# Chaos in Black Holes Surrounded by Electromagnetic Fields

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**Abstract**

In this paper we study the occurrence of chaos for charged particles moving around a Schwarzschild black hole, perturbed by uniform electric and magnetic fields. The appearance of chaos is analyzed resorting to the Poincaré-Melnikov method.

Keywords: Chaotic dynamics; Black holes; Ernst metric; Melnikov theory.

PACS numbers: 04.20.-q, 05.45.-a, 95.10.Fh, 95.30.Sf

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# 1 Introduction

In the last decade chaotic behaviour in general relativity started to be the subject of many interesting papers. Two main lines of research can be recognized. The first deals with chaoticity associated with inhomogeneous cosmological models, the second line assumes a given metric and looks for chaotic behaviour of geodesic motion in this background. An interesting selection of references can be found in [1]. In particular many papers devoted to the study of chaotic dynamics in general relativity resort to the Poincaré-Melnikov method (see e.g. [2, 3] for the general theory). The Melnikov method is an analytical criterion to determine the occurrence of chaos in integrable systems in which homoclinic (or heteroclinic) manifolds biasymptotic to unstable critical points or to periodic orbits (more generally to invariant tori) are subjected to small perturbations. Such perturbations may lead to the phenomenon of transversal intersections of the stable and unstable manifolds. This kind of dynamics can be then detected by the Melnikov functions, since they describe the transversal distance between the stable and the unstable manifolds of the critical point or periodic orbit.

The Melnikov method has been applied in many branches of physics and applied mathematics, so that it is impossible to give here even a partial account of the vast literature, and we will quote only some of the applications to general relativity that are more strictly related to the present paper (for a more complete list of references see e.g. [3, 4, 5]). Examples of applications of the Melnikov method in general relativity concern the study of the orbits around a black hole perturbed either by an external quadrupolar shell [4, 6] or gravitational radiation [1, 7].

In this work we take a slightly different approach, compared with the previous literature. Firstly, we do not analyze perturbations of the metric, but perturbations due to the interactions produced by uniform electric or magnetic fields. We consider these as perturbations to the Hamiltonian of a charged test particle in free fall in a Schwarzschild black hole. Secondly, we do not restrict ourselves to perturbations lying in the plane of the orbit, but we deal more in general with the full three dimensional problem, i.e. with perturbations which may change the plane of the orbit. Let us also remark that we will consider time-independent perturbations, and show that they produce chaos; this happens even when only the rotational invariance is broken.

The problem we study in this paper can be related to the astrophysical reality, since it is well known that magnetic fields can be associated with black holes. We will consider Schwarzschild black holes, i.e. non rotating ones, and we reserve to tackle the more interesting, but more complex problem of rotating black holes in a forthcoming paper. Isolated black holes cannot possess any properties other than mass, electric charge and angular momentum, but the medium surrounding the hole can be responsible for the magnetic field. For example, super massive black holes can acquire surrounding matter either by gravitationally pulling interstellar gas into its vicinity, or by the disruption of passing stars. The surrounded matter will be shaped in an accretion disk in a state of plasma, that produces a magnetic field.

Electric fields are less likely to be found near a black hole, although the Blandford-Znajek mechanism [8] is a process that can develop a potential difference (i.e. an electric field) between the poles and the equator of a black hole spinning in a magnetic field pointing along the axis of rotation.

On the other hand, it is not easy to account for the presence of charged particles since the ionized gas, that is shaped in an accretion disk surrounding a black hole, is in a state of plasma, and a plasma is electrically neutral. One way to account for the presence of charged particles is to consider again the Blandford-Znajek mechanism, that, if the field strength is large enough, can separate charges and accelerate them to relativistic velocities. Otherwise, we can model the motion of electromagnetic currents in a macroscopic piece of plasma considering the equivalent problem of the motion of a charged particle moving around a black hole surrounded by a magnetic field. This can be done since the magnetic field has the same effect on an electric current in a macroscopic piece of plasma as on a single charge.

As said before, we will deal in this paper only with uniform fields. Although the fields which can be found in the astrophysical reality, e.g. the Blanford-Znajek ones, are not uniform, some authors (see [9] and references therein) pointed out that models with uniform external fields are a fairly good approximation in order to explain the qualitative features of a black hole in a magnetic field, indeed qualitative arguments indicate the existence of a quasi-uniform field.

In the next section we discuss the equation of motion for the Schwarzschild solution, we find the homoclinic orbit and show that the perturbed systems we are considering are of type III, according to the classification given in [3]. In section 3 we present a summary of the Melnikov method for a system of this type. In section 4 we consider the perturbations given by uniform electric and magnetic fields. In the last section we apply the Melnikov method and we prove the occurrence of chaos in the perturbed system both in the case of electric and of magnetic field, but with a great difference: the occurrence of chaos is a first-order effect (in the strength of the field) in the case of the electric field, whereas it is a second-order effect in the case of the magnetic field. In particular, the Hamiltonian perturbed by a constant magnetic field turns out to be integrable at the first-order, and the first non-vanishing contribution to the Melnikov integral comes from the second-order term. On the other hand, the occurrence of chaos in this situation has been already proven [10] by means of numerical arguments based on the study of the trajectories in the Ernst metric; it can be interesting to point out that our approach confirms this result using a completely different (analytic) approach. As another interesting result, we obtain that in both cases (electric and magnetic fields) only the components of the fields on the plane of motion are responsible for the chaotic behavior, whereas the components normal to the plane are not.

It can be significant to remark that, although the occurrence of chaos can often be expected since the equations of motion are integrable only in very special cases, the consequences of chaos may be greatly relevant and sometimes far-reaching. Indeed, we can observe that chaotic systems may exhibit a very rich dynamics, including also regions of stability, periodic orbits, regions of ergodicity and so on, that can also lead to macroscopically observable phenomena. We recall, just to mention few examples, that chaotic defocusing of light might make black holes bright [11], (and hence observable) and that the depletion of the outer asteroidal belt in the Solar System might be explained with the existence of a chaotic region [12]. Moreover, this last example also reveals the richness of chaotic dynamic since it was shown that the only few asteroids found in the outer belt are resonant and dynamically protected [13].

## 2 Equations of Motion

In this section we study the motion of a relativistic (charged) particle in free fall in a gravitational field. To describe the motion of a particle moving in a space-time with a metric $g_{ab}$ it is convenient to consider the action (with $c = 1$):

$$S = \frac{m}{2}\int ds \qquad \text{or} \qquad S = \frac{m}{2}\int g_{ab}\dot{x}^a\dot{x}^b ds \tag{1}$$

where $x^a(s)$ denotes the world-line of a particle and $m$ denotes the mass. Using the same notations as in [1], the canonical conjugate momentum to $x^a$ is $p_a = mg_{ab}\dot{x}^b$ and satisfies the mass-shell constraint $g^{ab}p_ap_b = -m^2$. The Hamiltonian of the system can be defined as:

$$H_0 = \frac{1}{2m}g^{ab}p_ap_b \tag{2}$$

Let our background metric be the Schwarzschild metric, i.e. the metric of a non-rotating black hole:

$$ds^2 = -fdt^2 + f^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3}$$

where $f = 1 - 2M/r$. Then the Hamiltonian (2) becomes:

$$H_0 = \frac{1}{2m}\left(p_r^2 f + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) - f^{-1}\left(\frac{E}{m}\right)^2 \tag{4}$$

where

$$p_r = m f^{-1}\frac{dr}{ds} \quad , \quad p_\theta = m r^2 \frac{d\theta}{ds} \tag{5}$$

and

$$E = -p_t = m f \frac{dt}{ds} \quad , \quad p_\phi = m r^2 \sin^2\theta \frac{d\phi}{ds} \quad \text{and} \quad L^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} \tag{6}$$

are conserved quantities. We can introduce the following Hamiltonian, more suitable for our purposes (i.e. for finding the orbits of a particle in free fall in the Schwarzschild metric):

$$\mathcal{H}_0 = \frac{E^2 - m^2}{2m} = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) - \frac{Mm}{r} - \frac{Mp_\theta^2}{mr^3} - \frac{Mp_\phi^2}{mr^3\sin^2\theta}, \tag{7}$$

which can also be written in terms of $\mathcal{H}_0$ as:

$$\mathcal{H}_0 = H_0 f + \frac{E^2}{2m} - \frac{mM}{r} \tag{8}$$

This new Hamiltonian defines a problem similar to the one discussed in [1], but in our case we study the motion of a particle in the three dimensional space. Exactly as in [1], we can show that our Hamiltonian admits an unstable circular orbit $\gamma$ in the plane $\theta = \pi/2$, together with an homoclinic loop (or, to be more precise, a one parameter family of homoclinic loops) biasymptotic to this orbit; the spherical symmetry of the problem implies that the same happens for every plane for the origin.

Choosing for simplicity the plane $\theta = \pi/2$, it can be easily shown that this unstable circular orbit $\gamma$ has radius

$$r_u = \frac{6M}{1+\beta}, \tag{9}$$

where $\beta = \sqrt{1 - 12M^2m^2/p_\phi^2} < 1$, and that there are homoclinic orbits to this invariant set for $0 \le \beta < 1/2$. The equations of motion for the homoclinic orbits are:

$$\frac{dr}{ds} = \pm\sqrt{\frac{2}{m}\left(\frac{E^2 - m^2}{2m} - \frac{p_\phi^2}{r^2} + \frac{Mm}{r} + \frac{Mp_\phi^2}{mr^3}\right)} \quad , \quad \frac{d\phi}{ds} = \frac{p_\phi^2}{mr^2}, \tag{10}$$

where the sign $-$ (resp. $+$) holds for $s < 0$ (resp. $s > 0$). Let us denote by

$$R = R(s) \quad \text{and} \quad \Phi = \Phi(s) \tag{11}$$

the expressions of $r$ and $\phi$ with respect to $s$ for the homoclinic orbit with $R(0) = r_{max}$ (the turning point) and $\Phi(0) = \pi$ (i.e. the orbit having its axis coinciding with the $x$ axis and the point $r_u$ in the positive $x$). We can observe that the maximum value of $r$, along this homoclinic solution, is $r_{max} = 6M/(1 - 2\beta)$, but we will not need the explicit expression for those functions, that can be found in [1] (at least for $R = R(s)$), we only retain, as in [14, 15], the information that $R(s)$ is an even function and $\Phi(s)$ is an odd function of $s$.

In the following sections, we will study time-independent perturbations that destroy the spherical symmetry of the unperturbed system. This situation is another example, in a completely different setting, of symmetry-breaking perturbations that were already examined in [14, 15].

The perturbations that we consider in this work can be written in Hamiltonian form as

$$H = H_0 + \epsilon\, W(r, \phi, \theta, p_r, p_\phi, p_\theta) \tag{12}$$

We want to show that the Hamiltonian in (12) defines a system of type III (according to the classification given in [3]) with $n = 1$ and $m = 2$, so that we can apply the Melnikov theory developed for this kind of problems to prove that chaos occurs in the perturbed problem.

According to [3], systems of type III can be written in the following general form, where we have denoted by $\omega$ (to avoid any confusion with the azimuthal variable $\phi$) the angle (cyclic) variables conjugated to the action variables $I$:

$$\begin{cases} \dot{x} = JD_x H_0(x, I) + \epsilon J D_x W(x, I, \omega; \epsilon) \\ \dot{I} = -\epsilon D_\omega W(x, I, \omega; \epsilon) \\ \dot{\omega} = D_I H_0(x, I) + \epsilon D_I W(x, I, \omega; \epsilon) \end{cases} \qquad (x, I, \omega) \in \mathbb{R}^{2n} \times \mathbb{R}^m \times T^m \tag{13}$$

where $0 < \epsilon \ll 1$, the dot indicates differentiation with respect to $s$, and $J$ is the standard symplectic matrix which in our case $(n = 1)$ is simply

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{14}$$

and where the subsystem, for $\epsilon = 0$,

$$\dot{x} = JD_x H_0(x, I) \tag{15}$$

is assumed to be completely integrable, and to admit a hyperbolic fixed point with a homoclinic loop connecting this point to itself.

In our case we can write the equations of motion as

$$\begin{cases} \dot{r} = \frac{\partial H_0}{\partial p_r} + \epsilon \frac{\partial W}{\partial p_r} \\ \dot{p}_r = -\frac{\partial H_0}{\partial r} - \epsilon \frac{\partial W}{\partial r} \end{cases} \tag{16$a$}$$

and

$$\begin{cases} \dot{p}_\phi = -\epsilon \frac{\partial W}{\partial \phi} \\ \dot{p}_\theta = -\frac{\partial H_0}{\partial \theta} - \epsilon \frac{\partial W}{\partial \theta} \\ \dot{\phi} = \frac{\partial H_0}{\partial p_\phi} + \epsilon \frac{\partial W}{\partial p_\phi} \\ \dot{\theta} = \frac{\partial H_0}{\partial p_\theta} + \epsilon \frac{\partial W}{\partial p_\theta} \end{cases} \tag{16b}$$

where, again, the dot indicates differentiation with respect to $s$. The subsystem (16$a$) is, for $\epsilon = 0$, precisely as in (15) and in particular it admits an hyperbolic fixed point, given in (9). The subsystem (16$b$), here written in terms of the spherical variables $\phi, \theta$, should be transformed as in the second and third lines of (13) by introducing two action variables $I$ (e.g., $I_1 = p_\phi$, $I_2 = L$, the total angular momentum) with their conjugated cyclic variables $\omega_1, \omega_2$; we shall see that it is not necessary to explicitly perform this transformation.

# 3 The Melnikov Method

According to [3], the general expression of the Melnikov functions for a system of type III reduces in our case to

$$M_i^{\overline{I}}(\omega_0) = -\int_{-\infty}^{+\infty} D_{\omega_i} W(q_0^{\overline{I}}(s); 0)\ ds \qquad (i = 1, 2) \tag{17}$$

where

$$q_0^{\overline{I}}(s) \equiv (x^{\overline{I}}(s), \overline{I}, \int^t D_I H_0(x^{\overline{I}}(\xi), \overline{I}) d\xi + \omega_0) \tag{18}$$

is a generic homoclinic orbit of the unperturbed problem. $\overline{I}$ has to be chosen so that it defines a KAM torus (see [3] for more details). Actually, the convergence of (17) is a delicate matter: indeed the integrals converge only "conditionally", i.e. when the limits of integration are allowed to approach $+\infty$ and $-\infty$ along suitable sequences $T_j^s$ and $-T_j^u$ respectively. Such sequences must be chosen as in [3], i.e. for every $\epsilon$ sufficiently small we have to consider monotonely increasing sequences of real numbers, with $j = 1, 2, \ldots$ and $\lim_{j\to\infty} T_j^{s,u} = \infty$, such that $\lim_{j\to\infty} |q_\epsilon^s(T_j^s) - q_\epsilon^u(-T_j^u)| = 0$ and $\lim_{j\to\infty} |D_\omega(q_\epsilon^s(T_j^s), 0)| = \lim_{j\to\infty} |D_\omega(q_\epsilon^u(-T_j^u), 0)| = 0$, where $q_\epsilon^{s,u}$ are trajectories of the perturbed system ($\epsilon \neq 0$) in the stable and unstable manifolds (see [3]).

Moreover we recall that the existence of simultaneous zeroes of the Melnikov functions in (17) is sufficient to prove the occurrence of chaos. In fact it is known that the presence of simultaneous zeroes (and the periodicity of the perturbation) implies an infinite sequence of transversal intersections of the stable and unstable manifold leading to a chaotic dynamics (see e.g. [3]).

With $I_1 = p_\phi,\ I_2 = L$, and dropping the superscript over $\overline{I}$ to simplify the notation, we then find from (17) the two conditions

$$M_{p_\phi} = \int_{-\infty}^{+\infty} \{p_\phi, W\}\ ds = \ 0. \tag{19}$$

and

$$M_L = \int_{-\infty}^{+\infty} \{L, W\}\ ds \ = \ 0 \tag{20}$$

where both integrals, according to (17-18), are to be evaluated along a generic homoclinic orbit $q_0^I$ (18).

Apart from a rotation, which transforms the given perturbation $W$ into a new $\widetilde{W}$ (see below for more details), we can always choose the homoclinic orbit in the plane $\theta = \pi/2$, therefore it is easily seen that (19-20) become simply

$$M_{p_\phi} = \int_{-\infty}^{+\infty} \frac{\partial \widetilde{W}}{\partial \phi}\, ds \; = \; 0 \tag{21}$$

and

$$M_L = \int_{-\infty}^{+\infty} p_\phi \, \frac{\partial \widetilde{W}}{\partial \phi}\, ds \; = \; 0 \tag{22}$$

Condition (22) turns out to be the same as (21), and then we are left with only one condition. Hence, to find transversal intersections of the stable and unstable manifolds to the periodic orbit, and therefore a chaotic behavior, we have to find simple zeroes of the Melnikov function (21). Let us remark incidentally that the transversality of these intersections is not strictly necessary, indeed – to have chaos – it would be sufficient that the crossing is "topological", i.e. that there is really a crossing from one side to the other [16].

It is interesting to remark that we have found that, in our case, only one Melnikov condition has to be studied. The same result would be clearly obtained for the same problem restricted to the plane. This shows that to study the occurrence of chaos in this kind of systems (i.e. with conserved Hamiltonian and angular momentum) it is sufficient to consider the planar problem.

As already stated, to transform the conditions (19-20) into (21-22), a rotation is necessary, and in particular one may choose the rotation in such a way that the generic homoclinic orbit $q_0^{(I)}$ is transformed precisely into the homoclinic orbit in the plane $z = 0$ with axis coinciding with the $x$ axis, as chosen in Sect. 2 (see (10-11)). This rotation is defined by the following Euler angles (with the conventions and notations as in [17]):

$$-\Omega, \; -i, \; -\omega \tag{23}$$

where (with the language of celestial mechanics) $i$ the inclination of the plane of the orbit, $\omega$ the angle of the perihelion with the line of nodes in the orbital plane, and $\Omega$ is the longitude of the ascending node. Notice that, in terms of our previous variables, one has $i = \arccos\,(p_\phi/L)$ and

$$\Omega = \omega_{01}\ ,\ \ \omega = \omega_{02} \tag{24}$$

which then play the role of the arbitrary "phases" $\omega_0$ in eq. (18).

Denoting by $A$ the matrix of this rotation, the given perturbation $W(\mathbf{x}, \mathbf{p_x})$ ($\mathbf{x} = (x, y, z)$, etc.) will assume a new expression $\widetilde{W}$ obtained replacing $\mathbf{x}$ with $A\mathbf{x}$ and $\mathbf{p_x}$ with $A\mathbf{p_x}$. It is then easy to verify that the Melnikov condition (21) becomes

$$\begin{aligned} &M(\omega,\Omega) = \\ &\int_{-\infty}^{+\infty} R(s)\Big( - C_1(R(s),\Phi(s),\omega,\Omega)\,\sin\Phi(s) + C_2(\ldots)\cos\Phi(s)\Big)\ ds = 0 \end{aligned} \tag{25}$$

with

$$\begin{aligned} C_1 &= \left(\frac{\partial\widetilde{W}}{\partial x_1}\right)_0 (\cos\omega\cos\Omega - \cos i\sin\omega\sin\Omega) + \\ &+ \left(\frac{\partial\widetilde{W}}{\partial x_2}\right)_0 (-\cos\omega\sin\Omega - \cos i\sin\omega\cos\Omega) + \left(\frac{\partial\widetilde{W}}{\partial x_3}\right)_0 \sin i\sin\omega \\ C_2 &= \left(\frac{\partial\widetilde{W}}{\partial x_1}\right)_0 (\sin\omega\cos\Omega + \cos i\cos\omega\sin\Omega) + \\ &+ \left(\frac{\partial\widetilde{W}}{\partial x_2}\right)_0 (-\sin\omega\sin\Omega + \cos i\cos\omega\cos\Omega) - \left(\frac{\partial\widetilde{W}}{\partial x_3}\right)_0 \sin i\cos\omega \end{aligned} \tag{26}$$

and where $(\partial\widetilde{W}/\partial x_i)_0$ means that in the derivative of the given $W$ with respect to $x_i$ one has to replace $\mathbf{x}$ with $A\mathbf{x}$ and finally put $z = 0$ (or $\theta = \pi/2$).

If, e.g., $i = 0$, i.e. if the problem is completely planar, including the perturbation, or if the perturbation is "generic", i.e. has no "preferred" direction in the space (as often happens, see next Section for explicit examples), and therefore it is not restrictive to assume $i = 0$, then one gets

$$C_1 = \left(\frac{\partial \widetilde{W}}{\partial x_1}\right)_0 \cos\phi_0 + \left(\frac{\partial \widetilde{W}}{\partial x_2}\right)_0 \sin\phi_0 \ , \quad C_2 = \left(\frac{\partial \widetilde{W}}{\partial x_1}\right)_0 \sin\phi_0 - \left(\frac{\partial \widetilde{W}}{\partial x_2}\right)_0 \cos\phi_0 \tag{27}$$

where $\phi_0 = -(\omega + \Omega)$, the needed rotation is simply a rotation of angle $\phi_0$ around the $z-$axis and (25) becomes

$$M(\phi_0) = \int_{-\infty}^{+\infty} \frac{\partial W}{\partial \phi}(R(s), \Phi(s) + \phi_0, \pi/2, \dot{R}(s), L, 0)\, ds = 0 \tag{28}$$

In conclusion, it is clear that verifying Melnikov conditions for the appearance of chaotic behaviour amounts to verifying the existence of values of $\omega, \Omega$ for which (25) (or (28) in the above hypothesis) is satisfied.

# 4 The Perturbations

In this section we want to consider the motion of a relativistic charged particle in a gravitational field with an electromagnetic perturbation. To this end, we need the Hamiltonian describing a particle in an electromagnetic field, i.e.

$$H = \frac{1}{2m} g^{ab}(p_a - eA_a)(p_b - eA_b) \tag{29}$$

and hence, for weak fields:

$$H = H_0 - \frac{e}{m} g^{ab} p_a A_b \ + \ \text{higher order terms} \tag{30}$$

Let us remark that we neglect the effect of the electromagnetic fields back on the metric since direct effects of the magnetic field on a charge are generally very large compared to indirect gravitational effects on the mass arising from gravity of the field energy; we recall, incidentally, that the metric of a black hole immersed in a uniform magnetic field was found as an exact solution of the Einstein-Maxwell equations [18, 10].

Let us first consider a uniform electric field in a generic direction $\mathbf{l}$; then $A_i = (-\psi, \mathbf{A}) = (-\mathcal{E}\ \hat{n}\cdot\mathbf{l}, 0) = (-\mathcal{E}(l_1\cos\phi\sin\theta + l_2\sin\phi\sin\theta + l_3\cos\theta), 0)$, where $\hat{n}$ is the unit vector in the $\mathbf{r}$ direction. The perturbed Hamiltonian (30) is then

$$H = H_0 - \epsilon E f^{-1}(l_1\cos\phi\sin\theta + l_2\sin\phi\sin\theta + l_3\cos\theta) \tag{31}$$

where $\epsilon = e\mathcal{E}/m \ll 1$ .

More interesting, and a little more complicated, is the case where the perturbation is given by a magnetic field. Let $A_i = (0, \mathbf{A}) = (0, \frac{1}{2}\mathbf{B}\times\mathbf{r})$ where:

$$\mathbf{A} = \frac{B}{2}\ (k_2 z - k_3 y, k_3 x - k_1 z, k_1 y - k_2 x), \tag{32}$$

with $\mathbf{B} = B\ \mathbf{k} = B\ (k_1, k_2, k_3)$, where $\mathbf{k}$ is a unit vector in the direction of the magnetic field. Rewriting the vector potential in spherical coordinates we obtain

$$\begin{cases} A_r = 0 \\ A_\phi = \frac{Br^2}{2}(k_3\sin^2\theta - \sin\theta\cos\theta(k_1\cos\phi + k_2\sin\phi)). \\ A_\theta = \frac{Br^2}{2}(k_2\cos\phi - k_1\sin\phi) \end{cases} \tag{33}$$

In both cases the direction of the perturbing field is generic, therefore no rotation is required (it would simply change the directions $\mathbf{l}$, $\mathbf{k}$, which are not fixed; see the remark at the end of previous section).

The perturbed Hamiltonian (29) can be written as

$$
\begin{aligned}
H = H_0 - \frac{Ber^2}{2m}\left[g^{\phi\phi}p_\phi(k_3\sin^2\theta - \sin\theta\cos\theta(k_1\cos\phi + k_2\sin\phi))\right] - \\
-\frac{Ber^2}{2m}\left[(g^{\theta\theta}p_\theta(k_2\cos\phi - k_1\sin\phi))\right] + \\
+\frac{B^2e^2r^4}{8m}\left[g^{\phi\phi}(k_3\sin^2\theta - \sin\theta\cos\theta(k_1\cos\phi + k_2\sin\phi))\right]^2 + \\
+\frac{B^2e^2r^4}{8m}\left[(g^{\theta\theta}(k_2\cos\phi - k_1\sin\phi))\right]^2
\end{aligned} \tag{34}
$$

where this time we retain also the quadratic terms in the magnetic field, in view of the discussion in the next section. Since we need in (21) (or (28)) quantities evaluated along the homoclinic orbit in the plane $\theta = \pi/2$, we can rewrite, recalling that $g^{\phi\phi} = r^{-2}(\sin\theta)^{-2}$ and $g^{\theta\theta} = r^{-2}$, the Hamiltonian (34) as:

$$
H = H_0 - \epsilon k_3\ p_\phi + \epsilon^2\frac{mr^2}{2}\left[(k_1^2 - k_2^2)\sin 2\phi - k_1k_2\cos 2\phi\right] \tag{35}
$$

where $\epsilon = \frac{Be}{2m} \ll 1$.

# 5 The Melnikov Conditions

The Melnikov integral for the perturbation produced by the uniform electric field can be found from (21) (or (28)):

$$
M_\phi = \int_{-\infty}^{+\infty} Ef^{-1}(l_1\sin(\Phi(s) + \phi_0) - l_2\cos(\Phi(s) + \phi_0))\ ds \tag{36}
$$

since $\theta = \pi/2$. Now, using the fact that $R$ and $\Phi$ are respectively even and odd functions of $s$, we can write the integral as:

$$
M_\phi = (l_1\sin\phi_0 - l_2\cos\phi_0)\int_{-\infty}^{+\infty} Ef^{-1}\cos\Phi(s)\ ds. \tag{37}
$$

or, defining two constants $L$ and $\alpha$ such that $l_1 = L\cos\alpha$ and $l_2 = L\sin\alpha$, the Melnikov condition is

$$M_\phi = J_1 L \sin(\phi_0 - \alpha) = 0 \tag{38}$$

where

$$J_1 = J_1(p_\phi, M, m) = \int_{-\infty}^{+\infty} E f^{-1} \cos\Phi(s)\ ds \tag{39}$$

From numerical evaluations and general arguments for this type of (conditionally convergent) integrals (see also [1]), we can assume that $J_1 \neq 0$, or that it vanishes for at most some isolated values of the parameters $(p_\phi, m, M)$ involved. Therefore, the Melnikov function (37) has simple zeroes and hence, thanks to the periodicity of the perturbating term in the integral, there is a infinite sequence of transversal intersections of the asymptotic stable and unstable manifolds, leading as well known to a chaotic dynamics.

In the case of the magnetic field, it is clear from (35) and (28) that the first-order term in $\epsilon$ of the perturbation gives no contribution to the Melnikov integral: actually, it is easy to see that the perturbed Hamiltonian truncated at the first-order is integrable (indeed, it admits $L^2$ as an additional constant of motion). Then, if we consider the first non-vanishing contribution to the Melnikov integral, this gives the following condition

$$[(k_1^2 - k_2^2)\sin 2\phi_0 - 2k_1 k_2 \cos 2\phi_0] \int_{-\infty}^{+\infty} R^2(s)\cos 2\Phi(s) ds = 0 \tag{40}$$

which can also be written, with obvious notations, similar to the above (37-39),

$$J_2 K \sin(2\phi_0 - \delta) = 0 \tag{41}$$

As before, we can conclude that chaos occurs also in this case, but – unlike the case of electric field – this is now a "second-order effect". On the other hand, the motion of test particles moving around a black hole immersed in a magnetic field was studied in [10], where the authors, performing a numerical study of the orbits of a particle in Ernst space-time, presented strong evidence of the occurrence of chaos and nonintegrability. As the authors acknowledge, the numerical methods they use cannot give a rigorous proof of the nonintegrability or of the occurrence of chaos. Instead, the Melnikov method is able to detect the existence of a infinite sequence of transversal intersections of the stable and unstable manifolds and hence, via the Smale-Birkoff theorem, to manifest the equivalence to a symbolic dynamics expressed by the Smale horseshoe. Therefore, the Melnikov technique provides an analytic proof of the occurrence of chaos in the problem discussed here and in [10]. In particular our analysis shows that the chaotic dynamics appears even when the reaction of the magnetic field on the black hole is neglected, but does not when the terms quadratic in the magnetic field (29) are neglected.

Finally, it can be observed that, in both cases (i.e. for both the electric and magnetic field) whereas the component of the field on the plane of motion leads to a chaotic dynamics, the component normal to the plane does not. This behavior can be explained observing that the component normal to the plane of motion is a constant on such plane. Therefore it doesn't break the symmetry of the system on the plane of motion, and hence it doesn't lead to the appearance of chaos. Since the problem is spherically symmetric, the reasoning used for the $\theta = \pi/2$ plane can be applied to every plane for the origin of coordinates. Thus, given an electric or magnetic field, on each plane for the origin (except at most the one normal to the field), chaos appears for a suitable choice of initial conditions.

**Acknowledgments** The first author (MS) is grateful to Professor Werner Israel for his enlightening comments and suggestions, and to the University of Victoria for financial support (University of Victoria Fellowship).

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