi^{\mathrm{T}}_\delta(C)| \le c'_1 \exp[-(\beta - \bar{\beta}) c'_2 n]. \tag{4.4}$$

The constants $\bar{\beta}$, c'_1 and c'_2 do not depend on $\beta > 0$, $\delta \in [0,1]$ and $L \in \mathbb{N}$.

Proof. In the same way as we obtained (3.6), we can see that there exists a function Φ^{T}_δ satisfying (4.3). Set in Lemma 1.3, $a(\cdot) = 6\mathbb{M}^2(\cdot)$ and $d(\cdot) = (\beta - \bar{\beta})\mathbb{M}^2(\cdot)$ for sufficiently large $\bar{\beta} > 0$. We have from (A.6) (see Appendix) that for any $\gamma \in \mathcal{C}(-L', 2l)$,

$$\Phi_0(\gamma) \le \exp[-\frac{\beta}{2} \mathbb{M}^2(\gamma)]. \tag{4.5}$$

Therefore, we can see by Lemma 1.3 that there exist $c_1' > 0$ and $c_2' > 0$ such that for each $x \in \Lambda$ and any $n \in \mathbb{N}$,

$$\sum_{C \in \mathbf{C}(n); C \iota \partial Q(x)} |\Phi_\delta^{\mathrm{T}}(C)| \le c_1' \exp[-(\beta - \bar{\beta}) c_2' n].$$

Note that $\bar{\beta}$, c_1' and c_2' do not depend on $\beta > 0$, $\delta \in [0,1]$ and $L \in \mathbb{N}$.

For each $\mathbb{W} \in \mathcal{SW}(-L', 2l)$, we define

$$P_\Lambda^\delta(\mathbb{W}) = \mathrm{Z}(\mathcal{SW}(-L', 2l), \mathbb{J}_\delta)^{-1} \Phi_\delta(\mathbb{W}) \exp[-\sum_{C \iota \Gamma(\mathbb{W})} \mathbf{1}_{L', 2l}(C) \Phi_\delta^{\mathrm{T}}(C)]. \tag{4.6}$$

Proof of Lemma 4.1. We define the events

$$\mathcal{F}_1 = \{\sigma \in \Omega_\Lambda; \mathrm{h}^+(\Gamma_\Lambda^\omega(\sigma)) \ge l\}$$

and

$$\mathcal{F}_1' = \{\sigma \in \Omega_{\Lambda(-L',\infty)}; \mathrm{h}^+(\Gamma_{\Lambda(-L',\infty)}^\omega(\sigma)) \ge l\}.$$

We have by FKG inequality that

$$\mu_\Lambda^{\omega,\mathbb{J}_\delta}(\mathcal{F}_1) \le \mu_{\Lambda(-L',\infty)}^{\omega,\mathbb{J}_1}(\mathcal{F}_1'). \tag{4.7}$$

We also have by FKG inequality that

$$\mu_{\Lambda(-L',\infty)}^{\omega,\mathbb{J}_1}(\mathcal{F}_2') \le \mu_{\Lambda(-\infty,\infty)}^{\omega,\mathbb{J}_1}(\mathcal{F}_2''), \tag{4.8}$$

where

$$\mathcal{F}_2' = \{\sigma \in \Omega_{\Lambda(-L',\infty)}; \mathrm{h}^-(\Gamma_{\Lambda(-L',\infty)}^\omega(\sigma)) \le -l\}$$

and

$$\mathcal{F}_2'' = \{\sigma \in \Omega_{\Lambda(-\infty,\infty)}; \mathrm{h}^-(\Gamma_{\Lambda(-\infty,\infty)}^\omega(\sigma)) \le -l\}.$$

Let

$$\mathcal{SW}^+ = \{\mathbb{W} \in \mathcal{SW}(-L', \infty); \mathrm{h}^+(\mathbb{W}) \ge l,\ \mathrm{h}^-(\mathbb{W}) > -l\}$$

and let

$$\mathcal{SW}^- = \{\mathbb{W} \in \mathcal{SW}(-\infty, \infty); \mathrm{h}^-(\mathbb{W}) \le -l\}.$$

Then, we have from (3.8), (3.9), (4.7) and (4.8) that

$$\begin{aligned}
\mu_\Lambda^{\omega,\mathbb{J}_\delta}(\mathcal{F}_1) &\le \mu_{\Lambda(-L',\infty)}^{\omega,\mathbb{J}_1}(\mathcal{F}_1' \backslash \mathcal{F}_2') + \mu_{\Lambda(-\infty,\infty)}^{\omega,\mathbb{J}_1}(\mathcal{F}_2'') \\
&= \sum_{\mathbb{W} \in \mathcal{SW}^+} P_{\Lambda(-L',\infty)}^1(\mathbb{W}) + \sum_{\mathbb{W} \in \mathcal{SW}^-} P_{\Lambda(-\infty,\infty)}^1(\mathbb{W}).
\end{aligned} \tag{4.9}$$

By a similar argument as in (3.21)-(3.32), we can estimate the first and second terms in RHS of (4.9) from above, respectively, by

$$c_{13}\exp[-(\beta-c_{14})K_3\log L]. \tag{4.10}$$

Set

$$\mathcal{G}_1=\{\sigma\in\Omega_\Lambda;\mathrm{h}^-(\Gamma^\omega_\Lambda(\sigma))\le -L'+l/2\}$$

and

$$\mathcal{G}'_1=\{\sigma\in\Omega_{\Lambda(-\infty,2l)};\mathrm{h}^-(\Gamma^\zeta_{\Lambda(-\infty,2l)}(\sigma))\le -L'+l/2\},$$

where we write $\zeta=\omega_{-L''}$ (see (1.15)). We define $\Gamma^\zeta_{\Lambda(-\infty,2l)}(\cdot)$ in a similar way as $\Gamma^\omega_{\Lambda(-\infty,\infty)}(\cdot)$ (see Section 3). Then, we have by FKG inequality that

$$\mu^{\omega,\mathbb{J}_\delta}_\Lambda(\mathcal{G}_1)\le\mu^{\zeta,\mathbb{J}_1}_{\Lambda(-\infty,2l)}(\mathcal{G}'_1). \tag{4.11}$$

Therefore, we can see by a similar argument as in (4.7)-(4.9) that

$$\mu^{\omega,\mathbb{J}_\delta}_\Lambda(\mathcal{G}_1)\le c_{15}\exp[-(\beta-c_{16})K_3\log L]. \tag{4.12}$$

Now, we are going to estimate $\mu^{\omega,\mathbb{J}_\delta}_\Lambda((\mathcal{A}^\omega_\Lambda)^c\backslash(\mathcal{F}_1\cup\mathcal{G}_1))$. Let

$$\mathcal{SW}=\{\mathbb{W}\in\mathcal{SW}(-L',2l);\Gamma(\mathbb{W})=\Gamma^\omega_\Lambda(\sigma)\text{ for some }\sigma\in(\mathcal{A}^\omega_\Lambda)^c\backslash(\mathcal{F}_1\cup\mathcal{G}_1)\}.$$

We will introduce some definitions and notations. For each $\Theta\subset\mathbb{Z}^3$, $\partial\Theta$ will indicate $\partial Q(\Theta)$. For each $\Theta\subset\Lambda$, we define $T_\Theta:\Omega_\Lambda\to\Omega_\Lambda$ by

$$T_\Theta(\sigma)(x)=\begin{cases}-\sigma(x) & \text{if } x\in\Theta,\\ \sigma(x) & \text{otherwise.}\end{cases}$$

For each $\Gamma\in\mathcal{O}(-L',2l)$, there exists a unique finite l_∞-connected $\Theta(\Gamma)\subset\mathbb{Z}^3$ which satisfies the following conditions:

(i) $\{x\in\partial_{ex}\Lambda;x^3\ge 0\}\subset\Theta(\Gamma)\subset(\Lambda\cup\{x\in\partial_{ex}\Lambda;x^3\ge 0\})$,

(ii) $(\Theta(\Gamma))^c$ is l_∞-connected, and

(iii) $\Gamma=\partial\Theta(\Gamma)\backslash\partial(\Lambda\cup\partial_{ex}\Lambda)$.

Let $V\subset\mathbb{R}^3$. For each $i\in\mathbb{N}$, we define $\mathrm{shift}(V,i)=V+(0,0,i)$. For each $r_1\in\mathbb{R}$ and each $r_2\in\mathbb{R}$, we define

$$\mathrm{cyl}(V,r_1,r_2)=\{x\in\mathbb{R}^3;\pi(x)\in\partial^2\hat{\pi}(V),\ r_1-(1/2)\le x^3\le r_2-(1/2)\},$$

where for $\hat{V}\subset\mathrm{H}(-\frac{1}{2})$, $\partial^2\hat{V}$ indicates the boundary of $\hat{V}$ under the induced 2-dimensional topology of $\mathrm{H}(-\frac{1}{2})$. Especially, $\mathrm{cyl}(Q(V),r_1,r_2)$ will be indicated by $\mathrm{cyl}(V,r_1,r_2)$ if $V\subset\mathbb{Z}^3$.

Let us fix $\mathbb{W} \in \mathcal{SW}$ and let $\Gamma = \Gamma(\mathbb{W})$. We decompose $\Gamma \backslash \text{cyl}(\Lambda, -L'', 0)$ into connected components. We define $\mathbb{F} = \mathbb{F}(\Gamma) = \mathbb{F}(\mathbb{W})$ by

$$\left\{ F; \begin{array}{c} F \text{ is a connected component of } \Gamma \backslash \text{cyl}(\Lambda, -L'', 0) \text{ and} \\ \text{there exists a unique } l_\infty\text{-connected } \Theta \subset \Lambda \text{ which satisfies that} \\ \Theta^c \text{ is } l_\infty\text{-connected and } \ F \ominus \partial\Theta \subset \text{cyl}(\Lambda, -L'', 0) \end{array} \right\}, \tag{4.13}$$

where $\ominus$ denotes the symmetric difference. The set Θ is uniquely determined by each $F \in \mathbb{F}$, and hence is denoted by $\Theta(F)$. Let $L \in \mathbb{N}$ be sufficiently large ($\delta > 0$ is small enough). By the definition of $\mathbb{J}_\delta$ and (A.6), we can see that for each $F \in \mathbb{F}$,

$$\begin{aligned} & H_\Lambda^{\omega,\mathbb{J}_\delta}(\sigma) - H_\Lambda^{\omega,\mathbb{J}_\delta}(T_{\Theta(F)}\sigma) \\ & \quad = 2\mathbb{M}^2(F) - 2\delta\mathbb{M}^2(\partial\Theta(F) \cap \text{cyl}(\Lambda, -L'', 0)) \\ & \quad \geq \frac{1}{2}\mathbb{M}^2(\partial\Theta(F)) - 2\delta\mathbb{M}^2(\partial\Theta(F)) \\ & \quad \geq \frac{1}{3}\mathbb{M}^2(\partial\Theta(F)), \end{aligned} \tag{4.14}$$

where $\sigma \in \Omega_\Lambda$ satisfies that $\Gamma_\Lambda^\omega(\sigma) = \Gamma$. Therefore, we have by a part of standard Peierls' argument that for each $F \in \mathbb{F}$,

$$\mu_\Lambda^{\omega,\mathbb{J}_\delta}(\Gamma_\Lambda^\omega(\sigma) \text{ contains } F) \leq \exp[-\frac{\beta}{3}\mathbb{M}^2(\partial\Theta(F))]. \tag{4.15}$$

Let $K_4 > 0$ be a large constant to be specified later. We define

$$\mathcal{SW}' = \left\{ \mathbb{W} \in \mathcal{SW}; \begin{array}{l} \text{there exist no elements } F \in \mathbb{F}(\mathbb{W}) \\ \text{such that } \mathbb{M}^2(F) \geq K_4 \log L \end{array} \right\}. \tag{4.16}$$

Note that $\mathbb{M}^2(F) \leq \mathbb{M}^2(\partial\Theta(F))$ for each $F \in \mathbb{F}$. Then, we have by standard Peierls' counting argument, (4.15) and (4.16) that

$$\sum_{\mathbb{W} \in \mathcal{SW} \backslash \mathcal{SW}'} P_\Lambda^\delta(\mathbb{W}) \leq c_{17} \exp[-(\frac{\beta}{3} - c_{18})K_4 \log L]. \tag{4.17}$$

Due to $\mathbb{J}_\delta$, there can exist a standard wall having less energy than we expect from the size of it. For this reason, we will estimate $\sum_{\mathbb{W} \in \mathcal{SW}'} P_\Lambda^\delta(\mathbb{W})$ in separate four cases. We define

$$\mathcal{SW}_1 = \{\mathbb{W} \in \mathcal{SW}'; (\Phi_0(\{W\}))^5/\Phi_1(\{W\}) \leq 1 \text{ for all } W \in \mathbb{W}\}, \tag{4.18}$$

which corresponds to the first case where the energy of each standard wall is proportional to the size of it.

Let $\mathbb{W} \in \mathcal{SW}(-L', 2l)$ and let $z \in \mathbb{Z}$. For each $W \in \mathbb{W}$, we will denote by $\mathcal{B}(W, z)$ the collection of connected components of $W \cap \mathrm{H}(z)$ if it is not empty. We define for each $W \in \mathbb{W}$,

$$\rho(W, z) = \begin{cases} \inf\left\{\dfrac{\mathbb{M}^1(B \backslash \partial\Lambda)}{\mathbb{M}^1(B)}; B \in \mathcal{B}(W, z)\right\} & \text{if } W \cap \mathrm{H}(z) \neq \emptyset, \\ \infty & \text{otherwise,} \end{cases} \tag{4.19}$$

and define

$$\rho(\mathbb{W}, z) = \inf_{W \in \mathbb{W}} \rho(W, z). \tag{4.20}$$

By the definition of $\mathbb{J}_\delta$, we can see that $\mathbb{W} \in \mathcal{SW}_1$ if $\rho(\mathbb{W}, z) \geq 1/5$ for all $z \in \mathbb{Z}$ with $-L'' \leq z < 0$. Hence, we will consider

$$\mathcal{SW}'' = \{\mathbb{W} \in \mathcal{SW}'; \rho(\mathbb{W}, z) < 1/5 \text{ for some } z \in \mathbb{Z} \text{ with } -L'' \leq z < 0\}.$$

Lemma 4.3. *For each $\mathbb{W} \in \mathcal{SW}(-L', 2l)$, we have the following properties:*

(i) If $\rho(\mathbb{W}, z_0) < 1/5$ for some $z_0 \in \mathbb{Z}$, then there exists a unique standard wall $W_0 \in \mathbb{W}$ such that $\rho(W_0, z) < 1/5$ whenever $\rho(\mathbb{W}, z) < 1/5$.

(ii) For any $W \in \mathbb{W} \backslash \{W_0\}$, $(\Phi_0(\{W\}))^5 / \Phi_1(\{W\}) \leq 1$ holds.

Proof. We will first prove (i). Let $W \in \mathbb{W}$ and let $z_0 \in \mathbb{Z}$ be some integer such that $\rho(W, z_0) < 1/5$. Then, we can see by the definition of ρ and (A.11) that

$$\mathbb{M}^1(W \cap \mathrm{H}(z_0)) \geq 8L \qquad \text{and} \qquad \mathbb{M}^1(W \cap \mathrm{H}(z_0) \cap \partial\Lambda) \geq 6L. \tag{4.21}$$

Hence, if there exist $W_1 \in \mathbb{W}$, $W_2 \in \mathbb{W}$, $z_1 \in \mathbb{Z}$ and $z_2 \in \mathbb{Z}$ such that $\rho(W_1, z_1) < 1/5$ and $\rho(W_2, z_2) < 1/5$, we can see from (4.21) that $\pi(W_1) \cap \pi(W_2) \neq \emptyset$. By this and the definition of (standard) walls, it holds that $W_1 = W_2$.

We will next prove (ii). Note that for each $\delta > 0$,

$$\begin{aligned} \Phi_\delta(\{W\}) \;=\; & e^{8\beta L^2} \exp\Big[-2\beta\mathbb{M}^2\big(\Gamma(\{W\}) \backslash \mathrm{cyl}(\Lambda, -L'', 0)\big) \\ & \qquad -2\beta\delta\mathbb{M}^2\big(\Gamma(\{W\}) \cap \mathrm{cyl}(\Lambda, -L'', 0)\big)\Big]. \end{aligned} \tag{4.22}$$

By the definition of ρ, (i) and (4.22), we have that for any $W \in \mathbb{W} \backslash \{W_0\}$

$$\begin{aligned} (\Phi_0(\{W\}))^5 &\leq \exp\big[-10\beta\mathbb{M}^2\big((W \backslash \pi(W)) \backslash \mathrm{cyl}(\Lambda, -L'', 0)\big)\big] \\ &\leq e^{8\beta L^2} \exp[-2\beta\mathbb{M}^2(\Gamma(\{W\}))] \\ &= \Phi_1(\{W\}). \end{aligned} \tag{4.23}$$

We define

$$\mathcal{SW}_2 = \left\{ \mathbb{W} \in \mathcal{SW}''\backslash\mathcal{SW}_1;\ \begin{array}{l}\text{there exist some } W \in \mathbb{W}\backslash\{W_0\} \\ \text{with } \mathbb{M}^2(W) \geq K_5 \log L\end{array} \right\},$$

where W_0 is the unique standard wall in (i) of Lemma 4.3 and $K_5 > 0$ is a large constant to be specified later. $\mathcal{SW}_2$ corresponds to the second case. In the same way as in (3.27)-(3.29), we can see that

$$|\mathrm{h}^*(\mathbb{W}\backslash\{W_0\})| \leq (K_5 \log L)^2 \tag{4.24}$$

for $* = +$ and $-$ if $\mathbb{W} \in \mathcal{SW}''\backslash(\mathcal{SW}_1 \cup \mathcal{SW}_2)$. We define

$$z_{\min} = z_{\min}(\mathbb{W}) = \min\{z \in \mathbb{Z}; \rho(W_0, z) < 1/5,\ z \geq -L''\}, \tag{4.25}$$

for each $\mathbb{W} \in \mathcal{SW}''$. We define

$$\mathcal{SW}_3 = \{\mathbb{W} \in \mathcal{SW}''\backslash(\mathcal{SW}_1 \cup \mathcal{SW}_2); z_{\min}(\mathbb{W}) \leq -l/2\}$$

and

$$\mathcal{SW}_4 = \{\mathbb{W} \in \mathcal{SW}''\backslash(\mathcal{SW}_1 \cup \mathcal{SW}_2); z_{\min}(\mathbb{W}) > -l/2\},$$

which corresponds to the third and fourth cases, respectively.

First, we claim that for any $\mathbb{W} \in \mathcal{SW}''\backslash\mathcal{SW}_1$,

$$\mathbb{M}^1(W_0 \cap \mathrm{H}(z)) > 2L \tag{4.26}$$

holds for all negative integer z with $z \geq z_{\min}$. To see this, assume that $\mathbb{W} \in \mathcal{SW}''\backslash\mathcal{SW}_1$ and $\mathbb{M}^1(W_0 \cap \mathrm{H}(z_0)) \leq 2L$ for some negative integer z_0 with $z_0 > z_{\min}$. Then, we will show that

$$\begin{aligned}\mathbb{M}^2 &\left(\left\{ x \in \mathbb{R}^3;\ \begin{array}{l} x \in w \text{ for some face } w \in \Gamma(\{W_0\}) \\ \text{such that } \mathbb{M}^2(\pi(w)) = 1 \end{array} \right\} \right) - 4L^2 \\ &\geq 4L^2, \end{aligned} \tag{4.27}$$

which together with the definitions of W_0 and $z_{\min}$ implies that $(\Phi_0(\{W_0\}))^5/\Phi_1(\{W_0\}) \leq 1$ (see (4.22) and (4.23)). This together with (ii) of Lemma 4.3 shows that $\mathbb{W} \in \mathcal{SW}_1$, which contradicts $\mathbb{W} \in \mathcal{SW}''\backslash\mathcal{SW}_1$. Thus, we obtained (4.26). We will show (4.27) assuming that $\mathbb{W} \in \mathcal{SW}''\backslash\mathcal{SW}_1$ and $\mathbb{M}^1(W_0 \cap \mathrm{H}(z_0)) \leq 2L$ for some negative integer z_0 with $z_0 > z_{\min}$. Note that there exists a unique $B_0 \in \mathcal{B}(W_0, z_{\min})$ such that $\mathbb{M}^1(B_0\backslash\partial\Lambda)/\mathbb{M}^1(B_0) < 1/5$. From this and (A.12), we have that $\mathbb{M}^2(\hat{\pi}(B_0)) \geq 3(4L^2)/4$. Therefore, we have that

$$\mathbb{M}^2(\hat{\pi}(W_0)) \geq \frac{3}{4}(4L^2). \tag{4.28}$$

By the hypothesis that $\mathbb{M}^1(W_0 \cap \mathrm{H}(z_0)) \leq 2L$ and Lemma A.1, we have that

$$\mathbb{M}^2\left(\hat{\pi}(W_0 \cap \mathrm{H}(z_0))\right) \leq \frac{1}{4}L^2. \tag{4.29}$$

Therefore, we can see from (4.28) and (4.29) that

$$\begin{aligned}
&\mathbb{M}^2\left(\left\{x\in\mathbb{R}^3;\ \begin{array}{l}x\in w \text{ for some face } w\in\Gamma(\{W_0\})\\ \text{such that } \mathbb{M}^2(\pi(w))=1\end{array}\right\}\right)-4L^2\\
&\quad\geq \frac{2}{3}\mathbb{M}^2\left(\left\{x\in\mathbb{R}^3;\ \begin{array}{l}x\in w \text{ for some face } w\in\Gamma(\{W_0\})\\ \text{such that } \mathbb{M}^2(\pi(w))=1\end{array}\right\}\right)\\
&\quad\geq 2\Big(\frac{3}{4}(4L^2)-\frac{1}{4}L^2\Big)\\
&\quad\geq 4L^2.
\end{aligned}\tag{4.30}$$

In considering the four cases, we have to be aware that Φ_δ^{T} is not invariant under the vertical shift due to $\mathbb{J}_\delta$. Hence, we will introduce the following lemma before we proceed to the four cases.

Lemma 4.4. *Suppose that $\beta>0$ and $L\in\mathbb{N}$ are sufficiently large. Let $\mathbb{W}\in\mathcal{SW}$ and let $W\in\mathbb{W}$. Let $\mathbb{W}'=\mathbb{W}\backslash\{W\}$. Suppose that $\mathbb{W}'\in\mathcal{SW}(-L',2l)$, $\mathrm{h}^+(\mathbb{W}')\leq 3l/2$ and $\mathrm{h}^-(\mathbb{W}')\geq -L'+l/4$. Then, we have that*

$$\Big|\sum_{C\iota\Gamma(\mathbb{W})}\mathbf{1}_{L',2l}(C)\Phi_\delta^{\mathrm{T}}(C)-\sum_{C\iota\Gamma(\mathbb{W}')}\mathbf{1}_{L',2l}(C)\Phi_\delta^{\mathrm{T}}(C)\Big|\leq c_{19}\mathbb{M}^2(W).\tag{4.31}$$

Proof. Let $\beta>0$ and $L\in\mathbb{N}$ be sufficiently large. We will first prove (4.31) for the special case where $W=\mathrm{cyl}(W,-h,0)$ for some $h\in\mathbb{N}$. We have by the definition of $\mathbb{J}_\delta$ and (4.4) (see (3.15) and (3.23)) that

$$\begin{aligned}
&\Big|\sum_{C\iota\Gamma(\mathbb{W})}\mathbf{1}_{L',2l}(C)\Phi_\delta^{\mathrm{T}}(C)-\sum_{C\iota\Gamma(\mathbb{W}')}\mathbf{1}_{L',2l}(C)\Phi_\delta^{\mathrm{T}}(C)\Big|\\
\leq\ & 16L^2c_1'\exp[-(\beta-\bar\beta)c_{20}L^{\frac{2}{3}}(\log L)^2]\\
&+c_{21}\sum_{i=1}^{h}\sum_{x\in D(-i;W)}c_1'\exp\Big[-(\beta-\bar\beta)c_{20}\mathrm{dist}_1(x,\mathrm{cyl}(\Lambda,-i,-i+1))\Big]\\
&+c_{21}\sum_{i=1}^{h}\sum_{x\in D(-i;W)}c_1'\exp\Big[-(\beta-\bar\beta)\times\\
&\qquad c_{20}\mathrm{dist}_1(x,\mathrm{cyl}(\Lambda,-L''-i,-L''-i+1))\Big]\\
\leq\ & (c_{22}+c_{23})\mathbb{M}^2(W),
\end{aligned}\tag{4.32}$$

where $D(-i;W)=\{x\in\mathrm{cyl}(W\cap\mathrm{H}(-i),-L',2l+1);x+(1/2,1/2,0)\in\mathbb{Z}^3\}$.

We will next prove (4.31) for a general case. By the same argument as in (4.32), we have that

$$
\begin{aligned}
&\Big|\sum_{C\iota\Gamma(\mathbb{W})}\mathbf{1}_{L',2l}(C)\Phi_\delta^{\mathrm{T}}(C)-\sum_{C\iota\Gamma(\mathbb{W}')}\mathbf{1}_{L',2l}(C)\Phi_\delta^{\mathrm{T}}(C)\Big|\\
\le\ & c_{22}+c_{21}\sum_{i\in\mathbb{Z}}\sum_{x\in D(i;W)}c_1'\exp\Big[-(\beta-\bar\beta)c_{20}\mathrm{dist}_1(x,\mathrm{cyl}(\Lambda,i,i+1))\Big]\\
&+c_{21}\sum_{i\in\mathbb{Z}}\sum_{x\in D(i;W)}c_1'\exp\Big[-(\beta-\bar\beta)\times\\
&\qquad c_{20}\mathrm{dist}_1(x,\mathrm{cyl}(\Lambda,-L''+i,-L''+i+1))\Big]\\
\le\ & (c_{22}+c_{23})\mathbb{M}^2(W),
\end{aligned}
\tag{4.33}
$$

where $D(i;W)=\{x\in\mathrm{cyl}(W\cap\mathrm{H}(i),-L',2l+1);x+(1/2,1/2,0)\in\mathbb{Z}^3\}$ if $W\cap\mathrm{H}(i)\neq\emptyset$ and $D(i;W)=\emptyset$ otherwise.

From now on, we will estimate

$$\sum_{\mathbb{W}\in\mathcal{SW}_1}P_\Lambda^\delta(\mathbb{W}),\quad \sum_{\mathbb{W}\in\mathcal{SW}_2}P_\Lambda^\delta(\mathbb{W}),\quad \sum_{\mathbb{W}\in\mathcal{SW}_3}P_\Lambda^\delta(\mathbb{W})\quad\text{and}\quad \sum_{\mathbb{W}\in\mathcal{SW}_4}P_\Lambda^\delta(\mathbb{W}).$$

We assume that $L\in\mathbb{N}$ is sufficiently large ($\delta>0$ is sufficiently small) throughout this section.

Case 1. We will consider the case where $\mathbb{W}\in\mathcal{SW}_1$. Recall that

$$\mathcal{SW}_1=\{\mathbb{W}\in\mathcal{SW}';(\Phi_0(\{W\}))^5/\Phi_1(\{W\})\le 1\text{ for all }W\in\mathbb{W}\}.$$

Note that $\Phi_\delta(\{W\})\le\exp[-\beta\mathbb{M}^2(W)/5]$ for all $W\in\mathbb{W}$ if $\mathbb{W}\in\mathcal{SW}_1$ (see (3.24)). Thus, by a similar argument as in (3.22)-(3.30) together with (4.31), we have that

$$\sum_{\mathbb{W}\in\mathcal{SW}_1}P_\Lambda^\delta(\mathbb{W})\le c_{24}\exp[-(\frac{\beta}{5}-c_{25})K_5\log L]. \tag{4.34}$$

Case 2. We will consider the case where $\mathbb{W}\in\mathcal{SW}_2$. Recall that

$$\mathcal{SW}_2=\left\{\mathbb{W}\in\mathcal{SW}''\backslash\mathcal{SW}_1;\ \begin{matrix}\text{there exist some }W\in\mathbb{W}\backslash\{W_0\}\\ \text{with }\mathbb{M}^2(W)\ge K_5\log L\end{matrix}\right\},$$

where $\mathcal{SW}''=\{\mathbb{W}\in\mathcal{SW}';\rho(\mathbb{W},z)<1/5\text{ for some }z\in\mathbb{Z}\text{ with }-L''\le z<0\}$. In this case, by a similar argument as in (3.22)-(3.30) together with (ii) of Lemma 4.3 and (4.31), we also have that

$$\sum_{\mathbb{W}\in\mathcal{SW}_2}P_\Lambda^\delta(\mathbb{W})\le c_{26}\exp[-(\frac{\beta}{5}-c_{27})K_5\log L]. \tag{4.35}$$

Case 3. We will consider the case where $\mathbb{W}\in\mathcal{SW}_3$. Recall that

$$\mathcal{SW}_3=\{\mathbb{W}\in\mathcal{SW}''\backslash(\mathcal{SW}_1\cup\mathcal{SW}_2);z_{\min}(\mathbb{W})\le -l/2\}.$$

We define

$$\mathcal{SW}^\circ = \{\mathbb{W} \in \mathcal{SW}''\backslash(\mathcal{SW}_1 \cup \mathcal{SW}_2); \text{for any } W \in \mathbb{W}\backslash\{W_0\},\ W \cap \partial\Lambda = \emptyset\} \tag{4.36}$$

and $\mathcal{SW}_3^\circ = \mathcal{SW}_3 \cap \mathcal{SW}^\circ$. By a similar argument as in (3.22)-(3.30) together with (ii) of Lemma 4.3 and (4.31), we have that

$$\begin{aligned}
&\sum_{\mathbb{W}\in\mathcal{SW}_3} P_\Lambda^\delta(\mathbb{W}) \\
\leq\ & \Big(\sum_{\mathbb{W}\in\mathcal{SW}_3^\circ} P_\Lambda^\delta(\mathbb{W})\Big) \times \\
& \qquad \Big(\sum_{k=1}^{8L} \binom{4L^2}{k} \sum_{n_1\geq 1} \cdots \sum_{n_k\geq 1} \kappa^{n_1+\cdots+n_k} e^{-(\beta/5-c_{28})(n_1+\cdots+n_k)}\Big) \\
\leq\ & c_{29}\exp[c_{30}L\log L] \sum_{\mathbb{W}\in\mathcal{SW}_3^\circ} P_\Lambda^\delta(\mathbb{W}).
\end{aligned} \tag{4.37}$$

Let us fix $\mathbb{W} \in \mathcal{SW}_3^\circ$. Let $\Gamma = \Gamma(\mathbb{W})$ and $m = z_{\min}(\mathbb{W})$. We decompose $W_0\backslash\mathrm{cyl}(\Lambda, m, 0)$ into connected components. By $\mathbb{F}^1 = \mathbb{F}^1(\mathbb{W})$, we will denote the collection of such components which belong to $\mathbb{F}(\mathbb{W})$ and are included in $Q(\Lambda(m+1, 0))$. Note that $F \ominus \partial\Theta(F) \subset \mathrm{cyl}(\Lambda, m+1, 0)$ for each $F \in \mathbb{F}^1$. Let

$$F_0 = F_0(\mathbb{W}) = \mathrm{cyl}(\Lambda, m, 0) \ominus (\cup_{F\in\mathbb{F}^1}\partial\Theta(F)). \tag{4.38}$$

From (4.36) and (4.38), we can define the open contour $\Gamma'(\mathbb{W})$ which satisfies

$$\mathbb{M}^2\Big(\Gamma'(\mathbb{W}) \ominus \big(F_0(\mathbb{W}) \cup \mathrm{shift}(\Gamma(\mathbb{W}\backslash\{W_0\}), m)\big)\Big) = 0, \tag{4.39}$$

and define

$$\mathbb{W}' = \mathbb{W}'(\mathbb{W}) = \mathbb{W}(\Gamma'(\mathbb{W})). \tag{4.40}$$

Note that $\Gamma'(\mathbb{W}) = \Gamma(\mathbb{W}')$. We also define

$$\overline{\Gamma}(\mathbb{W}) = \Big(\Gamma(\{W_0\})\backslash(\cup_{F\in\mathbb{F}^1}\partial\Theta(F))\Big) \cup \Big(\cup_{F\in\mathbb{F}^1}(\partial\Theta(F) \cap \mathrm{cyl}(\Lambda, m+1, 0))\Big). \tag{4.41}$$

Let

$$\{F_i^2\}_{i=0}^{k_2} = \{F_i^2(\mathbb{W})\}_{i=0}^{k_2} = \mathbb{W}(\overline{\Gamma}(\mathbb{W})) \tag{4.42}$$

and let

$$F_0^2 = \text{the unique maximal standard wall such that } \mathbb{M}^2(\hat{\pi}(F_0^2)) \geq \frac{3}{4}(4L^2). \tag{4.43}$$

Then, we can see from the definitions of $\mathcal{SW}_3$ ($\mathcal{SW}'$), (standard) walls, and $\{F_i^2\}_{i=1}^{k_2}$ that for any $1 \le i \le k_2$,

$$\hat{\pi}(F_i^2) \cap \{x \in \mathrm{H}(-\tfrac{1}{2}); \mathrm{dist}_1(x, \partial\Lambda) \le K_4 \log L\} \neq \emptyset. \tag{4.44}$$

We decompose $F_0^2 \backslash \mathrm{cyl}(\Lambda, m, 0)$ into connected components. We will denote by $\{\tilde{F}_i^3\} = \{\tilde{F}_i^3(\mathbb{W})\}$ and $\{F_i^4\}_{i=1}^{k_4} = \{F_i^4(\mathbb{W})\}_{i=1}^{k_4}$, the collections of such components belonging to $\mathbb{F}(\mathbb{W})$ and not belonging to $\mathbb{F}(\mathbb{W})$, respectively. Note that $0 \le k_4 \le 8L$. Let

$$\{x_i\} = \{\pi(x); x \in \partial_{in}\Lambda \text{ and } x^3 = 0\}.$$

We define $\{F_i^3\}_{i=1}^{k_3} = \{F_i^3(\mathbb{W})\}_{i=1}^{k_3}$, inductively, by

$$F_1^3 = \cup_{i;\pi(\tilde{F}_i^3)\cap\{x_1\}\neq\emptyset} \tilde{F}_i^3, \tag{4.45}$$

and

$$F_j^3 = (\cup_{i;\pi(\tilde{F}_i^3)\cap\{x_j\}\neq\emptyset} \tilde{F}_i^3) \backslash (\cup_{i\le j-1} F_i^3). \tag{4.46}$$

Note that $k_3 \le 8L$.

We will show that the estimate

$$\begin{aligned} & P_\Lambda^\delta(\mathbb{W}) \Big/ P_\Lambda^\delta(\mathbb{W}') \\ = & \Big(\Phi_\delta(\mathbb{W}) \Big/ \Phi_\delta(\mathbb{W}')\Big) \times \\ & \qquad \Big(\exp[-\sum_{C \iota \Gamma(\mathbb{W})} \mathbf{1}_{L',2l}(C)\Phi_\delta^{\mathrm{T}}(C)] \Big/ \exp[-\sum_{C \iota \Gamma(\mathbb{W}')} \mathbf{1}_{L',2l}(C)\Phi_\delta^{\mathrm{T}}(C)]\Big) \\ \le & \; c_{31} \exp\Big[-(\frac{\beta}{3} - c_{32})\Big(\sum_{i=1}^{k_2} \mathbb{M}^2(F_i^2) + \sum_{i=1}^{k_3} \mathbb{M}^2(F_i^3) + \sum_{i=1}^{k_4} \mathbb{M}^2(F_i^4)\Big)\Big] \end{aligned} \tag{4.47}$$

holds, as follows.

We will first bound $\Phi_\delta(\mathbb{W}) / \Phi_\delta(\mathbb{W}')$ from above. For $j = 1$ or 2, let

$$S_{\pm j} = \{x \in \partial\Lambda; x_j = \pm L + (1/2)\}.$$

Take i with $-1 \ge i \ge m$. If for every $B \in \mathcal{B}(W_0, i)$, there exist some $j \in \{\pm 1, \pm 2\}$ such that $B \cap S_j = \emptyset$, then we can see from (4.16) and (4.26) that

$$\begin{aligned} \sum_{p=2}^{4} \sum_{j=1}^{k_p} \mathbb{M}^1(F_j^p \cap \mathrm{H}(i)) & \ge \inf_{n \in \mathbb{N}\cup\{0\}} \max\{2L,\ \frac{1}{4}(2L - nK_4 \log L),\ n\} \\ & \ge \frac{2L}{4 + K_4 \log L}. \end{aligned} \tag{4.48}$$

From this together with

$$\mathbb{M}^1\big((W_0 \backslash \partial\Lambda) \cap \mathrm{H}(i)\big) - \mathbb{M}^1\big((F_0 \backslash \partial\Lambda) \cap \mathrm{H}(i)\big) = \mathbb{M}^1\big((\cup_{p=2}^4 \cup_{j=1}^{k_p} F_j^p) \cap \mathrm{H}(i)\big), \tag{4.49}$$

we have that

$$\begin{aligned}
&\mathbb{M}^1\big((W_0\backslash\partial\Lambda)\cap \mathrm{H}(i)\big)+\delta\mathbb{M}^1\big((W_0\cap\partial\Lambda)\cap\mathrm{H}(i)\big)\\
&\qquad -\mathbb{M}^1\big((F_0\backslash\partial\Lambda)\cap\mathrm{H}(i)\big)-\delta\mathbb{M}^1\big((F_0\cap\partial\Lambda)\cap\mathrm{H}(i)\big)\\
\geq\;& \frac{1}{2}\mathbb{M}^1\big((\cup_{p=2}^4\cup_{j=1}^{k_p}F_j^p)\cap\mathrm{H}(i)\big)+\frac{L}{4+K_4\log L}-8\delta L\\
\geq\;& \frac{1}{2}\mathbb{M}^1\big((\cup_{p=2}^4\cup_{j=1}^{k_p}F_j^p)\cap\mathrm{H}(i)\big). \qquad (4.50)
\end{aligned}$$

If there exists a unique $B_0\in\mathcal{B}(W_0,i)$ such that $B_0\cap S_j\neq\emptyset$ for all $j\in\{\pm1,\pm2\}$, then we can see that

$$\begin{aligned}
&\mathbb{M}^1\big((\cup_{p=2}^4\cup_{j=1}^{k_p}F_j^p)\cap\mathrm{H}(i)\big)+\mathbb{M}^1\big((W_0\cap\partial\Lambda)\cap\mathrm{H}(i)\big)\\
\geq\;& \mathbb{M}^1\big(B_0\backslash(\cup_{F\in\mathbb{F}^1}F)\big)\\
\geq\;& \mathbb{M}^1\big(\partial\Lambda\cap\mathrm{H}(i)\big)-\mathbb{M}^1\big((\cup_{F\in\mathbb{F}^1}(\partial\Theta(F)\cap\partial\Lambda))\cap\mathrm{H}(i)\big)\\
=\;& \mathbb{M}^1\big((F_0\cap\partial\Lambda)\cap\mathrm{H}(i)\big). \qquad (4.51)
\end{aligned}$$

From this and (4.49), we have that

$$\begin{aligned}
\mathrm{LHS}(4.50)\;\geq\;& (1-\delta)\mathbb{M}^1\big((\cup_{p=2}^4\cup_{j=1}^{k_p}F_j^p)\cap\mathrm{H}(i)\big)\\
\geq\;& \frac{1}{2}\mathbb{M}^1\big((\cup_{p=2}^4\cup_{j=1}^{k_p}F_j^p)\cap\mathrm{H}(i)\big). \qquad (4.52)
\end{aligned}$$

From (3.24), (4.50), (4.52) and the definitions of $z_{\min}$, $\{F_j^2\}_{j=0}^{k_2}$, $\{F_j^3\}_{j=1}^{k_3}$ and $\{F_j^4\}_{j=1}^{k_4}$, we have that

$$\Phi_\delta(\mathbb{W})/\Phi_\delta(\mathbb{W}')\leq\exp\Big[-\frac{\beta}{3}\Big(\sum_{j=1}^{k_2}\mathbb{M}^2(F_j^2)+\sum_{j=1}^{k_3}\mathbb{M}^2(F_j^3)+\sum_{j=1}^{k_4}\mathbb{M}^2(F_j^4)\Big)\Big]. \qquad (4.53)$$

In order to estimate the effect of entropy, we will next bound from above

$$\Big|\sum_{C\iota\Gamma(\mathbb{W})}\mathbf{1}_{L',2l}(C)\Phi_\delta^{\mathrm{T}}(C)-\sum_{C\iota\Gamma(\mathbb{W}')}\mathbf{1}_{L',2l}(C)\Phi_\delta^{\mathrm{T}}(C)\Big|.$$

To do this, we have only to look at standard walls $W\in\mathbb{W}\backslash\{W_0\}$ such that (see (1.26))

$$\mathrm{b}(W,\mathbb{W})\neq\mathrm{b}(W,\mathbb{W}').$$

Note that

$$\mathrm{b}(W,\mathbb{W}')=m-1$$

for all $W\in\{W'\in\mathbb{W}\backslash\{W_0\};W'$ is a maximal standard wall$\}$. Let $v(W)=\mathrm{b}(W,\mathbb{W})-\mathrm{b}(W,\mathbb{W}')$. We will first consider standard walls such that $\hat{\pi}(W)\subset(\mathrm{H}(-\frac{1}{2})\backslash\hat{\pi}(W_0))$. We define

$$\mathbb{W}_{ex}=\{W\in\mathbb{W};\hat{\pi}(W)\subset(\mathrm{H}(-\tfrac{1}{2})\backslash\hat{\pi}(W_0))\}.$$

Note that $v(W) = m < 0$ for any $W \in \mathbb{W}_{ex}$. We decompose $\hat{\pi}(Q(\Lambda))\backslash\hat{\pi}(W_0)$ into connected components $\{R_k\}_{k=1}^r$. For each $k \le r$ and any $-1 \ge i \ge m$, there exist $D_k(i) \subset \mathrm{H}(-\frac{1}{2})$ such that $R_k \subset D_k(i)$ and

$$\partial^2 D_k(i) \ominus \partial^2 \hat{\pi}(Q(\Lambda)) + (0, 0, i + (1/2)) \subset F_j^4 \cap \mathrm{H}(i)$$

for some $j = j(k, i)$. By the definition of W_0, we can see that

$$\mathbb{M}^1(\partial^2 D_k(i)) \le 2\mathbb{M}^1(F_j^4 \cap \mathrm{H}(i)). \tag{4.54}$$

It may happen that $j(k_1, i) = j(k_2, i)$ even if $k_1 \ne k_2$. In this case, $D_{k_1}(i) = D_{k_2}(i)$. From this and (4.54), we can see that

$$\begin{aligned} \sum_{i=-1}^{m} \mathbb{M}^1(\cup_{k=1}^r \partial^2 D_k(i)) &\le 2\sum_{i=-1}^{m}\sum_{j=1}^{k_4} \mathbb{M}^1(F_j^4 \cap \mathrm{H}(i)) \\ &\le 2\sum_{j=1}^{k_4} \mathbb{M}^2(F_j^4). \end{aligned} \tag{4.55}$$

We will next consider standard walls W such that $v(W) \ne 0$ and $\hat{\pi}(W) \subset \hat{\pi}(W_0)$. We define

$$\mathbb{W}_{in}^- = \{W \in \mathbb{W}\backslash\{W_0\}; v(W) < 0 \text{ and } \hat{\pi}(W) \subset \hat{\pi}(W_0)\}$$

and

$$\mathbb{W}_{in}^+ = \{W \in \mathbb{W}\backslash\{W_0\}; v(W) > 0 \text{ and } \hat{\pi}(W) \subset \hat{\pi}(W_0)\}.$$

For each $W \in \mathbb{W}_{in}^-$ and any $m-1 \ge i \ge m + v(W)$, there exist $D_W(i) \subset \mathrm{H}(-\frac{1}{2})$ such that $\hat{\pi}(W) \subset D_W(i)$ and

$$\partial^2 D_W(i) + (0, 0, i + (1/2)) \subset (F_j^2 \cup F_k^4) \cap \mathrm{H}(i) \tag{4.56}$$

for some $j = j(W, i)$ and $k = k(W, i)$. It may happen that $j(W_1, i) = j(W_2, i)$ and $k(W_1, i) = k(W_2, i)$ even if $W_1 \ne W_2$. In this case, $D_{W_1}(i) = D_{W_2}(i)$. From this and (4.56), we can see that for any $i \le m - 1$,

$$\mathbb{M}^1(\cup_{W\in\mathbb{W}_{in}^-}\partial^2 D_W(i)) \le \sum_{j=1}^{k_2} \mathbb{M}^1(F_j^2 \cap \mathrm{H}(i)) + \sum_{j=1}^{k_4} \mathbb{M}^1(F_j^4 \cap \mathrm{H}(i)). \tag{4.57}$$

Hence, we have from (4.57) that

$$\sum_{i \le m-1} \mathbb{M}^1(\cup_{W\in\mathbb{W}_{in}^-}\partial^2 D_W(i)) \le \sum_{j=1}^{k_2} \mathbb{M}^2(F_j^2) + \sum_{j=1}^{k_4} \mathbb{M}^2(F_j^4). \tag{4.58}$$

Similarly, we have that

$$\sum_{i \ge m+1} \mathbb{M}^1(\cup_{W\in\mathbb{W}_{in}^+}\partial^2 D_W(i)) \le \sum_{j=1}^{k_2} \mathbb{M}^2(F_j^2) + \sum_{j=1}^{k_4} \mathbb{M}^2(F_j^4). \tag{4.59}$$

Therefore, in a similar way as in the proof of Lemma 4.4, we can see from (4.55), (4.58) and (4.59) that

$$\begin{aligned}
&\Big|\sum_{C\iota\Gamma(\mathbb{W})}\mathbf{1}_{L',2l}(C)\Phi_\delta^{\mathrm{T}}(C)-\sum_{C\iota\Gamma(\mathbb{W}')}\mathbf{1}_{L',2l}(C)\Phi_\delta^{\mathrm{T}}(C)\Big| \\
\le\;& c_{22}+2c_{23}\Big(\sum_{i=1}^{k_2}\mathbb{M}^2(F_i^2)+\sum_{i=1}^{k_4}\mathbb{M}^2(F_i^4)\Big)+2c_{23}\sum_{i=1}^{k_4}\mathbb{M}^2(F_i^4) \\
\le\;& c_{33}\Big(1+\sum_{i=1}^{k_2}\mathbb{M}^2(F_i^2)+\sum_{i=1}^{k_4}\mathbb{M}^2(F_i^4)\Big) \qquad (4.60)
\end{aligned}$$

in this time $\cup_{W'\in\mathbb{W}_{in}^-\cup\mathbb{W}_{in}^+}\partial^2 D_{W'}(i)$ and $\cup_{k=1}^r\partial^2 D_k(i)$ play the role of $W\cap \mathrm{H}(i)$. Thus, (4.47) follows from (4.53) and (4.60).

We will next show that

$$\sum_{\mathbb{W}'}P_\Lambda^\delta(\mathbb{W}')\le c_{34}\exp[-(\beta-c_{35})K_2L(\log L)^3], \qquad (4.61)$$

where $\sum_{\mathbb{W}'}$ stands for the summation over all families of the standard walls such that $\mathbb{W}'=\mathbb{W}'(\mathbb{W})$ for some $\mathbb{W}\in\mathcal{SW}_3^\circ$. Let us fix $\Gamma\in\mathcal{O}(-L',2l)$ such that for $*=+$ and $-$

$$|\mathrm{h}^*(\Gamma)|<(K_5\log L)^2, \qquad (4.62)$$

and such that

$$\{W\in\mathbb{W}(\Gamma);W\cap\partial\Lambda\ne\emptyset\}=\emptyset. \qquad (4.63)$$

For each $i\in\mathbb{Z}$ such that $\mathrm{shift}(\Gamma,i)\subset Q(\Lambda)$, let $\Gamma_i=\mathrm{shift}(\Gamma,i)$. We have from (4.63) that for any negative integer $i\ge -L''$,

$$\frac{\mu_\Lambda^{\omega,\mathbb{J}_\delta}(\Gamma_\Lambda^\omega(\sigma)\supset\Gamma_i)}{\mu_\Lambda^{\omega,\mathbb{J}_\delta}(\Gamma_\Lambda^\omega(\sigma)=\Gamma)}=\frac{\mathrm{Z}(R_{\Gamma_i}^+,+,\mathbb{J}_\delta)\mathrm{Z}(R_{\Gamma_i}^-,-,\mathbb{J}_\delta)}{\mathrm{Z}(R_\Gamma^+,+,\mathbb{J}_\delta)\mathrm{Z}(R_\Gamma^-,-,\mathbb{J}_\delta)}\cdot\frac{\mathrm{Z}(R_{\Gamma_i}^-,\omega,\mathbb{J}_\delta)}{\mathrm{Z}(R_{\Gamma_i}^-,-,\mathbb{J}_\delta)}. \qquad (4.64)$$

We can see by Jensen's inequality that

$$\left(\frac{\mathrm{Z}(R_{\Gamma_i}^-,\omega,\mathbb{J}_\delta)}{\mathrm{Z}(R_{\Gamma_i}^-,-,\mathbb{J}_\delta)}\right)^{-1}\ge\exp\Big[-2\beta\delta\sum_{x\in\partial_{in}R_{\Gamma_i}^-\cap\partial_{in}\Lambda;-L''\le x^3\le -1}\mu_{R_{\Gamma_i}^-}^{\omega,\mathbb{J}_\delta}[\sigma(x)]\Big]. \qquad (4.65)$$

By a similar argument as in (4.14) and (4.15), we can show that $\mu_{R_{\Gamma_i}^-}^{\omega,\mathbb{J}_\delta}[\sigma(x)]\le -1/4$. From this and (4.65), we have that

$$\mathrm{Z}(R_{\Gamma_i}^-,\omega,\mathbb{J}_\delta)/\mathrm{Z}(R_{\Gamma_i}^-,-,\mathbb{J}_\delta)\le\exp[-4\beta\delta L|i|]. \qquad (4.66)$$

By the definition of $\mathbb{J}_\delta$, we can see that

$$\Phi_\delta^{\mathrm{T}}(C) = \Phi_\delta^{\mathrm{T}}(\mathrm{shift}(C, i)) \tag{4.67}$$

if $C \cap \partial\Lambda = \emptyset$ and $\mathrm{shift}(C, i) \cap \partial\Lambda = \emptyset$. We can see from (4.4) (see (3.15) and (3.23)), (4.62) and (4.67) that

$$\begin{aligned} &\Big| \sum_{C \in \mathbf{C}; C \cap \Gamma_i \neq \emptyset} \mathbf{1}_{L',2l}(C)\Phi_\delta^{\mathrm{T}}(C) - \sum_{C \iota \Gamma} \mathbf{1}_{L',2l}(C)\Phi_\delta^{\mathrm{T}}(C) \Big| \\ \leq\ & 16L^2 c_1' \exp[-(\beta - \bar{\beta})c_{36} L^{\frac{2}{3}} (\log L)^2] \\ &\quad + c_{37}\mathbb{M}^2\big(\mathrm{cyl}(\Lambda, -(K_5 \log L)^2, (K_5 \log L)^2)\big) \\ &\quad + c_{37}\mathbb{M}^2\big(\mathrm{cyl}(\Lambda, -L'' - (K_5 \log L)^2, -L'' + (K_5 \log L)^2)\big), \end{aligned} \tag{4.68}$$

which together with (3.13) implies that

$$\frac{\mathrm{Z}(R_{\Gamma_i}^+, +, \mathbb{J}_\delta)\mathrm{Z}(R_{\Gamma_i}^-, -, \mathbb{J}_\delta)}{\mathrm{Z}(R_\Gamma^+, +, \mathbb{J}_\delta)\mathrm{Z}(R_\Gamma^-, -, \mathbb{J}_\delta)} \leq c_{38} \exp[c_{39} L(\log L)^2]. \tag{4.69}$$

Therefore, we have from (4.64), (4.66) and (4.69) that

$$\mu_\Lambda^{\omega,\mathbb{J}_\delta}(\Gamma_\Lambda^\omega(\sigma) \supset \Gamma_i) \leq c_{38} \exp[-4\beta\delta L|i| + c_{39} L(\log L)^2]\, \mu_\Lambda^{\omega,\mathbb{J}_\delta}(\Gamma_\Lambda^\omega(\sigma) = \Gamma). \tag{4.70}$$

Recall that we only consider the case where $i \leq -l/2$. Then, we can prove (4.61) from (4.24), (4.39), (4.40), (4.62) and (4.70).

Let $p = 2$ or 4 and let $1 \leq i \leq k_p$. Note that F_i^p is connected, and that we can regard F_i^p as the union of all faces w such that $\mathbb{M}^2(w \cap F_i^p) = 1$. For each $1 \leq i \leq k_3$, there exists a union of faces $G_i \supset F_i^3$ such that G_i is connected and that

$$\mathbb{M}^2(G_i) \leq 2\mathbb{M}^2(F_i^3). \tag{4.71}$$

We can see this, as follows. By the definition of $\{F_i^3\}_{i=1}^{k_3}$ (see (4.45) and (4.46)), we can connect all components of F_i^3 by using only faces in $\mathrm{cyl}(\partial\Lambda, -m, 0)$. Let G_i be the smallest one among such connected sets containing F_i^3. For each $k_2 \geq 0$, each $k_3 \geq 1$ and each $k_4 \geq 0$, we can see from (4.44) that the number of the combinations of the starting points of $\{F_i^2\}_{i=1}^{k_2}$, $\{G_i\}_{i=1}^{k_3}$ and $\{F_i^4\}_{i=1}^{k_4}$ is bounded from above, by

$$\binom{8K_4 L \log L}{k_2}\binom{16L^2}{k_3}\binom{16L^2}{k_4}. \tag{4.72}$$

Thus, we have from (4.37), (4.47), (4.61) and (4.72) that

$$\begin{aligned}
&\sum_{\mathbb{W}\in\mathcal{SW}_3} P^\delta_\Lambda(\mathbb{W}) \\
\le\ & c_{40}L\exp[-(\beta-c_{35})K_2L(\log L)^3+c_{30}L\log L]\times \\
& \Big(\sum_{k_2=1}^{8K_4L\log L}\sum_{k_3=1}^{8L}\sum_{k_4=1}^{8L}\sum_{n_1\ge1}\cdots\sum_{n_{k_2+k_3+k_4}\ge1}\binom{8K_4L\log L}{k_2}\binom{16L^2}{k_3}\binom{16L^2}{k_4} \\
&\quad \kappa^{n_1+\cdots+n_{k_2+k_3+k_4}}e^{-(\beta/3-c_{32})(n_1+\cdots+n_{k_2+k_3+k_4})}\Big) \\
\le\ & c_{41}\exp[-(\beta-c_{42})K_2L(\log L)^3]. \qquad (4.73)
\end{aligned}$$

Case 4. We will consider the case where $\mathbb{W}\in\mathcal{SW}_4$. Recall that

$$\mathcal{SW}_4=\{\mathbb{W}\in\mathcal{SW}''\backslash(\mathcal{SW}_1\cup\mathcal{SW}_2); z_{\min}(\mathbb{W})>-l/2\}.$$

We define $\overline{\Gamma}(\mathbb{W})$, $\{F^2_i(\mathbb{W})\}_{i=0}^{k_2}$, $\{F^3_i(\mathbb{W})\}_{i=1}^{k_3}$ and $\{F^4_i(\mathbb{W})\}_{i=1}^{k_4}$ for each $\mathbb{W}\in\mathcal{SW}_4$ in the same way as in (4.38)-(4.46). Consider the set (see (1.27))

$$\mathcal{SW}_{41}=\left\{\mathbb{W}\in\mathcal{SW}_4;\ \begin{array}{l}\text{for some } i\ge1,\ \mathbb{M}^2(F^2_i(\mathbb{W}))\ge K_5\log L\\ \text{and } \mathrm{b}(F^2_i(\mathbb{W}),\overline{\Gamma}(\mathbb{W}))<z_{\min}(\mathbb{W})-l/8\end{array}\right\}.$$

Then, by a similar argument as in case 1 together with the definition of $z_{\min}$, we have that

$$\sum_{\mathbb{W}\in\mathcal{SW}_{41}} P^\delta_\Lambda(\mathbb{W})\le c_{43}\exp[-(\frac{\beta}{5}-c_{44})K_5\log L]. \qquad (4.74)$$

Let $\mathbb{W}\in\mathcal{SW}_4\backslash\mathcal{SW}_{41}$ and let $\Gamma=\Gamma(\mathbb{W})$. We can see by the definition of $\mathcal{SW}_4$ (or $\mathcal{SW}$) that $\Theta(\Gamma)\ni x$ for some $x\in\Lambda$ with $x^3=-l$. For such a point $x\in\Lambda$ and each $z\in\mathbb{Z}$ with $-3l/4\le z\le -5l/8$, we define

$$\Theta(\Gamma,x,z)=\{y\in\Theta_z(\Gamma); y \text{ is } l_\infty\text{-connected to } x \text{ in } \Theta_z(\Gamma)\}\cap \mathrm{H}(z), \qquad (4.75)$$

where

$$\Theta_z(\Gamma)=\{y\in\Theta(\Gamma); y^3\le z\}. \qquad (4.76)$$

We define $\Theta(\Gamma,x,z)=\emptyset$ for all $z\in\mathbb{Z}$ with $-3l/4\le z\le -5l/8$ if $\Theta(\Gamma)\not\ni x$. For each $x\in\Lambda$ with $x^3=-l$, we define

$$\mathcal{SW}_{42}(x)=\left\{\mathbb{W}\in\mathcal{SW}_4\backslash\mathcal{SW}_{41};\ \begin{array}{l}|\Theta(\Gamma(\mathbb{W}),x,z)|\ge L^{\frac{2}{3}} \text{ for all } z\in\mathbb{Z}\\ \text{with } -3l/4\le z\le -5l/8\end{array}\right\}.$$

For each $\mathbb{W}\in\mathcal{SW}_{42}(x)$, we can see by the definition of $\mathcal{SW}_{42}(x)$ that for $p=2$ or 4, there exists some $1\le i\le k_p$ such that

$$\mathbb{M}^2(F^p_i(\mathbb{W}))\ge L^{\frac{1}{3}}l/8\ge K_2L(\log L)^2/8. \qquad (4.77)$$

We define

$$\mathcal{SW}_{42} = \cup_{x \in \Lambda; x^3 = -l} \mathcal{SW}_{42}(x)$$

and

$$\mathcal{SW}_{43} = \mathcal{SW}_4 \backslash (\mathcal{SW}_{41} \cup \mathcal{SW}_{42}).$$

Then, by a similar argument as in case 3, the definition of $\mathcal{SW}_{42}$ and (4.77), we have that for $N = \lceil K_2 L (\log L)^2 / 8 \rceil$ and $N' = \lceil 8 K_4 L (\log L) \rceil$

$$\begin{aligned}
& \sum_{\mathbb{W} \in \mathcal{SW}_{42}} P_\Lambda^\delta(\mathbb{W}) \\
\leq \; & 4L^2 \Big(\sum_{k_2=1}^{N'} \sum_{k_3=1}^{8L} \sum_{k_4=1}^{8L} \sum_{n_1 \geq N} \sum_{n_2 \geq 1} \cdots \sum_{n_{k_2+k_3+k_4} \geq 1} \binom{N'}{k_2} \binom{16L^2}{k_3} \binom{16L^2}{k_4} \\
& \qquad \kappa^{n_1 + \cdots + n_{k_2+k_3+k_4}} e^{-(\beta/3 - c_{32})(n_1 + \cdots + n_{k_2+k_3+k_4})} \Big) \\
\leq \; & c_{45} \exp[-\frac{1}{24}(\beta - c_{46}) K_2 L (\log L)^2]. \qquad (4.78)
\end{aligned}$$

Finally, we will consider the case where $\mathbb{W} \in \mathcal{SW}_{43}$. Let us fix $\mathbb{W} \in \mathcal{SW}_{43}$ and let $\Gamma = \Gamma(\mathbb{W})$. In this case, there exists some $x \in \Lambda$ with $x^3 = -l$ such that $|\Theta(\Gamma, x, z)| < L^{\frac{2}{3}}$ for some $z \in \mathbb{Z}$ with $-3l/4 \leq z \leq -5l/8$. Let us fix such a point $x \in \Lambda$ and an integer $z \in \mathbb{Z}$. We define

$$\mathcal{A}(\Theta(\Gamma, x, z)) = \left\{ \sigma \in \Omega_\Lambda; \begin{array}{l} \mathbb{W}(\Gamma_\Lambda^\omega(\sigma)) \in \mathcal{SW}_{43}, \; \Theta(\Gamma_\Lambda^\omega(\sigma)) \ni x \\ \text{and } \Theta(\Gamma_\Lambda^\omega(\sigma), x, z) = \Theta(\Gamma, x, z) \end{array} \right\},$$

$$\mathcal{A}^\pm(\Theta(\Gamma, x, z)) = \{ \sigma \in \Omega_\Lambda; \text{for some } \eta \in \Omega_{\Theta(\Gamma,x,z)}, \; \eta_{\Theta(\Gamma,x,z)} \sigma \in \mathcal{A}(\Theta(\Gamma, x, z)) \}$$

and

$$\mathcal{A}^+(\Theta(\Gamma, x, z)) = \{ \sigma \in \mathcal{A}^\pm(\Theta(\Gamma, x, z)); \sigma(y) = +1 \text{ for all } y \in \Theta(\Gamma, x, z) \}.$$

Then, we have by the finite energy property that

$$\mu_\Lambda^{\omega, \mathbb{J}\delta}\Big(\mathcal{A}(\Theta(\Gamma, x, z))\Big) \leq \exp[12\beta |\Theta(\Gamma, x, z)|] \, \mu_\Lambda^{\omega, \mathbb{J}\delta}\Big(\mathcal{A}^+(\Theta(\Gamma, x, z))\Big). \qquad (4.79)$$

By (A.6) and standard Peierls' argument, we have that

$$\mu_\Lambda^{\omega, \mathbb{J}\delta}\Big(\mathcal{A}^+(\Theta(\Gamma, x, z))\Big) \leq c_{47} \exp[-(\beta - c_{48}) l/2],$$

which together with (4.79) implies that

$$\begin{aligned}
\mu_\Lambda^{\omega, \mathbb{J}\delta}\Big(\mathcal{A}(\Theta(\Gamma, x, z))\Big) \; \leq \; & c_{47} \exp[12\beta L^{\frac{2}{3}} - (\frac{\beta}{2} - c_{49}) K_2 L^{\frac{2}{3}} (\log L)^2] \\
\leq \; & c_{50} \exp[-(\frac{\beta}{2} - c_{51}) K_2 L^{\frac{2}{3}} (\log L)^2]. \qquad (4.80)
\end{aligned}$$

For each $x \in \Lambda$ with $x^3 = -l$ and each collection $\Theta(z)$ of l_∞-connected components in $\mathrm{H}(z) \cap \Lambda$, we define

$$\mathcal{A}(x, \Theta(z)) = \left\{ \sigma \in \Omega_\Lambda; \begin{array}{l} \mathbb{W}(\Gamma^\omega_\Lambda(\sigma)) \in \mathcal{SW}_{43},\ \Theta(\Gamma^\omega_\Lambda(\sigma)) \ni x \\ \text{and } \Theta(\Gamma^\omega_\Lambda(\sigma), x, z) = \Theta(z) \end{array} \right\}.$$

We have from (4.80) that

$$\begin{aligned} \sum_{\mathbb{W} \in \mathcal{SW}_{43}} P^\delta_\Lambda(\mathbb{W}) &\leq \sum_x \sum_z \sum_{\Theta(z)} \mu^{\omega, \mathbb{J}\delta}_\Lambda\Big(\mathcal{A}(x, \Theta(z))\Big) \\ &\leq c_{50} \exp[-(\frac{\beta}{2} - c_{51}) K_2 L^{\frac{2}{3}} (\log L)^2] \sum_x \sum_z \sum_{N=1}^{L^{\frac{2}{3}}} \binom{4L^2}{N} \\ &\leq c_{52} \exp[-(\frac{\beta}{2} - c_{53}) K_2 L^{\frac{2}{3}} (\log L)^2], \end{aligned} \tag{4.81}$$

where $\sum_x$ and $\sum_z$ stand for $\sum_{x \in \Lambda; x^3 = -l}$ and $\sum_{z \in \mathbb{Z}; -3l/4 \leq z \leq -5l/8}$, respectively, and $\sum_{\Theta(z)}$ stands for the summation over all collections $\Theta(z)$ of l_∞-connected components in $\mathrm{H}(z) \cap \Lambda$ such that $|\Theta(z)| < L^{\frac{2}{3}}$.

Let us fix $\varepsilon > 0$. From (4.9), (4.10), (4.12), (4.17), (4.34), (4.35), (4.73), (4.74), (4.78) and (4.81), we have that for some K_3, K_4, K_5 and K_2 large enough and for sufficiently large $L \in \mathbb{N}$,

$$\mu^{\omega, \mathbb{J}\delta}_\Lambda((\mathcal{A}^\omega_\Lambda)^c) \leq \varepsilon L^{-3},$$

and we finished the proof of Proposition 2.2.

Appendix

In this appendix, we will prove the claims which we used in Section 4. By $\partial\Theta$, we will denote $\partial Q(\Theta)$ if $\Theta \subset \mathbb{Z}^d$. Let 1

$$\mathcal{C}(L, d) = \left\{ \gamma; \begin{array}{l} \gamma = \partial\Theta \text{ for some } l_\infty\text{-connected } \Theta \subset \Lambda_d(L) \\ \text{which satisfies that } \Theta^c \text{ is } l_\infty\text{-connected} \end{array} \right\}. \tag{A.1}$$

The set Θ is uniquely determined by each $\gamma \in \mathcal{C}(L, d)$, and hence is denoted by $\Theta(\gamma)$. We define for each $i \leq d$,

$$\mathcal{C}_i(L, d) = \left\{ \gamma \in \mathcal{C}(L, d); \begin{array}{l} \text{if } x \in \Theta(\gamma) \text{ and } y \in \mathrm{L}^i(x) \\ \text{with } y^i \leq x^i, \text{ then } y \in \Theta(\gamma) \end{array} \right\},$$

where $\mathrm{L}^i(x) = \{y \in \Lambda_d(L); y^j = x^j \text{ for any } j \neq i\}$. We also define

$$\mathcal{S}(L, d) = \cap_{i=1}^d \mathcal{C}_i(L, d).$$

By $\hat{\partial}\Lambda_d(L)$, we will denote

$$\partial\Lambda_d(L) \backslash \{x \in \mathbb{R}^d; x^d = L + (1/2) \text{ or } -L + (1/2)\}.$$

Lemma A.1. *Suppose that $d \geq 2$. For each $i \leq d$, consider the map $\varphi_i : \mathcal{C}(L,d) \ni \gamma \mapsto \varphi_i(\gamma) \in \mathcal{C}_i(L,d)$ which satisfies that for any $x \in \Lambda_d(L)$*

$$|\Theta(\gamma) \cap \mathrm{L}^i(x)| = |\Theta(\varphi_i(\gamma)) \cap \mathrm{L}^i(x)|. \tag{A.2}$$

Then, for each $i \leq d$ and any $\gamma \in \mathcal{C}(L,d)$

$$\mathbb{M}^{d-1}(\gamma) \geq \mathbb{M}^{d-1}(\varphi_i(\gamma)) \tag{A.3}$$

holds. Moreover, for each $i \leq d$ and any $\gamma \in \mathcal{C}(L,d)$ with $\mathbb{M}^{d-1}(\gamma \backslash \hat{\partial}\Lambda_d(L))/\mathbb{M}^{d-1}(\gamma) \leq 1/2$

$$\frac{\mathbb{M}^{d-1}(\gamma \backslash \partial\Lambda_d(L))}{\mathbb{M}^{d-1}(\gamma)} \geq \frac{\mathbb{M}^{d-1}(\varphi_i(\gamma) \backslash \partial\Lambda_d(L))}{\mathbb{M}^{d-1}(\varphi_i(\gamma))} \tag{A.4}$$

and

$$\frac{\mathbb{M}^{d-1}(\gamma \backslash \hat{\partial}\Lambda_d(L))}{\mathbb{M}^{d-1}(\gamma)} \geq \frac{\mathbb{M}^{d-1}(\varphi_i(\gamma) \backslash \hat{\partial}\Lambda_d(L))}{\mathbb{M}^{d-1}(\varphi_i(\gamma))} \tag{A.5}$$

hold.

See Section 3 of [S,02] for the proof of (A.3) and (A.4). To obtain (A.5), we have only to replace ∂ with $\hat{\partial}$ in Section 3 of [S,02].

Corollary A.2. *Suppose that $d = 3$ and $\gamma \in \mathcal{C}(L,3)$. Then, it holds that*

$$\mathbb{M}^2(\gamma \backslash \hat{\partial}\Lambda_3(L)) \geq \frac{1}{4}\mathbb{M}^2(\gamma). \tag{A.6}$$

Proof. From (A.5), we have only to show (A.6) for all $\gamma \in \mathcal{S}(L,3)$. For each $\gamma \in \mathcal{S}(L,3)$ and each $i \leq 3$, we set

$$S_{\pm i}(\gamma) = \gamma \cap \{x \in \mathbb{R}^3; x^i = \pm L + (1/2)\}.$$

Let $\lambda > 0$ be a constant to be specified later. Let us fix $\gamma \in \mathcal{S}(L,3)$. If $\mathbb{M}^2(S_{-3}(\gamma)) \geq \lambda\mathbb{M}^2(\gamma)$, then we have that

$$\begin{aligned} &\frac{\mathbb{M}^2(\gamma \backslash \hat{\partial}\Lambda_3(L))}{\mathbb{M}^2(\gamma)} \\ = \ &\frac{1}{\mathbb{M}^2(\gamma)}\Big(2\mathbb{M}^2(S_{-3}(\gamma)) + \sum_{i=1}^{2}\big[\mathbb{M}^2(S_{-i}(\gamma)) - \mathbb{M}^2(S_{+i}(\gamma))\big]\Big) \\ \geq \ &2\lambda. \end{aligned} \tag{A.7}$$

Hence, we assume that $\mathbb{M}^2(S_{-3}(\gamma)) < \lambda\mathbb{M}^2(\gamma)$. Since we can see by the definition of $\mathcal{S}(L,3)$ that for any $i \leq 3$,

$$2L\mathbb{M}^2(S_{+i}(\gamma)) \leq |\Theta(\gamma)| \leq 2L\mathbb{M}^2(S_{-3}(\gamma)),$$

we have that for any $i \le 3$,

$$\mathbb{M}^2(S_{+i}(\gamma)) < \lambda \mathbb{M}^2(\gamma). \tag{A.8}$$

We have from (A.8) that

$$\begin{aligned} \frac{\mathbb{M}^2(\gamma \backslash \hat{\partial}\Lambda_3(L))}{\mathbb{M}^2(\gamma)} &\ge \frac{1}{\mathbb{M}^2(\gamma)}\Big(\frac{\mathbb{M}^2(\gamma)}{2} - \sum_{i=1}^{2} \mathbb{M}^2(S_{+i}(\gamma))\Big) \\ &\ge (1-4\lambda)/2. \end{aligned} \tag{A.9}$$

Therefore, from (A.7), (A.9) and $\lambda = 1/8$, we obtained (A.6).

Corollary A.3. *Suppose that $d = 2$ and $\gamma \in \mathcal{C}(L, 2)$. If*

$$\frac{\mathbb{M}^1(\gamma \backslash \partial\Lambda_2(L))}{\mathbb{M}^1(\gamma)} < \frac{1}{4} \tag{A.10}$$

holds, then we have that

$$\mathbb{M}^1(\gamma) \ge 8L \qquad \textit{and} \qquad \mathbb{M}^1(\gamma \cap \partial\Lambda_2(L)) \ge 6L, \tag{A.11}$$

and that

$$|\Theta(\gamma)| \ge 3(4L^2)/4. \tag{A.12}$$

Proof. Let us fix $\gamma \in \mathcal{C}(L, 2)$. Let $\gamma' = \varphi_2 \circ \varphi_1(\gamma)$. Assume that $\mathbb{M}^1(\gamma) < 8L$. Then, we have from (A.3) that $\mathbb{M}^1(\gamma') < 8L$. By this and the definition of γ', we can see that

$$S_{+1}(\gamma') = \emptyset \qquad \text{or} \qquad S_{+2}(\gamma') = \emptyset, \tag{A.13}$$

which implies that

$$\mathbb{M}^1(\gamma' \backslash \partial\Lambda_2(L))/\mathbb{M}^1(\gamma') \ge 1/4. \tag{A.14}$$

From this and (A.4), we have that

$$\mathbb{M}^1(\gamma \backslash \partial\Lambda_2(L))/\mathbb{M}^1(\gamma) \ge 1/4, \tag{A.15}$$

which contradicts (A.10). Thus, we obtained the first inequality in (A.11). Since

$$\mathbb{M}^1(\gamma \backslash \partial\Lambda_2(L))/\mathbb{M}^1(\gamma) \ge 1 - \mathbb{M}^1(\gamma \cap \partial\Lambda_2(L))/\mathbb{M}^1(\gamma),$$

we can see that the second inequality in (A.11) is also true.

We will show (A.12). Assume that $|\Theta(\gamma)| < 3(4L^2)/4$. Then, we can see from (A.2) that $|\Theta(\gamma')| < 3(4L^2)/4$, which implies that $\mathbb{M}^1(\gamma' \backslash \partial\Lambda_2(L)) \ge 2L$. By this and the definition of $\mathcal{S}(L, 2)$, we have that

$$\mathbb{M}^1(\gamma' \backslash \partial\Lambda_2(L))/\mathbb{M}^1(\gamma') \ge 2L/8L = 1/4,$$

which together with (A.4) contradicts (A.10). Thus, we obtained (A.12).

Added in proof. Let $d=3$. We define the surface tension in $(0,0,1)$ direction τ_β by (see [MMR,92])

$$\tau_\beta = - \lim_{L\to\infty} \frac{1}{\beta(2L)^2} \log \frac{\mathrm{Z}(\Lambda_3(L),\omega_0)}{\mathrm{Z}(\Lambda_3(L),+)}. \tag{A.16}$$

Here, ω_0 is the boundary condition defined in (1.15).

Theorem A.4. *Let $d=3$. Consider a stochastic Ising model on the square $\Lambda_3(L)$ with the free boundary condition. Then, there exists $\beta_0>0$ such that for any $\beta \geq \beta_0$ and any $L \in \mathbb{N}$*

$$\mathrm{gap}(\Lambda_3(L),\phi) \geq B \exp\bigl(-4\beta\tau_\beta L^2 - \beta C L^{\frac{5}{3}} (\log L)^2\bigr) \tag{A.17}$$

holds, where $B = B(c_m, c_M) > 0$ and $C = C(\beta) > 0$.

From the proof of Proposition 2.2, we can see under the same hypothesis as in Proposition 2.2 that for any $\varepsilon > 0$ and any $\kappa > 0$ there exists $K_6 = K_6(\beta,\varepsilon,\kappa) > 0$ such that for sufficiently large $L \in \mathbb{N}$ and any $x \in \Lambda(-L, -L+M-3l)$,

$$\mu^{+,\hat{\mathbb{J}}_\delta}_{\Lambda(-L,-L+M)}(\sigma(x)=+1) - \mu^{\eta,\hat{\mathbb{J}}_\delta}_{\Lambda(-L,-L+M)}(\sigma(x)=+1) \leq \varepsilon L^{-\kappa}. \tag{A.18}$$

From this and the same argument as in Section 4 and Appendix in [Ma,94] (or Appendix in [CGMS,96]), we can obtain Theorem A.4.

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