# INVERSE SPECTRAL THEORY FOR SYMMETRIC OPERATORS WITH SEVERAL GAPS: SCALAR-TYPE WEYL FUNCTIONS

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**Abstract**

Let $S$ be the orthogonal sum of infinitely many pairwise unitarily equivalent symmetric operators with non-zero deficiency indices. Let $J$ be an open subset of $\mathbb{R}$. If there exists a self-adjoint extension $S_0$ of $S$ such that $J$ is contained in the resolvent set of $S_0$ and the associated Weyl function of the pair $\{S, S_0\}$ is monotone with respect to $J$, then for any self-adjoint operator $R$ there exists a self-adjoint extension $\widetilde{S}$ such that the spectral parts $\widetilde{S}_J$ and $R_J$ are unitarily equivalent. The proofs relies on the technique of boundary triples and associated Weyl functions which allows in addition, to investigate the spectral properties of $\widetilde{S}$ within the spectrum of $S_0$. So it is shown that for any extension $\widetilde{S}$ of $S$ the absolutely continuous spectrum of $S_0$ is contained in that one of $\widetilde{S}$. Moreover, for a wide class of extensions the absolutely continuous parts of $\widetilde{S}$ and $S$ are even unitarily equivalent.

**Keywords**: symmetric operators, self-adjoint extensions, abstract boundary conditions, Weyl function.

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# Contents

# 1 Introduction

Let $S$ be a densely defined symmetric operator in a separable Hilbert space $\mathfrak{H}$ with deficiency indices $n_+(S) = n_-(S) \leq \infty$. We recall that a bounded open interval $J = (\alpha, \beta)$ is called a gap for $S$ if

$$\|2Sf - (\alpha+\beta)f\| \geq \quad (\beta - \alpha)\|f\|, \quad f \in \mathrm{dom} S. \tag{1.1}$$

If $\alpha \to -\infty$, then 1.1 turns into $(Sf, f) \geq \beta \|f\|^2$, for all $f \in \mathrm{dom}S$, meaning that $(-\infty, \beta)$ is a gap for $A$ if $S$ is semi-bounded below with the lower bound $\beta$. The problem whether there exist self-adjoint extensions $\widetilde{S}$ of $S$ preserving the gap $(\alpha, \beta)$ has been extensively investigated in the middle of the thirties. It has been positively solved by M. Stone, K. Friedrichs and H. Freidental for operators semi-bounded from below ($\alpha = -\infty$) (see, [1, 27]) and by M.G.Krein [22] for the case of a finite gap. The problem to describe completely the set $\mathrm{Ext}_S(\alpha, \beta)$ of all self-adjoint extensions $\widetilde{S}$ of $S$ preserving the gap has been solved by M.G.Krein [22], [23](see also [1],[27]) in the case $J = (-\infty, \beta)$ and in [17] for a finite gap $J = (\alpha, \beta)$.

M. G. Krein [22] has investigated the spectrum of self-adjoint extensions $\widetilde{S}$ within a gap $J$ of a densely defined symmetric operator $S$ with finite deficiency indices. Namely, Krein has shown that if $R$ is any self-adjoint operator on some auxiliary separable Hilbert space such that $\dim(E_R(J)\mathfrak{K}) \leq n$, then there exists a self-adjoint extension $\widetilde{S}$ such that the part $R_J := R \upharpoonright E_R(J)\mathfrak{K}$ of $R$ is unitarily equivalent to $\widetilde{S}_J := \widetilde{S} \upharpoonright E_{\widetilde{S}}(J)\mathfrak{K}$., i.e $\widetilde{S}_J \cong R_J$, where $E_R(\cdot)$ and $E_{\widetilde{S}}(\cdot)$ are the spectral measures of $R$ and $\widetilde{S}$, respectively.

The result was generalized in [8] to the case of infinite deficiency indices. In this case it was shown that if $R$ is any self-adjoint operator with pure point spectrum, then there exists a self-adjoint extension $\widetilde{S}$ such that $\widetilde{S}_J \cong R_J$. Naturally, the question arises whether we can put other kind of spectra into $J$, for instance, absolutely continuous or singular continuous spectrum. This problem has been investigated in a series of papers [2, 7, 8, 9, 10, 11]. For the class of (weakly) significant deficient symmetric operators (for the definition see [2, 9]) it was shown [2, Theorem 6.2] that for any auxiliary self-adjoint operator $R$ and any open subset $J_0 \subseteq J$ there exists a self-adjoint extension $\widetilde{S}$ such that

$$\widetilde{S}^{pp} \cong R_J^{pp}, \tag{1.2}$$

$$\widetilde{S}_J^{ac} \cong R_J^{ac}, \tag{1.3}$$

$$\sigma_{sc}(\widetilde{S}) \cap J = \overline{J_0} \cap J \tag{1.4}$$

where $R^{ac}$, $\widetilde{S}^{ac}$ and $R^{pp}$, $\widetilde{S}^{pp}$ denote the absolutely continuous and pure point parts of $R, \widetilde{S}$, respectively. Notice that the deficiency indices of (weakly) significant deficient symmetric operators are always infinite. The assumption that $S$ is a (weakly) significant deficient symmetric operator was essentially used in the first proof of (1.3) and (1.4). Later on this assumption was dropped for the third relation (1.4), see [10]. However, one has to mention that the singular continuous spectrum obtained in [10] belongs to a certain class of sets which excludes a wide class of possible sets, for instance, Cantor sets.

In [11] an attempt was made to remove all these restrictions assuming that the symmetric operator $S$ has a special structure, namely,

$$S = \bigoplus_{k=1}^{\infty} S_k \quad \text{on} \quad \mathfrak{K} = \bigoplus_{k=1}^{\infty} \mathfrak{K}_k, \tag{1.5}$$

where each of the operators $S_k$ is unitarily equivalent to a fixed (i.e. $k$-independent) densely defined closed symmetric operator $A$ in a separable Hilbert space and $A$ has positive deficiency indices. If $J$ is a gap of $A$ (and therefore of $S_k$ for every $k$), then for any self-adjoint operator on any separable Hilbert space $\mathfrak{R}$ there exists a self-adjoint extension $\widetilde{S}$ of $S$ in $\mathfrak{K}$ such that the relations (1.2) and (1.3) hold as well as $\sigma_{sc}(\widetilde{S}) \cap J = \sigma_{sc}(R) \cap J$, cf. [11, Theorem 10]. We remark that if $n_\pm(A) < \infty$, then the operator $S$ is not (weakly) significant deficient. Thus [11, Theorem 10] weakens considerable the property (1.4) for the special case (1.5). The proof relies on a technique which is quite different from that of [2, 8, 9, 10] and which is called the method of boundary triples and associated Weyl functions. We describe the method briefly in the next section.

The previous results advise the assertion that for any densely defined closed symmetric operator $S$ with infinite deficiency indices and gap $J$ there is a self-adjoint extension $\widetilde{S}$ such that the conditions (1.2), (1.3) and $\widetilde{S}^{sc}_J \cong R^{sc}_J$ are satisfied for any auxiliary self-adjoint operator $R$. Indeed, this is true and was proved in [7, Theorem 27]. In particular, $\widetilde{S}$ has the same spectrum, the same absolutely continuous and singular continuous spectrum and the same eigenvalues inside $J$ as $R$.

Since for one gap the problem on the spectral properties of self-adjoint extensions is completely solved, naturally the question arises whether is it possible to extend the results to the case of several gaps. It turns out that an analogous statement is wrong if $J$ is the union of disjoint gaps. In general, there does not even exist a self-adjoint extension $\widetilde{S}$ of $S$ such that $J \subset \rho(\widetilde{S})$.

In 1947 M.G. Krein posed the problem to find necessary and sufficient conditions for a symmetric operator with several gaps such that there is an exit space self-adjoint extension or canonical self-adjoint extension preserving the gaps. This problem has been solved in [17], where a criterion for the existence of such types of extensions has been found and a complete description of those extensions has been obtained.

In the following we always assume that there exists a self-adjoint extension $S_0$ in the original space such that $J \subseteq \rho(S_0)$ where $\rho(S_0)$ denotes the resolvent set of $S_0$. Under this assumption we are interested in the following problem: Let $S$ be a closed symmetric operator with equal deficiency indices $n_\pm(S)$ and let $J \subseteq \rho(S_0)$ be an open subset of $\mathbb{R}$. Further, let $R$ be a self-adjoint operator in a separable Hilbert space $\mathfrak{R}$ satisfying the condition $\dim(E_R(J)\mathfrak{R}) \leq n$. Does there exist a self-adjoint extension $\widetilde{S}$ of $S$ such that $\widetilde{S}_J \cong R_J$? In general, the answer to this question is no, see Example 6.1, which means, that the solution of this problem requires additional assumptions. To formulate these additional assumptions we rely on the theory of abstract boundary conditions. Using this framework for each pair $\{S, S_0\}$ there is a boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, cf. Section 2.2, such that $S_0 = S^* \upharpoonright \ker(\Gamma_0)$. To each boundary triple one associates a Weyl function $M(\cdot)$, cf. Section 2.3, which is the main tool in this approach. We demand that the Weyl function $M(\cdot)$ is monotone with respect to $J$, cf. Definition 2.3.

In the present paper we restrict ourselves to the case (1.5) which was already treated in [11]. Under this assumption we present a complete solution of the inverse spectral problem for symmetric operators with gaps and monotone Weyl function.

**THEOREM 1.1** *Let $\{S_k\}_{k=1}^{\infty}$ be a family of closed symmetric operators $S_k$ defined in the separable Hilbert spaces $\mathfrak{K}_k$ such that the operators $S_k$ are unitarily equivalent to a closed symmetric operator $A$ in $\mathfrak{H}$ with equal positive deficiency indices. If there exists a boundary triple $\Pi_0 = \{\mathcal{H}_0, \Gamma_0^0, \Gamma_1^0\}$ for $A^*$ such that the corresponding Weyl function $M(\cdot)$ is monotone with respect to the open set $J \subseteq \rho(A_0)$, $A_0 := A^* \upharpoonright \ker(\Gamma_0^0)$, then for any auxiliary self-adjoint operator $R$ in some separable Hilbert space $\mathfrak{R}$ the closed symmetric operator $S$ defined by* (1.5) *admits a self-adjoint extension $\widetilde{S}$ such that that the spectral parts $\widetilde{S}_J$ and $R_J$ are unitarily equivalent, i.e. $\widetilde{S}_J \cong R_J$.*

The proof of Theorem 1.1, given at the end of Section 4, has the advantage that the extension $\widetilde{S}$ is constructed explicitly which allows to draw conclusions on the spectral properties outside the gaps. In more detail, let as assume for the moment that $n_\pm(S_j) = 1$. If $\Pi_j = \{\mathbb{C}, \Gamma_0^j, \Gamma_1^j\}$ is a boundary triple for $S_j^*$, then $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\} := \bigoplus_{j=1}^{\infty} \Pi_j$ performs a boundary triple for $S = \bigoplus_{j=1}^{\infty} S_j$ which is associated with the pair $\{S, S_0\}$, i.e $S_0 = S^* \upharpoonright \ker(\Gamma_0)$. Using this boundary triple we indicate explicitly a self-adjoint boundary operator $B$ in $\mathcal{H}$ such that the self-adjoint extension $\widetilde{S} = S_B$ determined by

$$\widetilde{S} = S_B = S^* \upharpoonright \mathrm{dom}(S_B), \quad \mathrm{dom}(S_B) := \ker(\Gamma_0 - B\Gamma_1), \tag{1.6}$$

cf. [11], has the required spectral properties.

We note that Theorem 1.1 essentially complements the results of [2, 7, 8, 9, 10, 11] for symmetric operators of the special form (1.5) even for one gap $J$ because in contrast to the existing results the extension $\widetilde{S}$ is constructed explicitly and the approach allows to obtain spectral information on parts outside the gaps, cf. Section 5, which was until now not possible in this general form in this case.

The paper is organized as follows. In Section 2 we summarize definitions and statements which are necessary in the following. In particular, we define spectral measures which are non-orthogonal in general, Nevanlinna functions, boundary triples, Weyl functions and $\gamma$-fields.

In Section 3 we consider the important case of a symmetric operator $A$ with several gaps which admits a self-adjoint extension $A_0$ preserving the gaps such that the Weyl function $M(\cdot)$ corresponding to the pair $\{A, A_0\}$ is monotone and of scalar-type. We calculate (see Theorem 3.3) the non-orthogonal spectral measures (bounded and unbounded) in the gaps of $A$ for every self-adjoint extension $A_B = A_B^*$ which is disjoint from $A_0$.

In Section 4 we apply Theorem 3.3 to obtain a complete solution of the inverse spectral problem for a symmetric operator of the form (1.5) with several gaps and monotone Weyl functions and prove finally Theorem1.1.

In Section 5 we complement the main results on the spectrum of the operator $S_B$ (see (1.6)) outside the gaps. Namely, applying the Weyl function technique elaborated in [12] we show that if $S$ is simple, then for any self-adjoint extension $\widetilde{S}$ of $S$ the absolutely continuous spectrum of $\widetilde{S}$ contains that one of $S_0$ where $S_0 := S^* \restriction \ker(\Gamma_0)$, cf. Theorem 5.2 and Corollary 5.4. Moreover, it turns out that if $B$ is singular, then the absolutely continuous parts of $S_B$ and $S_0$ are unitarily equivalent, cf. Theorem 5.6.

In Section 6 we consider three examples of symmetric operators of the form (1.5). Using the Weyl function technique we calculate explicitly the non-orthogonal spectral measure $\Sigma_B(\cdot)$ of any extension $S_B = S_B^*$. We rely on the fact that it is much easier to calculate the non-orthogonal spectral measure $\Sigma_B(\cdot)$ of $S_B$ than the corresponding orthogonal one $E_{A_B}(\cdot)$. However, since both measures are spectrally equivalent in the sense of [26] the knowledge of $\Sigma_B(\cdot)$ allows to recover the spectral properties of $E_{A_B}(\cdot)$. We also remark that our first example concerns a symmetric operator with periodic scalar-type Weyl function, and the Weyl function technique allows us to show that any self-adjoint extension $\widetilde{S}$ is periodic.

We conjecture that Theorem 1.1 remains true for any symmetric operator $S$ admitting a boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ such that the associated Weyl function $M(\cdot)$ is monotone with respect to $J \subseteq \rho(S_0)$ but not necessarily of scalar-type. In a forthcoming paper we confirm this hypothesis for a wide class of symmetric operators with gaps.

Throughout the paper we use the following notations: $\mathcal{B}(X)$ denotes the Borel $\sigma$-algebra of a topological space $X$ while $\mathcal{B}_b(\mathbb{R})$ denotes the set of all bounded $\delta \in \mathcal{B}(\mathbb{R})$. $\mathrm{mes}(\delta)$ stands for the Lebesgue measure of $\delta \in \mathcal{B}(\mathbb{R})$. By $\mathfrak{H}$, $\mathfrak{R}$, $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{K}'$ we denote separable Hilbert spaces. The set of all bounded linear operators from $\mathfrak{H}$ to $\mathfrak{R}$ is denoted by $[\mathfrak{H}, \mathfrak{R}]$ or $[\mathfrak{H}]$ if $\mathfrak{H} = \mathfrak{R}$. $\mathcal{C}(\mathfrak{H})$ stands for the set of closed densely defined operators in $\mathfrak{H}$.

If $A$ is a symmetric operator, we denote by $\mathcal{N}_z := \ker(A^* - z)$ the deficiency subspaces of $A$ and by $n_\pm(A) := \dim \mathcal{N}_{\pm i}$ its deficiency indices. The set of all self-adjoint extensions of a closed symmetric operator $A$ is denoted by $\mathrm{Ext}_A$. As usual $E_T(\cdot)$ stands for the spectral measure (resolution of the identity) of a self-adjoint operator $T$ in $\mathfrak{H}$. We denote by $\sigma_{ac}(T)$, $\sigma_s(T)$, $\sigma_{sc}(T)$ and $\sigma_{pp}(T)$ the absolutely continuous, singular, singular continuous and the pure point spectrum of the operator $T = T^*$, respectively. By $\sigma_p(T)$ the set of eigenvalues of $T$ is indicated, $\overline{\sigma_p(T)} = \sigma_{pp}(T)$. Finally, we denote the resolvent set of an operator by $\rho(\cdot)$.

# 2 Preliminaries

A mapping $\Sigma(\cdot) : \mathcal{B}_b(\mathbb{R}) \longrightarrow [\mathcal{H}]$ is called an operator (operator-valued) measure if

(i) $\Sigma(\cdot)$ is $\sigma$-additive, in the strong sense.

(ii) $\Sigma(\delta) = \Sigma(\delta)^* \geq 0$ for $\delta \in \mathcal{B}_b(\mathbb{R})$.

The operator measure is called bounded if it extends to the Borel algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$, i.e $\Sigma(\mathbb{R}) \in [\mathcal{H}]$. Otherwise, the operator measure is called unbounded. A bounded operator measure $\Sigma(\cdot) = E(\cdot)$ is called orthogonal if, in addition, the following conditions are satisfied:

(iii) $E(\delta_1)E(\delta_2) = E(\delta_1 \cap \delta_2)$ for $\delta_1, \delta_2 \in \mathcal{B}(\mathbb{R})$,

(iv) $E(\mathbb{R}) = I_{\mathcal{H}}$.

Setting in (iii) $\delta_1 = \delta_2$, one concludes that an orthogonal measure $E(\cdot)$ takes its values in the set of orthogonal projections on the Hilbert space $\mathcal{H}$.

Every orthogonal measure $E(\cdot)$ determines the operator $T = T^* = \int_{\mathbb{R}} \lambda dE(\lambda)$ in $\mathcal{H}$ with $E(\cdot)$ being its resolution of the identity. Conversely, by the spectral theorem, every operator $T = T^*$ in $\mathcal{H}$ admits the above representation with the orthogonal spectral measure $E =: E_T$.

The following result is known (see [13]) as a generalized Naimark dilation theorem.

**PROPOSITION 2.1** *If $\Sigma(\cdot) : \mathcal{B}(\mathbb{R}) \longrightarrow [\mathcal{H}]$ is a bounded operator measure, then there exist a Hilbert space $\mathcal{K}$, a bounded operator $K \in [\mathcal{H}, \mathcal{K}]$ and an orthogonal measure $E(\cdot) : \mathcal{B}(\mathbb{R}) \longrightarrow [\mathcal{K}]$ (an orthogonal dilation) such that*

$$\Sigma(\delta) = K^* E(\delta) K, \quad \delta \in \mathcal{B}(\mathbb{R}). \tag{2.1}$$

*If the orthogonal dilation is minimal, i.e.*

$$\mathrm{span}\{E(\delta)\mathrm{ran}(K) : \ \delta \in \mathcal{B}(\mathbb{R})\} = \mathcal{K}, \tag{2.2}$$

*then it is uniquely determined up to unitary equivalence. That is, if one has two bounded operators $K \in [\mathcal{H}, \mathcal{K}]$ and $K' \in [\mathcal{H}, \mathcal{K}']$ as well as two minimal orthogonal dilations $E(\cdot) : \mathcal{B}(\mathbb{R}) \longrightarrow [\mathcal{K}]$ and $E'(\cdot) : \mathcal{B}(\mathbb{R}) \longrightarrow [\mathcal{K}']$ obeying $\Sigma(\delta) = K^* E(\delta) K = K'^* E'(\delta) K'$, $\delta \in \mathcal{B}(\mathbb{R})$, then there exists an isometry $V : \mathcal{K}' \longrightarrow \mathcal{K}$ such that $E'(\delta) = V^* E(\delta) V$, $\delta \in \mathcal{B}(\mathbb{R})$.*

Note that a short and simple proof of the Naimark dilation theorem as well as of Proposition 2.1 has recently been obtained in [26].

**DEFINITION 2.2** We call $E(\cdot)$, satisfying (2.1) and (2.2), the minimal orthogonal measure associated to $\Sigma(\cdot)$, or the minimal orthogonal dilation of $\Sigma(\cdot)$.

Every operator measure $\Sigma(\cdot)$ admits the Lebesgue-Jordan decomposition $\Sigma = \Sigma^{ac} + \Sigma^{s}$, $\Sigma^{s} = \Sigma^{sc} + \Sigma^{pp}$ where $\Sigma^{ac}, \Sigma^{s}, \Sigma^{sc}$ and $\Sigma^{pp}$ are the absolutely continuous, singular, singular continuous and pure point components (measures) of $\Sigma(\cdot)$, respectively. Non-topological supports of measures $\Sigma^{\tau} (\tau \in \{ac, sc, pp\})$ can be chosen to be mutually disjoint (see [12]). Therefore, if an operator measure $\Sigma$ is orthogonal, $\Sigma(\cdot) = E_T(\cdot)$, then the ortho-projections $P^{\tau} := E_T^{\tau}(\mathbb{R}) (\tau \in \{ac, sc, pp\})$ are pairwise orthogonal. Every subspace $\mathfrak{H}_T^{\tau} := P^{\tau}\mathfrak{H}$ reduces the operator $T = T^*$ and the Lebesgue-Jordan decomposition yields

$$\mathfrak{H} = \mathfrak{H}_T^{ac} \oplus \mathfrak{H}_T^{sc} \oplus \mathfrak{H}_T^{pp}, \qquad T = T^{ac} \oplus T^{sc} \oplus T^{pp}, \tag{2.3}$$

where $T^{\tau} := P^{\tau} T \upharpoonright \mathfrak{H}_T^{\tau}$, $\tau \in \{ac, sc, pp\}$.

## 2.1 Nevanlinna functions

Let $\mathcal{H}$ be a separable Hilbert space. We recall that an operator-valued function $F : \mathbb{C}_+ \longrightarrow [\mathcal{H}]$ is said to be a Nevanlinna (or Herglotz or $R_{\mathcal{H}} -$ ) one [1, 24, 28] if it is holomorphic and takes values in the set of dissipative operators on $\mathcal{H}$, i.e.

$$\Im\mathrm{m}(F(z)) := \frac{F(z) - F(z)^*}{2i} \geq 0, \quad z \in \mathbb{C}_+.$$

Usually, one considers a continuation of $F$ in $\mathbb{C}_-$ by setting $F(z) := F(\overline{z})^*$, $z \in \mathbb{C}_-$. Notice that this does not necessarily coincide with a holomorphic continuation of $F$ to $\mathbb{C}_-$ if it exists.

If $F(\cdot)$ is a Nevanlinna function, $F \in R_{\mathcal{H}}$, then there exists a bounded operator measure $\Sigma_F^0(\cdot) : \mathcal{B}(\mathbb{R}) \longrightarrow [\mathcal{H}]$, which is non-orthogonal in general, and operators $C_k = C_k^* \in [\mathcal{H}]$, $k \in \{0, 1\}$, $C_1 \geq 0$, such that the representation

$$F(z) = C_0 + C_1 z + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} d\Sigma_F^0(t), \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-, \tag{2.4}$$

holds. The representation (2.4) is an operator generalization (see [13]) of a well-known result for scalar Nevanlinna (Herglotz) functions (cf. [1, 4, 24, 28]). The integral in (2.4) is understood in the strong sense. In the following the bounded measure $\Sigma_F^0(\cdot)$ is called the bounded spectral measure of $F(\cdot)$. The measure $\Sigma_F^0(\cdot)$ is uniquely determined by the Nevanlinna function $F(\cdot)$. Its associated orthogonal spectral measure is denoted by $E_F(\cdot)$.

By Proposition 2.1, there exists an auxiliary Hilbert space $\mathcal{K}_F$ and a bounded operator $K \in [\mathcal{H}, \mathcal{K}_F]$ obeying $\ker(K) = \ker(\Sigma^0_F(\mathbb{R}))$ and $\Sigma^0_F(\delta) = K^* E_F(\delta) K$, $\delta \in \mathcal{B}(\mathbb{R})$. By

$$\Sigma_F(\delta) := \int_\delta (1+t^2) d\Sigma^0_F(t), \quad \delta \in \mathcal{B}_b(\mathbb{R}), \tag{2.5}$$

one defines an operator measure which, in general, is non-orthogonal and unbounded. It is called the unbounded spectral measure of $F(\cdot)$ Using $\Sigma_F$ the representation (2.4) transforms into

$$F(z) = C_0 + C_1 z + \int_{-\infty}^{+\infty} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\Sigma_F(t), \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{2.6}$$

$F$ determines uniquely the unbounded spectral measure $\Sigma_F(\cdot)$ by means of the Stieltjes inversion formula (see [1]):

$$\Sigma_F((a,b)) = s - \lim_{\delta \to +0} s - \lim_{\epsilon \to +0} \frac{1}{\pi} \int_{a+\delta}^{b-\delta} \Im\mathrm{m}(F(x+i\epsilon))dx. \tag{2.7}$$

By $\mathrm{supp}(F)$ we denote the topological (minimal closed) support of the spectral measure $\Sigma_F$. Since $\mathrm{supp}(F)$ is closed the set $\mathcal{O}_F := \mathbb{R} \setminus \mathrm{supp}(F)$ is open. The Nevanlinna function $F(\cdot)$ admits an analytic continuation to $\mathcal{O}_F$ given by

$$F(\lambda) = C_0 + C_1 \lambda + \int_{-\infty}^{+\infty} \left( \frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\Sigma_F(t), \quad \lambda \in \mathcal{O}_F.$$

Using this representation we immediately find that $F(\cdot)$ is monotone on each component interval $\Delta$ of $\mathcal{O}_F$, i.e. $F(\lambda) \le F(\mu), \quad \lambda < \mu, \quad \lambda, \mu \in \Delta$. In general, this relation is not satisfied if $\lambda$ and $\mu$ belong to different component intervals .

**DEFINITION 2.3** *Let $F(\cdot)$ be a Nevanlinna function. The Nevanlinna function is monotone with respect to the open set $J \subseteq \mathcal{O}_F$ if for any two component intervals $J_1$ and $J_2$ of $J$ one has $F(\lambda_1) \le F(\lambda_2)$ for all $\lambda_1 \in J_1$ and $\lambda_2 \in J_2$ or $F(\lambda_1) \ge F(\lambda_2)$ for all $\lambda_1 \in J_1$ and $\lambda_2 \in J_2$.*

Let $L \in \mathbb{N} \cup \infty$ be the number of component intervals of $J$. Obviously, if $F(\cdot)$ is monotone with respect to $J$ and $L < \infty$, then there exists an enumeration $\{J_k\}^L_{k=1}$ of the components of $J$ such that

$$F(\lambda_1) \le F(\lambda_2) \le \ldots \le F(\lambda_L)$$

holds for $\{\lambda_1, \lambda_2, \ldots, \lambda_L\} \in J_1 \times J_2 \times \ldots \times J_L$. If $L = \infty$, then it can happen that such an enumeration does not exist. If $F(\cdot)$ is a scalar Nevanlinna function, then $F(\cdot)$ is monotone with respect to $J$ if and only if the condition $F(J_1) \cap F(J_2) = \emptyset$ is satisfied for any two component intervals $J_1$ and $J_2$ of $J$.

## 2.2 Boundary triples

In what follows $A$ will always denote a closed symmetric operator with deficiency indices $n_{\pm}(A) \le \infty$. Without loss of generality we may assume that $A$ is simple. This means that $A$ has no self-adjoint reducing subspaces.

Our approach to the inverse spectral theory of self-adjoint extensions is based on the concept of boundary triples (see [21] and references therein) and the corresponding Weyl functions ([16, 17, 18]). We start with the definition of a boundary triple which may be considered as an abstract version of the second Green's formula.

**DEFINITION 2.4** A triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of an auxiliary Hilbert space $\mathcal{H}$ and linear mappings $\Gamma_i : \mathrm{dom}(A^*) \longrightarrow \mathcal{H}$, $i = 0, 1$, is called a boundary triple for the adjoint operator $A^*$ of $A$ if the following two conditions are satisfied:

(i) The second Green's formula takes place:

$$(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g), \quad f, g \in \mathrm{dom}(A^*).$$

(ii) The mapping $\Gamma := \{\Gamma_0, \Gamma_1\} : \mathrm{dom}(A^*) \longrightarrow \mathcal{H} \oplus \mathcal{H}$, $\quad \Gamma f := \{\Gamma_0 f, \Gamma_1 f\}$, is surjective.

The above definition allows one to describe the set $\mathrm{Ext}_A$ in the following way (see ([16, 17, 25]).

**PROPOSITION 2.5** *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$. Then the mapping $\Gamma$ establishes a bijective correspondence $\widetilde{A} \to \Theta := \Gamma(\mathrm{dom}(\widetilde{A}))$ between the set $\mathrm{Ext}_A$ of self-adjoint extensions of $A$ and the set of self-adjoint linear relations in $\mathcal{H}$.*

By Proposition 2.5, the following definition is natural.

**DEFINITION 2.6** Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$.

(i) We put $A_\Theta = \widetilde{A}$, if $\Theta := \Gamma(\mathrm{dom}(\widetilde{A}))$ that is

$$A_\Theta := A^* \upharpoonright D_\Theta, \text{ where } \mathrm{dom}(A_\Theta) = D_\Theta := \{f \in \mathrm{dom}(A^*) : \{\Gamma_0 f, \Gamma_1 f\} \in \Theta\}. \quad (2.8)$$

(ii) If $\Theta = G(B)$ is the graph of an operator $B = B^* \in \mathcal{C}(\mathcal{H})$, then $\mathrm{dom}(A_\Theta)$ is determined by the equation $\mathrm{dom}(A_B) = D_B := D_\Theta = \ker(\Gamma_1 - B\Gamma_0)$. We set $A_B := A_\Theta$.

**REMARK 2.7** We note the following (see ([16, 17, 25]):

1. The deficiency indices $n_{\pm}(A)$ are equal to the dimension of $\mathcal{H}$, i.e $\dim(H) = n_{\pm}(A)$.

2. There exist two self-adjoint extensions $A_i := A^* \upharpoonright \ker(\Gamma_i)$ which are naturally associated to a boundary triple. According to Definition 2.6 $A_i := A_{\Theta_i}$, $i \in \{0, 1\}$, where $\Theta_0 = \{0\} \times \mathcal{H}$ and $\Theta_1 = \mathcal{H} \times \{0\}$. Conversely, if $A_0$ is a self-adjoint extension of $A$, then there exists a boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ such that $A_0 = A^* \upharpoonright \ker(\Gamma_0)$.

3. If $B = B^* \in [\mathcal{H}]$, then one defines a new boundary triple $\Gamma_B = \{\mathcal{H}, \Gamma_0^B, \Gamma_1^B\}$ for $A^*$ with $\Gamma_1^B := \Gamma_0$, $\quad \Gamma_0^B := B\Gamma_0 - \Gamma_1$. It is clear that $A_B = A^* \upharpoonright \ker(\Gamma_0^B)$.

4. $\Theta$ is the graph of an operator $B = B^* \in \mathcal{C}(\mathcal{H})$ iff the extensions $A_\Theta$ and $A_0$ are disjoint, i.e. $\mathrm{dom}(A_\Theta) \cap \mathrm{dom}(A_0) = \mathrm{dom}(A)$.

5. $\Theta = G(B)$ with $B = B^* \in [\mathcal{H}]$ iff $A_\Theta$ and $A_0$ are transversal, i.e. $\mathrm{dom}(A_\Theta)+\mathrm{dom}(A_0) = \mathrm{dom}(A^*)$.

## 2.3 Weyl functions

It is well known that Weyl functions are an important tool in the direct and inverse spectral theory of singular Sturm-Liouville operators. In [16, 17, 18] the concept of Weyl function was generalized to an arbitrary symmetric operator $A$ with infinite deficiency indices $(n, n)$. Let us recall the basic facts on Weyl functions.

**DEFINITION 2.8** ([16, 17]) Let $A$ be a densely defined closed symmetric operator and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$. The unique mapping $M(\cdot) : \rho(A_0) \longrightarrow [\mathcal{H}]$ defined by

$$\Gamma_1 f_z = M(z)\Gamma_0 f_z, \quad f_z \in \mathcal{N}_z = \ker(A^* - z), \quad z \in \mathbb{C}_+,$$

is called the Weyl function corresponding to the boundary triple $\Pi$.

It is well known (cf. [16, 17]) that the above implicit definition of the Weyl function is correct and that $M(\cdot)$ is a strict Nevanlinna function, i.e. an Nevanlinna function obeying $0 \in \rho(\Im\mathrm{m}(M(i)))$. Moreover, if $A$ is simple, then the Weyl function $M(\cdot)$ corresponding to $\Pi$ determines the pair $\{A, A_0\}$ uniquely up to unitary equivalence (cf. [16, 17]). Sometimes it is said for brevity that $M(\cdot)$ is the Weyl function of the pair $\{A, A_0\}$.

Since $A$ is densely defined the integral representation (2.4) for $M$ simplifies to

$$M(z) = C_0 + \int_{-\infty}^{+\infty} \frac{1 + tz}{t - z} d\Sigma_M^0(t), \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-, \tag{2.9}$$

i.e. $C_1 = 0$. The condition $0 \in \rho(\Im\mathrm{m}(M(i)))$ is equivalent to $0 \in \rho(\Sigma^0(\mathbb{R}))$. By $E_M(\cdot)$ we denote the minimal orthogonal dilation associated to $\Sigma_M^0(\cdot)$ on the Hilbert space $\mathcal{K}_M$. Using the unbounded spectral measure $\Sigma_M(\cdot)$, $\Sigma_M(\delta) = \int_\delta (1+t^2) d\Sigma_M^0(t)$, $\delta \in \mathcal{B}_b(\mathbb{R})$, (cf. (2.5)), we arrive at the representation

$$M(z) = C_0 + \int_{-\infty}^{+\infty} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\Sigma_M(t), \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{2.10}$$

Taking into account the Stieltjes inversion formula (2.7) one recovers $\Sigma_M((a,b))$ for finite open intervals $(a,b) \subseteq \mathbb{R}$. The Weyl function allows one to describe the spectrum of self-adjoint extensions (cf. [17]).

**PROPOSITION 2.9** *Let $A$ be a simple closed symmetric operator and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$ with Weyl function $M(\lambda)$. Suppose that $\Theta$ is a self-adjoint linear relation in $\mathcal{H}$ and $\lambda \in \rho(A_0)$. Then*

(i) $\sigma(A_0) = \mathrm{supp}(M)$.

(ii) $\lambda \in \rho(A_\Theta)$ *if and only if* $0 \in \rho(\Theta - M(\lambda))$.

(iii) $\lambda \in \sigma_\tau(A_\Theta)$ *if and only if* $0 \in \sigma_\tau(\Theta - M(\lambda))$, $\tau \in \{p, c\}$.

In what follows we need the following simple proposition (cf. [17]).

**PROPOSITION 2.10** *Let $A$ be a closed symmetric operator and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$.*

(i) *If $A$ is simple and $\Pi_1 = \{\mathcal{H}_1, \Gamma_0^1, \Gamma_1^1\}$ is another boundary triple for $A^*$ such that $\ker(\Gamma_0) = \ker(\Gamma_0^1)$, then the Weyl functions $M(\cdot)$ and $M_1(\cdot)$ of $\Pi$ and $\Pi_1$, respectively, are related by*

$$M_1(z) = K^* M(z) K + D, \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-,$$

*where $D = D^* \in [\mathcal{H}_1]$ and $K \in [\mathcal{H}_1, \mathcal{H}]$ is boundedly invertible.*

(ii) *If $\Theta = G(B)$, $B = B^* \in [\mathcal{H}]$, then the Weyl function $M_B(\cdot)$ corresponding to the boundary triple $\Pi_B = \{\mathcal{H}, \Gamma_0^B, \Gamma_1^B\} := \{\mathcal{H}, B\Gamma_0 - \Gamma_1, \Gamma_0\}$ is given by*

$$M_B(z) = (B - M(z))^{-1}, \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{2.11}$$

Since $M_B(\cdot)$ is a Weyl function it admits a representation (2.9) with $C_0$ and $\Sigma_M^0$ replaced by $C_B = C_B^*$ and a spectral measure $\Sigma_B^0 := \Sigma_{M_B}^0$, respectively. The associated orthogonal spectral measure is denoted by $E_B(\cdot)$ on the Hilbert space $\mathcal{K}_B := \mathcal{K}_{M_B}$. Similarly to (2.5) one can introduce the unbounded spectral measure $\Sigma_B(\cdot) := \Sigma_{M_B}(\cdot)$ which leads to the representation (2.10) with $\Sigma_M$ replaced by $\Sigma_B$.

## 2.4 $\gamma$-fields

With each boundary triple we associate a so-called $\gamma$-field.

**DEFINITION 2.11** Let $A$ be a densely defined closed symmetric operator and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$. The mapping $\rho(A_0) \ni z \longrightarrow \gamma(z) \in [\mathcal{H}, \mathcal{N}_z]$,

$$\gamma(z) := (\Gamma_0 \upharpoonright \mathcal{N}_z)^{-1} : \mathcal{H} \longrightarrow \mathcal{N}_z, \quad z \in \rho(A_0),$$

is called the $\gamma$-field of the boundary triple $\Pi$.

One can easily check that

$$\gamma(z) = (A_0 - z_0)(A_0 - z)^{-1}\gamma(z_0), \quad z, z_0 \in \rho(A_0). \tag{2.12}$$

The $\gamma$-field and the Weyl function $M(\cdot)$ are related by

$$M(z) - M(z_0)^* = (z - \bar{z}_0)\gamma(z_0)^*\gamma(z), \quad z, z_0 \in \rho(A_0). \tag{2.13}$$

The latter formula allows us to relate the orthogonal spectral measure $E_M(\cdot)$ associated to the Weyl function $M(\cdot)$ with the orthogonal spectral measure $E_{A_0}(\cdot)$ of the self-adjoint extension $A_0$ (cf. Lemma 3.2 from [12], and Theorem 1 from [25]).

**LEMMA 2.12** *Let $A$ be a simple densely defined closed symmetric operator on a separable Hilbert space $\mathfrak{H}$ with equal deficiency indices. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$ with Weyl function $M(\cdot)$. If $E_{A_0}(\cdot)$ is the orthogonal spectral measure of $A_0$ defined on $\mathfrak{H}$ and $E_M(\cdot)$ the associated minimal orthogonal dilation of $\Sigma^0_M(\cdot)$ defined on $\mathcal{K}_M$, then both measures are unitarily equivalent, that is, there is an isometry $W : \mathfrak{H} \longrightarrow \mathcal{K}_M$ such that $E_{A_0}(\delta) = W^* E_M(\delta) W$ for any Borel set $\delta \in \mathcal{B}(\mathbb{R})$.*

**PROOF.** By (2.13) one obtains

$$\Im\mathrm{m}(M(x+iy)h, h) = y(\gamma(x+iy)h, \gamma(x+iy)h), \quad h \in \mathcal{H}. \tag{2.14}$$

Further, it follows from (2.12) that

$$\gamma(x+iy) = [I + (x + i(y-1))(A_0 - x - iy)^{-1}]\gamma(i). \tag{2.15}$$

Inserting (2.15) into (2.14) one gets

$$\Im\mathrm{m}(M(x+iy)h, h) = y\int_{-\infty}^{\infty} \frac{1+t^2}{(t-x)^2 + y^2} d(E_{A_0}(t)\gamma(i)h, \gamma(i)h), \quad h \in \mathcal{H}.$$

On the other hand we obtain from (2.10) that

$$\Im\mathrm{m}(M(x+iy)h, h) = y\int_{-\infty}^{\infty} \frac{d(\Sigma_M(t)h, h)}{(t-x)^2 + y^2}, \quad h \in \mathcal{H}.$$

Applying the Stieltjes inversion formula (2.7) we find

$$(\Sigma_M((a,b))h,h) = \int_{(a,b)} (1+t^2)d(E_{A_0}(t)\gamma(i)h,\gamma(i)h), \quad h \in \mathcal{H},$$

which yields

$$\Sigma_M^0((a,b)) = \gamma(i)^* E_{A_0}((a,b))\gamma(i) \tag{2.16}$$

for any bounded open interval $(a,b) \subseteq \mathbb{R}$. Since $A$ is simple, it follows from (2.15) that

$$\mathrm{span}\{(A_0-\lambda)^{-1}\mathrm{ran}(\gamma(i)) : \ \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-\} = \mathfrak{H}. \tag{2.17}$$

By (2.16) and (2.17), $E_{A_0}(\cdot)$ is a minimal orthogonal dilation of $\Sigma_M^0(\cdot)$. By Proposition 2.1 we find that the spectral measures $E_{A_0}(\cdot)$ and $E_M(\cdot)$ are unitarily equivalent. $\square$

By Lemma 2.12, the following definition is natural.

**DEFINITION 2.13** Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$ with corresponding Weyl function $M(\cdot)$. We will call $\Sigma_M^0$ (resp. $\Sigma_M$) the bounded (resp. unbounded) non-orthogonal spectral measure of the extension $A_0 (= A^* \upharpoonright \ker(\Gamma_0))$.

We note that in contrast to orthogonal spectral measures, which are defined up to unitary equivalence for given self-adjoint operators, a non-orthogonal bounded spectral measure $\Sigma_M^0$ for a given Weyl function is not unique up to unitary equivalence. According to Proposition 2.10 two such measures $\Sigma_M^0$ and $\Sigma_{M_1}^0$ being the bounded spectral measures of the corresponding Weyl functions $M$ and $M_1$, are connected by $\Sigma_{M_1}^0(\delta) = K^*\Sigma_M^0(\delta)K$, $\delta \in \mathcal{B}(\mathbb{R})$, where $K \in [\mathfrak{H}', \mathfrak{H}]$ and is boundedly invertible.

**COROLLARY 2.14** *Let $A$ be a simple densely defined closed symmetric operator in a separable Hilbert space $\mathfrak{H}$ with equal deficiency indices. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$ and $M(\cdot)$ the corresponding Weyl function. Then*

$$\sigma(A_0) = \mathrm{supp}(M) := \mathrm{supp}(\Sigma_M), \quad \sigma_\tau(A_0) = \mathrm{supp}(\Sigma_M^\tau), \quad \tau \in \{ac, s, sc, pp\}.$$

**PROOF.** The first statement follows either from Proposition 2.9(i), or from Lemma 2.12. Further, it follows from (2.4) and the Lebesgue-Jordan decompositions of the measures $\Sigma_M(\cdot)$ and $E_M(\cdot)$, that $\Sigma_M^\tau(\cdot) = K^* E_M^\tau(\cdot) K$, $\tau \in \{ac, s, sc, pp\}$. To complete the proof it remains to apply Lemma 2.12. $\square$

By Corollary 2.14 one gets, in particular, that $\mathcal{O}_M = \rho(A_0) \cap \mathbb{R}$.

**REMARK 2.15** If $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triple for $A^*$ and $B = B^* \in \mathcal{C}(\mathcal{H}) \setminus [\mathcal{H}]$, then the extensions $A_B$ and $A_0$ are disjoint but not transversal. In this case a triple $\Pi_B = \{\mathcal{H}, \Gamma_0^B, \Gamma_1^B\} := \{\mathcal{H}, B\Gamma_0 - \Gamma_1, \Gamma_0\}$ forms a generalized boundary triple for $A_* := A^* \upharpoonright (\mathrm{dom}A_0 + \mathrm{dom}A_B)$ in the sense of [18], Definition 6.1. Note that $A_*$ is not closed, but $\overline{A}_* = A^*$. Now the Nevanlinna function $M_B(z) := (B - M(z))^{-1}$ can be treated as the Weyl function corresponding to the triple $\Pi_B$ (see [18], Definition 6.2). Both Lemma 2.12 and Corollary 2.14 can easily be extended to the case of generalized boundary triples.

# 3 Scalar-type Weyl functions

Let $A$ be a densely defined closed symmetric operator on $\mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$ with the Weyl function $M(\cdot)$. The Weyl function is said to be of scalar-type if there exists a scalar Nevanlinna function $m(\cdot)$ such that the representation

$$M(z) = m(z) I_{\mathcal{H}}, \quad z \in \mathbb{C}_+,$$

holds. In accordance with (2.6) the function $m(\cdot)$ admits the representation

$$m(z) = c_0 + c_1 z + \int_{-\infty}^{+\infty} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t), \quad z \in \mathbb{C}_+, \tag{3.1}$$

where $c_0, c_1 \in \mathbb{R}$, $c_1 \geq 0$ and $\mu(\cdot)$ is a scalar Radon measure obeying $(1+t^2)^{-1} \in L^1(\mathbb{R}, \mu)$. Since $M(\cdot)$ is a Weyl function we find $c_1 = 0$. Obviously, we have $\mathrm{supp}(M) = \mathrm{supp}(m)$. Further, the Weyl function $M(\cdot)$ is monotone with respect to $J \subseteq \mathcal{O}_M$ if and only if $m(\cdot)$ is monotone with respect to $J \subseteq \mathcal{O}_m := \mathbb{R} \setminus \mathrm{supp}(m)$.

If $B = B^* \in [\mathcal{H}]$, then the Weyl function $M_B(\cdot)$ of the boundary triple $\Pi_B$ is given by

$$M_B(z) := (B - M(z))^{-1} = (B - m(z) \cdot I_{\mathcal{H}})^{-1}, \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-, \tag{3.2}$$

cf. Proposition 2.10. If $B = B^* \in \mathcal{C}(\mathcal{H}) \setminus [\mathcal{H}]$, then (see Remark 2.15) $M_B(\cdot)$ of the form (3.2) is the Weyl function of the generalized boundary triple $\Pi_B$.

Being a Weyl function, $M_B(\cdot)$ admits the representation (cf. (2.10))

$$M_B(z) = C_0 + \int_{-\infty}^{+\infty} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\Sigma_B(t), \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-, \tag{3.3}$$

where $\Sigma_B(\cdot) := \Sigma_{M_B}(\cdot)$ is the (unbounded) non-orthogonal spectral measure of $M_B(\cdot)$. In accordance with the Stieltjes inversion formula (2.7), the spectral measure can be re-obtained by

$$\Sigma_B((a,b)) = s - \lim_{\delta \downarrow 0} s - \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} (M_B(x+i\epsilon) - M_B(x-i\epsilon))\, dx \tag{3.4}$$

with $M(z) := M(\overline{z})^*$, $z \in \mathbb{C}_-$. Taking into account (3.2) we find

$$M_B(x+i\epsilon) - M_B(x-i\epsilon) = \int_{-\infty}^{+\infty} \left( (\lambda - m(x+i\epsilon))^{-1} - (\lambda - m(x-i\epsilon))^{-1} \right) dE_B(\lambda), \tag{3.5}$$

which leads to the expression

$$\frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} (M_B(x+i\epsilon) - M_B(x-i\epsilon))\, dx = \int_{-\infty}^{+\infty} k_\Delta(\lambda, \delta, \epsilon) dE_B(\lambda), \quad \epsilon > 0, \tag{3.6}$$

where

$$k_\Delta(\lambda, \delta, \epsilon) := \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} \left( (\lambda - m(x+i\epsilon))^{-1} - (\lambda - m(x-i\epsilon))^{-1} \right) dx, \tag{3.7}$$

$\lambda \in \mathbb{R}$, $\Delta = (a,b) \subseteq \mathbb{R}$ and $\epsilon > 0$ with $m(z) := \overline{m(\overline{z})}$, $z \in \mathbb{C}_-$.

We denote by $\{\Delta_l\}_{l=1}^L$ ($L \in \mathbb{N}$ or $L = \infty$) the family of the component intervals $\Delta_l = (a_l, b_l)$ of $\mathcal{O}_m := \mathbb{R} \setminus \mathrm{supp}(m)$. This family is unique up to the (henceforth fixed) enumeration. Further, the function $m(\cdot)$ admits an analytic continuation to $\mathcal{O}_m$ such that

$$m(x) = c_0 + \int_{-\infty}^{+\infty} \left(\frac{1}{t-x} - \frac{t}{1+t^2}\right) d\mu(t), \qquad x \in \mathcal{O}_m.$$

Hence the function $m(\cdot)$ restricted to $\mathcal{O}_m$ is analytic. Moreover, one easily verifies that for every component interval $\Delta$ of $\mathcal{O}_m$

$$m(x) < m(y), \quad x < y, \quad x, y \in \Delta.$$

Therefore, for every component interval $\Delta$ of $\mathcal{O}_m$ the set $\Delta' := m(\Delta)$ is again an open interval. Thus $\mathcal{O}'_m := m(\mathcal{O}_m)$ is also open and the (not necessarily disjoint) union of the sets $\Delta' = m(\Delta)$ where the union is taken over all component intervals $\Delta$ of $\mathcal{O}_m$.

**LEMMA 3.1** *Let $m(\cdot)$ be a scalar Nevanlinna function. If $\Delta = (a,b)$ is contained in a component interval $\Delta_l$ of $\mathcal{O}_m$, then*

$$C_\Delta(\delta) := \sup_{\lambda \in \mathbb{R},\, \epsilon \in (0,1]} |k_\Delta(\lambda, \delta, \epsilon)| < \infty \tag{3.8}$$

*for each $\delta \in (0, (b-a)/2)$.*

**PROOF.** We have

$$m(x + i\epsilon) = m(x) - \epsilon^2 \tau_0(\epsilon, x) + i\epsilon\tau_1(\epsilon, x), \quad x \in \mathcal{O}_m, \tag{3.9}$$

where

$$\tau_0(\epsilon, x) := \int_{-\infty}^{+\infty} \frac{1}{y-x} \cdot \frac{1}{(y-x)^2 + \epsilon^2} d\mu(y) \tag{3.10}$$

and

$$\tau_1(\epsilon, x) := \int_{-\infty}^{+\infty} \frac{1}{(y-x)^2 + \epsilon^2} d\mu(y). \tag{3.11}$$

Using (3.10) and (3.11) we find constants $\varkappa_0(\delta)$, $\varkappa_1(\delta)$ and $\omega_1(\delta)$ such that

$$|\tau_0(\epsilon, x)| \le \varkappa_0(\delta) \quad \text{and} \quad 0 < \omega_1(\delta) \le \tau_1(\epsilon, x) \le \varkappa_1(\delta), \quad x \in (a+\delta, b-\delta), \tag{3.12}$$

for $\epsilon \in [0,1]$. Further we get from (3.9)

$$\begin{aligned} &p(\lambda, x, \epsilon) = \\ &\frac{1}{\lambda - m(x+i\epsilon)} - \frac{1}{\lambda - m(x) - i\epsilon\tau_1(\epsilon, x)} = \\ &\frac{\epsilon^2 \tau_0(\varepsilon, x)}{(\lambda - m(x+i\epsilon))(\lambda - m(x) - i\epsilon\tau_1(\epsilon, x))}, \quad \lambda \in \mathbb{R}, \quad x \in \mathcal{O}_m, \quad \epsilon > 0. \end{aligned} \tag{3.13}$$

Since both $m(x)$ and $\tau_0(\varepsilon, x)$ are real for $x \in \mathcal{O}_m$ (see (3.10)) we have from (3.9) that $|\lambda - m(x + i\varepsilon)| \geq \varepsilon\tau_1(\varepsilon, x)$ and $|\lambda - m(x) - i\varepsilon\tau_1(\varepsilon, x)| \geq \varepsilon\tau_1(\varepsilon, x)$, $\lambda \in \mathbb{R}$.

In view of (3.13) these inequalities yield

$$|p(\lambda, x, \epsilon)| \leq \left|\frac{\tau_0(\epsilon, x)}{\tau_1(\epsilon, x)^2}\right|, \quad \lambda \in \mathbb{R}, \quad x \in \mathcal{O}_m, \quad \epsilon > 0. \tag{3.14}$$

Combining (3.12) with (3.14) we obtain the estimate

$$|p(\lambda, x, \epsilon)| \leq \frac{\varkappa_0(\delta)}{\omega_1(\delta)^2}, \quad \lambda \in \mathbb{R}, \quad x \in (a + \delta, b - \delta), \quad \epsilon \in (0, 1]. \tag{3.15}$$

We set

$$r_\Delta(\lambda, \delta, \epsilon) := \frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\frac{1}{\lambda - m(x) - i\epsilon\tau_1(\epsilon, x)} - \frac{1}{\lambda - m(x) + i\epsilon\tau_1(\epsilon, x)}\right)dx,$$

for $\lambda \in \mathbb{R}$ and $\epsilon > 0$. By the representation

$$r_\Delta(\lambda, \delta, \epsilon) = \frac{1}{\pi}\int_{a+\delta}^{b-\delta}\frac{\epsilon\tau_1(\epsilon, x)}{(\lambda - m(x))^2 + \epsilon^2\tau_1(\epsilon, x)^2}dx$$

and the estimates (3.12) we obtain

$$r_\Delta(\lambda, \delta, \epsilon) \leq \frac{1}{\pi}\int_{a+\delta}^{b-\delta}\frac{\epsilon\varkappa_1(\delta)}{(\lambda - m(x))^2 + \epsilon^2\omega_1^2(\delta)}dx, \quad \lambda \in \mathbb{R}, \quad \epsilon \in (0, 1]. \tag{3.16}$$

The derivative $m'(x)$, $x \in \mathcal{O}_m$, admits the representation

$$m'(x) = \int_{-\infty}^{+\infty}\frac{1}{(t - x)^2}d\mu(t), \quad x \in \mathcal{O}_m. \tag{3.17}$$

Obviously, there exist constants $\omega_2(\delta)$ and $\varkappa_2(\delta)$ such that

$$0 < \omega_2(\delta) \leq m'(x) \leq \varkappa_2(\delta), \quad x \in (a + \delta, b - \delta). \tag{3.18}$$

Combining (3.16) with (3.18) we get

$$r_\Delta(\lambda, \delta, \epsilon) \leq \frac{\varkappa_1(\delta)}{\pi\omega_2(\delta)}\int_{a+\delta}^{b-\delta}\frac{\epsilon \cdot m'(x)}{(\lambda - m(x))^2 + \epsilon^2\omega_1^2(\delta)}dx, \quad \lambda \in \mathbb{R}, \quad \epsilon \in (0, 1].$$

Using the substitution $y = m(x)$ we derive

$$r_\Delta(\lambda, \delta, \epsilon) \leq \frac{\varkappa_1(\delta)}{\pi\omega_2(\delta)}\int_{m(a+\delta)}^{m(b-\delta)}\frac{\epsilon}{(\lambda - y)^2 + \epsilon^2\omega_1^2(\delta)}dy, \quad \lambda \in \mathbb{R}, \quad \epsilon \in (0, 1].$$

Finally, we get

$$r_\Delta(\lambda, \delta, \epsilon) \leq \frac{\varkappa_1}{\omega_1\omega_2}, \quad \lambda \in \mathbb{R}, \quad \epsilon \in (0, 1]. \tag{3.19}$$

Obviously, we have

$$k_\Delta(\lambda,\delta,\epsilon) = \frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(p(\lambda,x,\epsilon) - \overline{p(\lambda,x,\epsilon)}\right)dx + r_\Delta(\lambda,\delta,\epsilon), \quad \lambda \in \mathbb{R} \quad \epsilon > 0.$$

Hence we find the estimate

$$|k_\Delta(\lambda,\delta,\epsilon)| \le \frac{1}{\pi}\int_{a+\delta}^{b-\delta}|p(\lambda,x,\epsilon)|\,dx + r_\Delta(\lambda,\delta,\epsilon), \quad \lambda \in \mathbb{R}, \quad \epsilon > 0.$$

Taking into account (3.15) and (3.19) we arrive at the estimate

$$|k_\Delta(\lambda,\delta,\epsilon)| \le \frac{\varkappa_0(\delta)}{\pi\omega_1(\delta)}(b-a) + \frac{\varkappa_1(\delta)}{\omega_1(\delta)\omega_2(\delta)}, \quad \lambda \in \mathbb{R}, \quad \epsilon \in (0,1],$$

which proves (3.8). $\square$

Since the function $m(\cdot)$ is strictly monotone on each component interval $\Delta_l$ of $\mathcal{O}_m$, the inverse function $\varphi_l(\cdot)$ exists there. The function $\varphi_l(\cdot)$ is analytic and also strictly monotone. Its first derivative $\varphi_l'(\cdot)$ exists, is analytic and non-negative.

**LEMMA 3.2** *Suppose that $m(\cdot)$ is a scalar Nevanlinna function. Let $\Delta = (a,b)$ be contained in some component interval $\Delta_l$ of $\mathcal{O}_m := \mathbb{R}\setminus \operatorname{supp}(m)$. Then (with $k_\Delta$ defined as in (3.7))*

$$\lim_{\epsilon\to+0} k_\Delta(\lambda,\delta,\epsilon) = \theta_l(\lambda,\delta) := \begin{cases} 0 & \lambda \in \mathbb{R}\setminus[m(a+\delta), m(b-\delta)], \\ \frac{1}{2}\varphi_l'(\lambda) & \lambda \in \{m(a+\delta), m(b-\delta)\}, \\ \varphi_l'(\lambda) & \lambda \in (m(a+\delta), m(b-\delta)) \end{cases} \tag{3.20}$$

*for $\delta \in (0,(b-a)/2)$ and*

$$\lim_{\delta\to+0}\lim_{\epsilon\to+0} k_\Delta(\lambda,\delta,\epsilon) = \begin{cases} 0 & \lambda \in \mathbb{R}\setminus(m(a),m(b)) \\ \varphi_l'(\lambda) & \lambda \in (m(a),m(b)). \end{cases} \tag{3.21}$$

**PROOF.** At first let us show that

$$\lim_{\epsilon\downarrow 0}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta} p(\lambda,x,\epsilon)dx = 0, \quad \lambda \in \mathbb{R}. \tag{3.22}$$

By (3.13) one immediately gets that

$$\lim_{\epsilon\downarrow 0} p(\lambda,x,\epsilon) = 0, \quad \lambda \in \mathbb{R}, \quad x \in \mathcal{O}_m.$$

Now (3.22) is implied by (3.15) and the Lebesgue dominated convergence theorem.

Next we set

$$\tau_3(\epsilon,x) := \int_{-\infty}^{+\infty}\frac{1}{(y-x)^2+\epsilon^2}\cdot\frac{1}{(y-x)^2}d\mu(y), \quad x \in \mathcal{O}_m, \quad \epsilon \ge 0. \tag{3.23}$$

Obviously, there is a constant $\varkappa_3(\delta) > 0$ such that

$$0 \le \tau_3(\epsilon, x) \le \varkappa_3(\delta), \quad x \in (a+\delta, b-\delta), \quad \epsilon \in [0,1]. \tag{3.24}$$

Let

$$p_0(\lambda, x, \epsilon) := \frac{1}{\lambda - m(x) - i\epsilon\tau_1(\epsilon, x)} - \frac{1}{\lambda - m(x) - i\epsilon\tau_1(0, x)}, \quad \lambda \in \mathbb{R}, \quad x \in \mathcal{O}_m, \tag{3.25}$$

for $\epsilon > 0$. It follows from (3.11), (3.23) and (3.25) that

$$p_0(\lambda, x, \epsilon) = -i\frac{\epsilon^3\tau_3(\epsilon, x)}{(\lambda - m(x) - i\epsilon\tau_1(\epsilon, x))(\lambda - m(x) - i\epsilon\tau_1(0, x))}, \quad \lambda \in \mathbb{R}, \quad x \in \mathcal{O}_m, \tag{3.26}$$

for $\epsilon > 0$. Since $\lambda \in \mathbb{R}$ and $m(x)$ is real for $x \in \mathcal{O}_m$, we get from (3.26)

$$|p_0(\lambda, x, \epsilon)| \le \epsilon\frac{\tau_3(\epsilon, x)}{\tau_1(\epsilon, x)\tau_1(0, x)}, \quad \lambda \in R, \quad x \in \mathcal{O}_m, \quad \epsilon > 0.$$

Using (3.12) and (3.24) we obtain the estimate

$$|p_0(\lambda, x, \epsilon)| \le \epsilon\frac{\varkappa_3(\delta)}{\omega_1(\delta)^2}, \quad \lambda \in \mathbb{R}, \quad x \in (a+\delta, b-\delta), \quad \epsilon \in (0,1],$$

which immediately yields

$$\lim_{\epsilon\downarrow 0}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta} p_0(\lambda, x, \epsilon)dx = 0, \quad \lambda \in \mathbb{R}, \quad \delta > 0. \tag{3.27}$$

Finally, let us introduce

$$q_\Delta(\lambda, \delta, \epsilon) := \frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\frac{1}{\lambda - m(x) - i\epsilon\tau_1(0, x)} - \frac{1}{\lambda - m(x) + i\epsilon\tau_1(0, x)}\right)dx \tag{3.28}$$

for $\lambda \in \mathbb{R}$ and $\epsilon > 0$. Using the representation

$$q_\Delta(\lambda, \delta, \epsilon) = \frac{1}{\pi}\int_{a+\delta}^{b-\delta}\frac{\epsilon\tau_1(0, x)}{(\lambda - m(x))^2 + \epsilon^2\tau_1(0, x)^2}dx, \quad \lambda \in \mathbb{R}, \quad \epsilon > 0,$$

and the relation

$$m'(x) = \tau_1(0, x), \quad x \in \mathcal{O}_m,$$

(see (3.11) and (3.17)) we find after change of variable $y = m(x)$ that

$$q_\Delta(\lambda, \delta, \epsilon) = \frac{1}{\pi}\int_{m(a+\delta)}^{m(b-\delta)}\frac{\epsilon}{(\lambda - y)^2 + \epsilon^2\tau_1(0, \varphi_l(y))^2}dy, \quad \lambda \in \mathbb{R}, \quad \epsilon > 0.$$

By $\tau_1(0, \varphi_l(y)) = m'(\varphi_l(y)) = 1/\varphi_l'(y)$, $y \in \Delta_l$, we finally obtain that

$$q_\Delta(\lambda, \delta, \epsilon) = \frac{1}{\pi}\int_{m(a+\delta)}^{m(b-\delta)}\frac{\epsilon\varphi_l'(y)^2}{\varphi_l'(y)^2(\lambda - y)^2 + \epsilon^2}dy, \quad \lambda \in \mathbb{R}, \quad \epsilon > 0. \tag{3.29}$$

Next we prove the relation

$$\lim_{\epsilon\downarrow 0} q_\Delta(\lambda,\delta,\epsilon) = \theta_l(\lambda,\delta), \quad \delta \in (0,(b-a)/2), \quad \lambda \in \mathbb{R}. \tag{3.30}$$

We consider only the case when $\lambda \in (m(a+\delta), m(b-\delta))$. The other cases can be treated in a similar way.

Noting that $\varphi_l'(\lambda) > 0$ choose an arbitrary $c \in (0, \varphi_l'(\lambda))$. Since $\varphi_l'$ is continuous we can choose $\eta > 0$ such that $m(a+\delta) < \lambda - \eta < \lambda + \eta < m(b+\delta)$ and

$$0 < \varphi_l'(\lambda) - c \le \varphi_l'(y) \le \varphi_l'(\lambda) + c, \quad \lambda - \eta \le y \le \lambda + \eta. \tag{3.31}$$

Let $a, b > 0$. The change of variables $x = b(y-\lambda)/\epsilon$ yields

$$\int_{\lambda-\eta}^{\lambda+\eta} \frac{a^2\epsilon}{b^2(\lambda-y)^2+\epsilon^2}dy = \frac{a^2}{\epsilon}\int_{\frac{-b\eta}{\epsilon}}^{\frac{b\eta}{\epsilon}} \frac{1}{1+x^2}\cdot\frac{\epsilon}{b}dx \longrightarrow \frac{\pi a^2}{b}, \quad \text{as } \epsilon \downarrow 0. \tag{3.32}$$

Setting $a = \varphi_l'(\lambda) - c$ and $b = \varphi_l'(\lambda) + c$ resp. $a = \varphi_l'(\lambda) + c$ and $b = \varphi_l'(\lambda) - c$ in (3.32) and using (3.31) we obtain

$$\begin{aligned}
\pi\,\frac{(\varphi_l'(\lambda)-c)^2}{\varphi_l'(\lambda)+c} &\le \liminf_{\epsilon\downarrow 0}\int_{\lambda-\eta}^{\lambda+\eta}\frac{\epsilon\varphi_l'(y)^2}{\varphi_l'(y)^2(\lambda-y)^2+\epsilon^2}dy \\
&\le \limsup_{\epsilon\downarrow 0}\int_{\lambda-\eta}^{\lambda+\eta}\frac{\epsilon\varphi_l'(y)^2}{\varphi_l'(y)^2(\lambda-y)^2+\epsilon^2}dy \le \pi\,\frac{(\varphi_l'(\lambda)+c)^2}{\varphi_l'(\lambda)-c}.
\end{aligned} \tag{3.33}$$

Setting $G := (m(a+\delta), m(b-\delta)) \setminus (\lambda-\eta, \lambda+\eta)$ and applying the Lebesgue dominated convergence theorem we get

$$\lim_{\epsilon\downarrow 0}\int_G \frac{\epsilon\varphi_l'(y)^2}{\varphi_l'(y)^2(\lambda-y)^2+\epsilon^2}dy = 0. \tag{3.34}$$

By (3.33) and (3.34),

$$\begin{aligned}
\pi\,\frac{(\varphi_l'(\lambda)-c)^2}{\varphi_l'(\lambda)+c} &\le \liminf_{\epsilon\downarrow 0}\int_{m(a+\delta)}^{m(b-\delta)}\frac{\epsilon\varphi_l'(y)^2}{\varphi_l'(y)^2(\lambda-y)^2+\epsilon^2}dy \\
&\le \limsup_{\epsilon\downarrow 0}\int_{m(a+\delta)}^{m(b-\delta)}\frac{\epsilon\varphi_l'(y)^2}{\varphi_l'(y)^2(\lambda-y)^2+\epsilon^2}dy \le \pi\,\frac{(\varphi_l'(\lambda)+c)^2}{\varphi_l'(\lambda)-c}.
\end{aligned} \tag{3.35}$$

Since (3.35) holds for every $c \in (0, \varphi_l'(\lambda))$, (3.35) in combination with (3.29) imply (3.30). Combining (3.7), (3.13), (3.25) and (3.28) we derive the representation

$$\begin{aligned}
&k_\Delta(\lambda,\delta,\epsilon) = \\
&\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(p(\lambda,x,\epsilon) - \overline{p(\lambda,x,\epsilon)}\right) + \frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(p_0(\lambda,x,\epsilon) - \overline{p_0(\lambda,x,\epsilon)}\right) + q_\Delta(\lambda,\delta,\epsilon)
\end{aligned} \tag{3.36}$$

where $\lambda \in \mathbb{R}$ and $\epsilon > 0$. Now combining the relations (3.22), (3.27) and (3.30) with (3.36), we arrive at (3.20). The relation (3.21) immediately follows from (3.20). $\square$

Now we are ready to calculate a non-orthogonal spectral measure $\Sigma_B^0$ in a gap of any self-adjoint extension $A_B = A_B^* \in \mathrm{Ext}_A$ if only $A$ admits a boundary triple with a scalar-type Weyl function.

**THEOREM 3.3** *Let $m$ be a scalar Nevanlinna function, $B = B^*$ a self-adjoint (not necessarily bounded) operator in $\mathcal{H}$ and $\Sigma_B(\cdot)$ (resp. $\Sigma_B^0(\cdot)$) the unbounded (bounded) non-orthogonal spectral measure of $M_B(z) = (B - m(z)I_{\mathcal{H}})^{-1}$ (see (3.3)). Then for every component interval $\Delta_l$ of $\mathcal{O}_m$*

$$\Sigma_B(\delta) = \varphi_l'(B_{m(\Delta_l)})E_B(m(\delta)), \quad \delta \in \mathcal{B}_b(\Delta_l), \tag{3.37}$$

*and*

$$\Sigma_B^0(\delta) = \varphi_l'(B_{m(\Delta_l)})(I + \varphi_l(B_{m(\Delta_l)})^2)^{-1}E_B(m(\delta)), \quad \delta \in \mathcal{B}(\Delta_l). \tag{3.38}$$

**PROOF.** 1. First we prove (3.37) for $\delta$ which are really contained in $\Delta_l$, i.e $\overline{\delta} \subset \Delta_l$. If $\delta = \Delta = (a, b)$ is such an interval, then by (3.4), (3.5), (3.6) and the Stieltjes inversion formula (2.7), we obtain that

$$\Sigma_B(\Delta) = s - \lim_{\varepsilon \downarrow 0} s - \lim_{\epsilon \downarrow 0} \int_{-\infty}^{+\infty} k_\Delta(\lambda, \varepsilon, \epsilon) dE_B(\lambda). \tag{3.39}$$

On the other hand, combining Lemma 3.1 with Lemma 3.2 and applying the Lebesgue dominated convergence theorem we get that for every $h \in \mathcal{H}$

$$\begin{aligned} &\lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} k_\Delta(\lambda, \varepsilon, \epsilon) dE_B(\lambda)h = \\ &\quad \int_{-\infty}^{\infty} \lim_{\epsilon \downarrow 0} k_\Delta(\lambda, \varepsilon, \epsilon) dE_B(\lambda)h = \varphi_l'(B)E_B(m((a+\varepsilon, b-\varepsilon))h + \\ &\quad \frac{1}{2}\left[\varphi_l'(m(a+\varepsilon))E_B(\{m(a+\varepsilon)\}) + \varphi_l'(m(b-\varepsilon))E_B(\{m(b-\varepsilon)\})\right]h. \end{aligned} \tag{3.40}$$

Combining (3.39) with (3.40), we arrive at (3.37) with $\delta = (a, b)(\subset \Delta_l)$.

2. Passing to an arbitrary $\delta \in \mathcal{B}_b(\Delta_l)$ we observe that $\Sigma_B(\delta)$ is bounded for any $\delta \in \mathcal{B}_b(\Delta_l)$ but $T_l := \varphi_l'(B_{m(\Delta_l)})$ in general not. However, one has $T_l E_B\big(m(\delta)\big) \in [\mathcal{H}]$ for $\delta \in \mathcal{B}_b(\Delta_l)$ if the closure $\overline{\delta}$ obeys $\overline{\delta} \subseteq \Delta_l$. Therefore, the equality (3.37) is valid for any $\delta \in \mathcal{B}_b\big((\alpha + 1/n, \beta - 1/n)\big)$, $n \in \mathbb{N}$.

Let now $\delta \in \mathcal{B}_b(\Delta_l)$. Setting $\delta_n := \delta \cap (\alpha + 1/n, \beta - 1/n)$ we get $\overline{\delta_n} \subset \Delta_l$, $n \in \mathbb{N}$, and

$$\lim_{n \to \infty} E_B\big(m(\delta_n)\big)T_l h = E_B\big(m(\delta)\big)T_l h \tag{3.41}$$

for any $h \in \mathrm{dom}(T_l)$. Since $\overline{\delta_n} \subset \Delta_l$, $n \in \mathbb{N}$, we find

$$\lim_{n\to\infty} T_l E_B\big(m(\delta_n)\big)h = \lim_{n\to\infty} \Sigma_B(\delta_n)h = \Sigma_B(\delta)h. \tag{3.42}$$

Thus, $T_l E_B(m(\delta)) = \Sigma_B(\delta) \in [\mathcal{H}]$ which proves the identity (3.37) for any $\delta \in \mathcal{B}_b(\Delta_l)$.

3. Formula (3.38) follows from (3.37). Indeed, one has

$$\begin{aligned}
\Sigma_B^0(\delta) &= \int_\delta (1+t^2)^{-1} d\Sigma_B(t) = \int_\delta \varphi_l'(B)E_B(m(\delta))(1+t^2)^{-1} dE_B(m(t)) \\
&= \varphi_l'(B)E_B(m(\delta)) \int_{m(\delta)} (1+\varphi_l^2(s))^{-1} dE_B(t) = \varphi_l'(B)(1+\varphi_l^2(B))^{-1}E_B(m(\delta)).
\end{aligned}$$

for $\delta \in \mathcal{B}_b(\Delta_l)$. □

The following corollary follows easily from Proposition 2.9 but we prefer to obtain it directly from Theorem 3.3.

**COROLLARY 3.4** *Let $A$ be a symmetric operator in $\mathcal{H}$, $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ a boundary triple for $A^*$ and $B = B^* \in \mathcal{C}(\mathcal{H})$. If the corresponding Weyl function $M(\cdot)$ is of scalar-type, i.e. $M(z) = m(z) \cdot I_{\mathcal{H}}$, then for every component interval $\Delta_l$ of $\mathcal{O}_m$ the operator $A_B E_{A_B}(\Delta_l)$ is purely absolutely continuous (resp. singular, singular continuous, purely point) if the operator $B$ is so, i.e.*

$$B = B^\tau \Longrightarrow A_B E_{A_B}(\Delta_l) = (A_B E_{A_B}(\Delta_l))^\tau, \qquad \tau \in \{ac, s, sc, pp\}. \tag{3.43}$$

**PROOF.** Let $B$ be absolutely continuous, i.e. $B = B^{ac}$. Then for every $\delta \in \mathcal{B}(\Delta_l)$ with $\mathrm{mes}(\delta) = 0$ one has $\mathrm{mes}(m(\delta)) = 0$ since $m$ is absolutely continuous on $\Delta_l$. Hence $E_B(m(\delta)) = 0$. Applying Theorem 3.3 we get $\Sigma_B(\delta) = \Sigma_B^0(\delta) = 0$.

If $B$ is singular, i.e. $B = B^s$, then the measure $E_B(\cdot)E_B(\Delta_l')$ admits a (non-topological) support of the form $m(\Delta_l^s) \subset \Delta_l'$ where $\Delta_l' = m(\Delta_l)$. By definition, this means that $\mathrm{mes}(m(\Delta_l^s)) = 0$ and $E_B(m(\Delta_l^s)) = E_B(\Delta_l')$. Note that $\mathrm{mes}(\Delta_l^s) = 0$ since $\Delta_l^s = \varphi(m(\Delta_l^s))$ and $\varphi_l$ is absolutely continuous on $m(\Delta_l)$. By (3.37) and (3.38) both measures $\Sigma_B(\cdot)$ and $\Sigma_B^0(\cdot)$, restricted to the interval $\Delta_l$, are supported on $\Delta_l^s$ and therefore are singular within the gap $\Delta_l$.

The cases of singular continuous and pure point spectrum can be treated quite similar. □

**REMARK 3.5**

1. We note that if for some $l$

$$BE_B(m(\Delta_l)) = (BE_B(m(\Delta_l)))^\tau, \qquad \tau \in \{ac, s, sc, pp\},$$

then the implication (3.43) remains true.

2. If the equality $m(\Delta_l) = m(\mathcal{O}_m)$ holds for some $l$, then the implication (3.43) is in fact an equivalence, in particular, if $m(\Delta_l) = \mathbb{R}$.

**REMARK 3.6** Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$. Then every self-adjoint extension $\widetilde{A} \in \mathrm{Ext}_A$ is of the form (2.8), that is $\widetilde{A} = A_\Theta$ for $\Theta := \{\Gamma_0, \Gamma_1\}\mathrm{dom}(\widetilde{A})$. Note that formulas (3.37) and (3.38) remain valid after the corresponding modification for the Nevanlinna function $M_\Theta(z) := (\Theta - m(z) \cdot I_\mathcal{H})^{-1}$ with $\Theta = \Theta^*$ being a linear relation. Thus Theorem 3.3 presents an explicit form of a part $(\Sigma^0_\Theta)_J$ of a non-orthogonal spectral measure $\Sigma_\Theta$ restricted to the gap $J$ of $A$, for any operator $A_\Theta = A^*_\Theta \in \mathrm{Ext}_A$.

# 4 Inverse spectral problem for direct sums of symmetric operators

## 4.1 The case of scalar-type Weyl function

Throughout this section we suppose in addition that $m(\cdot)$ is monotone with respect to $J \subseteq \mathcal{O}_m$. We apply Theorem 3.3 to obtain a solution of the inverse spectral problem for a symmetric operator $A$ satisfying the above assumptions. Namely, we indicate a boundary operator $B = B^*$ in $\mathcal{H}$ such that the corresponding extension $A_B \in \mathrm{Ext}_A$ yields an explicit solution of the above problem.

We recall that if $E_T(\cdot)$ is the orthogonal spectral measure of a self-adjoint operator $T$ in $\mathfrak{H}$ and $\delta \in \mathcal{B}(\mathbb{R})$, then the underlying Hilbert space $\mathfrak{H}$ admits an orthogonal decomposition $\mathfrak{H} = \mathrm{ran}(E_T(\delta)) \oplus \mathrm{ran}(E_T(\mathbb{R}\setminus\delta))$. According to this decomposition $T$ itself can be decomposed as $T = T_\delta \oplus T_{\mathbb{R}\setminus\delta}$ where $T_\delta$ and $T_{\mathbb{R}\setminus\delta}$ is a self-adjoint operator in the Hilbert space $\mathrm{ran}(E_T(\delta))$ and $\mathrm{ran}(E_T(\mathbb{R} \setminus \delta))$, respectively. For every Borel-measurable function $f$ defined on $\delta$ we set $f(T) := f(T_\delta)$.

We start with a simple result being a corollary to Theorem 3.3.

**PROPOSITION 4.1** *Suppose that a scalar Nevanlinna function $m(\cdot)$ is monotone with respect to the open set $J \subseteq \mathcal{O}_m$ and $T = T^*$ is a self-adjoint operator in $\mathcal{H}$ satisfying $E_T(\mathbb{R} \setminus J) = 0$. Let $\Sigma_{m(T)}$ (resp. $\Sigma^0_{m(T)}$) be the unbounded (resp. bounded) non-orthogonal spectral measure of the Nevanlinna function $(m(T) - m(\cdot) \cdot I_\mathcal{H})^{-1}$. Then*

$$\Sigma_{m(T)}(\delta) = (m'(T))^{-1} E_T(\delta), \quad \delta \in \mathcal{B}_b(J), \tag{4.1}$$

$$\Sigma^0_{m(T)}(\delta) = (m'(T))^{-1}(1 + T^2)^{-1} E_T(\delta), \quad \delta \in \mathcal{B}(J). \tag{4.2}$$

**PROOF.** By the $\sigma$-additivity and outer regularity of the involved measures it suffices to prove the assertion in the special case when $\delta$ is really contained in a component interval $\Delta$ of $\mathcal{O}_m$, i.e $\overline{\delta} \subset \Delta$. We set $B = m_J(T) = B^*$. Then

$$B = m(T_\Delta) \oplus m(T_{\mathbb{R}\setminus\Delta}) = m(T_\Delta) \oplus m(T_{J\setminus\Delta}). \tag{4.3}$$

In the last step we have used that $E_T(\mathbb{R} \setminus J) = 0$. Moreover,

$$E_B(m(\delta)) = E_{m(T_\Delta)}(m(\delta)) \oplus E_{m(T_{J\setminus\Delta})}(m(\delta)) = E_{T_\Delta}(\delta) \oplus 0 = E_T(\delta), \qquad \delta \in \mathcal{B}(\Delta) \tag{4.4}$$

where we have used the fact that $m(\Delta) \cap m(J \setminus \Delta) = \emptyset$ which follows from the monotonicity of $m$ on $J$. In particular, one gets

$$\mathrm{ran}(E_B(m(\Delta))) = \mathrm{ran}(E_T(\Delta)). \tag{4.5}$$

By (4.3) and (4.5) we obtain that

$$B_{m(\Delta)} := BE_B(m(\Delta)) = m(T_\Delta). \tag{4.6}$$

Let $\Delta := \Delta_l$ for some $l$. We note that

$$\varphi_l'(m(\lambda)) = \frac{1}{m'(\lambda)}, \quad \lambda \in \Delta_l. \tag{4.7}$$

Combining (3.37) with (4.4), (4.6) and (4.7) we get

$$\Sigma_B(\delta) = \varphi_l'(B_{m(\Delta_l)})E_B(m(\delta)) = \varphi_l'(m(T_{\Delta_l}))E_T(\delta) = (m'(T_{\Delta_l}))^{-1}E_T(\delta), \qquad \delta \in \mathcal{B}_b(\Delta_l).$$

Thus we have proved (4.1). Formula (4.2) follows from (4.1) just in the same way as (3.38) follows from (3.37). $\square$

**REMARK 4.2** We note that even though the right hand sides of (4.1) and (4.2) make sense without the assumption of monotonicity of $m(\cdot)$ with respect to $J$, nevertheless, the equalities (4.1) and (4.2) might be false without this assumption.

To prove the main theorem of this section the following lemma is helpful.

**LEMMA 4.3** *Let $A$ be a densely defined closed symmetric operator on a separable Hilbert space $\mathfrak{H}$ with equal deficiency indices. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$ and $M(\cdot)$ the corresponding Weyl function. Further, let $\widehat{A}$ be a closed symmetric extension of $A$ obeying*

$$A \subseteq \widehat{A} \subseteq A_0, \quad (\mathrm{dom}(A_0) = \ker(\Gamma_0)).$$

*Then there is a boundary triple $\widehat{\Pi} = \{\widehat{\mathcal{H}}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ such that $A_0 = \widehat{A}_0 := \widehat{A}^* \upharpoonright \ker(\widehat{\Gamma}_0)$ and the corresponding Weyl function $\widehat{M}(\cdot)$*

(i) *is of scalar-type provided $M(\cdot)$ is of scalar-type,*

(ii) *is monotone with respect to $J \subseteq \mathcal{O}_M$ provided $M(\cdot)$ is monotone with respect to $J$,*

(iii) *is of scalar-type and monotone with respect to $J \subseteq \mathcal{O}_M$ provided $M(\cdot)$ is of scalar-type and monotone with respect to $J$.*

**PROOF.** We put $\mathcal{H}_1 := \overline{\Gamma_1 \mathrm{dom}(\widehat{A})} \subset \Gamma_1 \mathrm{dom}(A_0) = \mathcal{H}$. Let $\pi$ be the orthogonal projection from $\mathcal{H}$ onto $\widehat{\mathcal{H}} := \mathcal{H} \ominus \mathcal{H}_1$. Setting $\widehat{\Gamma}_0 := \Gamma_0 \upharpoonright \mathrm{dom}(\widehat{A}^*)$ and $\widehat{\Gamma}_1 := \pi\Gamma_1 \upharpoonright \mathrm{dom}(\widehat{A}^*)$, we easily check that $\widehat{\Pi} := \{\widehat{\mathcal{H}}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ forms a boundary triple for $\widehat{A}^*$ such that $\widehat{A}_0 := \widehat{A}^* \upharpoonright \ker(\widehat{\Gamma}_0) = A_0$ (cf. [15]). The corresponding Weyl function is $\widehat{M}(\cdot) = \pi M(\cdot) \upharpoonright \widehat{\mathcal{H}}$ is monotone with respect to $J$, because $M(\cdot)$ is monotone with respect to $J$. Obviously, if $M(\cdot)$ is of scalar-type, then $\widehat{M}(\cdot)$ is also of scalar-type. $\square$

We come now to the main theorem of this section.

**THEOREM 4.4** *Let $A$ be a densely defined closed symmetric operator in a separable Hilbert space $\mathfrak{H}$ with equal deficiency indices $n_\pm(A) =: n(A)$. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$ with scalar-type Weyl function $M(\cdot) = m(\cdot)\, I_{\mathcal{H}}$. If the Weyl function $M(\cdot)$ is monotone with respect to the open set $J \subseteq \mathcal{O}_M (\subset \rho(A_0))$, then for any auxiliary self-adjoint operator $R$ on some separable Hilbert space $\mathfrak{R}$ obeying $\dim(E_R(J)\mathfrak{R}) \le n(A)$ there exists a self-adjoint extension $\widetilde{A}$ of $A$ such that $\widetilde{A}_J \cong R_J$.*

**PROOF.** Let us assume that $A$ is simple. If $n(A) = \dim(\mathcal{H}) = \dim(E_R(J)\mathfrak{R})$, then there exists a partial isometry $U : \mathcal{H} \longrightarrow \mathfrak{R}$ such that $U^*U = I_{\mathcal{H}}$ and $UU^* = E_R(J)$. We set $T := U^*RU$. Obviously, we have $E_T(\mathbb{R} \setminus J) = 0$. Notice that $T = T_J \cong R_J$.

We put $B := m(T)$ and consider the self-adjoint extension $\widetilde{A} := A_B \in \mathrm{Ext}_A$ defined by

$$A_B = A^* \upharpoonright \mathrm{dom}(A_B), \qquad \mathrm{dom}(A_B) = \ker(\Gamma_1 - B\Gamma_0), \quad B = U^* m(R_J) U, \tag{4.8}$$

cf. Remark 2.7 and Definition 2.6. By Proposition 2.10 and Remark 2.15 the Weyl function $M_B(\cdot)$ of the pair $\{A, A_B\}$, which corresponds either to the boundary triple $\Pi_B = \{\mathcal{H}, B\Gamma_0 - \Gamma_1, \Gamma_0\}$ if $B = B^* \in [\mathcal{H}]$ or to the generalized boundary triple $\Pi_B$ if $B = B^* \in \mathcal{C}(\mathcal{H}) \setminus [\mathcal{H}]$, is given by (2.11), that is, $M_B(z) = (B - M(z))^{-1}$. Let $\Sigma_B(\cdot)$, $(\Sigma^0_B(\cdot))$ be the corresponding unbounded (bounded) non-orthogonal spectral measure. Then by Proposition 4.1 we get

$$\Sigma_B(\delta \cap J) = (m'_J(T))^{-1} E_T(\delta \cap J) = (m'_J(T))^{-1} E_T(\delta), \quad \delta \in \mathcal{B}(\mathbb{R}). \tag{4.9}$$

Hence, setting

$$D := m'_J(T)^{-1/2}(I + T^2)^{-1/2} (\in [\mathcal{H}]), \tag{4.10}$$

we obtain from (4.9) and (2.5) that

$$\begin{aligned}\Sigma^0_B(\delta\cap J) &= \int_{\delta\cap J}(1+t^2)^{-1}d\Sigma_B(t) = \int_{\delta\cap J}(1+t^2)^{-1}m'_J(T)^{-1}dE_T(t) \\ &= m'_J(T)^{-1}(I+T^2)^{-1}E_T(\delta) = D^*E_T(\delta)D = D^*E_T(\delta\cap J)D, \quad \delta\in\mathcal{B}(\mathbb{R}).\end{aligned} \tag{4.11}$$

Identity (4.11) means that the spectral measure $E_T(=E_{T_J})$ is the orthogonal dilation of the measure $\Sigma^0_{B,J}:\mathcal{B}(\mathbb{R})\ni\delta\mapsto\Sigma^0_B(\delta\cap J)$. Since $\ker(D)=\{0\}$ this dilation is minimal. Let further $E_B(\cdot)$ be the orthogonal spectral measure in $\mathcal{K}_B$ associated to $\Sigma^0_B(\cdot)$ (see Definition 2.2). Then $E_B(\cdot)E_B(J)$ is the orthogonal spectral measure in $\mathcal{K}_{B,J}:=E_B(J)\mathcal{K}_B$ associated to $\Sigma^0_{B,J}(\cdot)$. By Proposition 2.1, the measures $E_T(\cdot)$ and $E_B(\cdot)E_B(J)$ are unitarily equivalent.

Finally, by Lemma 2.12 and Remark 2.15 the spectral measures $E_B(\cdot)E_B(J)$ and $E_{\widetilde{A}}(\cdot)E_{\widetilde{A}}(J)$, where $\widetilde{A}:=A_B$, are also unitarily equivalent. Hence the spectral measures $E_T(\cdot)$ and $E_{\widetilde{A}}(\cdot)E_{\widetilde{A}}(J)$ are unitarily equivalent. Thus, $\widetilde{A}_J\cong T\cong R_J$.

If $\dim(E_R(J)\mathfrak{R})<\dim(\mathcal{H})$, then there is a closed symmetric extension $\widehat{A}$, $A\subset\widehat{A}\subset A_0$, such that $n(\widehat{A})=\dim(E_R(J)\mathfrak{R})$. By Lemma 4.3 there is a boundary triple $\widehat{\Pi}=\{\widehat{\mathcal{H}},\widehat{\Gamma}_0,\widehat{\Gamma}_1\}$ such that the corresponding Weyl function $\widehat{M}(\cdot)$ is monotone with respect to $J$ and of scalar-type. Following now the line of reasoning of the first part we complete the proof.

If $A$ is not simple, then the operator $A$ admits a decomposition $A=A_s\oplus A'$ where $A_s$ is a self-adjoint and $A'$ is a simple closed symmetric operator which has the same gaps as $A$. Setting $\Pi'=\{\mathcal{H},\Gamma'_0,\Gamma'_1\}$, $\Gamma'_i:=\Gamma_i\upharpoonright\mathrm{dom}(A'^*)$, $i=0,1$, one performs a boundary triple for $A'^*$ such that the corresponding Weyl functions $M'(\cdot)$ and $M(\cdot)$ coincides. By $n(A)=n(A')$ the condition $\dim(E_R(J)\mathfrak{R})\le n(A')$ is satisfied. Hence applying the considerations above to the simple closed symmetric operator $A'$ we get a self-adjoint extension $\widetilde{A}'$ of $A'$ obeying $\widetilde{A}'\cong R_J$. Setting $\widetilde{A}=A_s\oplus\widetilde{A}'$ we obtain the desired extension for $A$. $\square$

**REMARK 4.5** If the deficiency indices of the closed symmetric operator $A$ are infinite, then, of course, the condition $\dim(E_R(J)\mathfrak{R})\le n(A)=\infty$ is always satisfied.

## 4.2 General case: Proof of Theorem 1.1

In this subsection we apply Theorem 4.4 to the case of direct sums of pairwise unitarily equivalent symmetric operators. We start with the following simple lemma.

**LEMMA 4.6** *Let $A$ be a densely defined closed symmetric operator on the separable Hilbert space $\mathfrak{H}$. Further, let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple of $A^*$ with the corresponding Weyl function $M(\cdot)$. If the densely defined closed symmetric operator $S$ on the separable Hilbert space $\mathfrak{K}$ is unitarily equivalent to $A$, then there exists a boundary triple*

$\Pi_1 = \{\mathcal{H}, \Gamma_0^1, \Gamma_1^1\}$ *with Weyl function* $M_1(\cdot) : \mathbb{C}_+ \longrightarrow [\mathcal{H}]$ *such that the self-adjoint extensions* $S_0$, $A_0$ *are unitarily equivalent and the corresponding Weyl functions* $M_1(\cdot)$ *and* $M(\cdot)$ *coincide.*

**PROOF.** Since $S$ is unitarily equivalent to $A$ there is an isometric operator $U : \mathfrak{H} \longrightarrow \mathfrak{K}$ such that $S = UAU^{-1}$. Obviously, one has $S^* = UA^*U^{-1}$. We set $\Gamma_i^1 := \Gamma_i U^{-1}$, $\quad i \in \{0,1\}$. One easily checks that $\Pi_1 = \{\mathcal{H}, \Gamma_0^1, \Gamma_1^1\}$ is a boundary triple for $S^*$. In particular, one finds that $S_0 = UA_0U^{-1}$. By Definition 2.8 one immediately gets that $M_1(z) = M(z)$ for $z \in \mathbb{C}_+$. $\square$

**LEMMA 4.7** *Let* $\{S_k\}_{k=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$, *be a sequence of closed symmetric operators* $S_k$ *defined on the separable Hilbert spaces* $\mathfrak{K}_k$. *If the operators* $S_k$ *are unitarily equivalent to a given closed symmetric operator* $A$ *on* $\mathfrak{H}$, *then there exists a closed symmetric extension* $\widehat{S}$ *of* $S = \oplus_{k=1}^N S_k$ *on* $\mathfrak{K} = \oplus_{k=1}^N \mathfrak{K}_k$ *such that* $\widehat{S}^*$ *admits a boundary triple* $\widehat{\Pi}$ *with a scalar-type Weyl function* $\widehat{M}(\cdot)$.

**PROOF.** Let $\Pi_0 = \{\mathcal{H}_0, \Gamma_0^0, \Gamma_1^0\}$ be a boundary triple of $A^*$ with corresponding Weyl function $M_0(\cdot)$. If the operators $S_k$ are unitarily equivalent to $A$, then by Lemma 4.6 we find a sequence of boundary triples $\Pi_k := \left\{\mathcal{H}_k, \Gamma_0^k, \Gamma_1^k\right\}$ for $S_k^*$ such that $\mathcal{H}_k = \mathcal{H}$, the self-adjoint extensions $S_{k,0}$ are unitarily equivalent to $A_0$ and the corresponding Weyl functions $M_k(\cdot)$ and $M_0(\cdot)$ coincide. Furthermore, one verifies that $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$,

$$\mathcal{H} := \bigoplus_{k=1}^N \mathcal{H}_k, \quad \mathcal{H}_k = \mathcal{H}_0 \quad \text{and} \quad \Gamma_i := \bigoplus_{k=1}^N \Gamma_i^k, \quad i \in \{0,1\}, \tag{4.12}$$

defines a boundary triple for $S^*$ where $S := \bigoplus_{k=1}^N S_k$. The Weyl function $M(\cdot)$ of this boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is given by

$$M(z) = \bigoplus_{k=1}^N M_k(z) \quad \text{where} \quad M_k(z) = M_0(z), \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-.$$

By $\widehat{A}$ we denote a closed symmetric extension of $A$ such that $n_\pm(\widehat{A}) = 1$. Consider a boundary triple $\widehat{\Pi} = \{\widehat{\mathcal{H}}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ for $\widehat{A}^*$. Since $\dim \widehat{\mathcal{H}} = 1$ (see Remark 2.7(i)) the space $\widehat{\mathcal{H}}$ can be identified with $\mathbb{C}$ and the corresponding Weyl function $\widehat{M}(\cdot) : \mathbb{C}_+ \longrightarrow [\widehat{\mathcal{H}}]$ can be identified with a scalar Nevanlinna function $\widehat{m}(\cdot) : \mathbb{C}_+ \longrightarrow [\mathbb{C}] = \mathbb{C}$.

Now, if each operator of $\{S_k\}_{k=1}^N$ is unitarily equivalent to $A$, then there is a new sequence of closed symmetric extensions $\{\widehat{S}_k\}_{k=1}^N$ such that each operator $\widehat{S}_k$ is unitarily equivalent to $\widehat{A}$ with deficiency indices $n_\pm(\widehat{A}) = 1$. Applying the construction from above we find a sequence of boundary triples $\left\{\widehat{\Pi}_k := \{\widehat{\mathcal{H}}_0^k, \widehat{\Gamma}_0^k, \widehat{\Gamma}_1^k\}\right\}_{k=1}^N$ such that $\widehat{\mathcal{H}}_k = \mathbb{C}$, the self-adjoint extensions $\widehat{S}_{k,0} = \widehat{S}_k^* \upharpoonright \ker(\widehat{\Gamma}_0^k)$, and $\widehat{A}_0 = \widehat{A} \upharpoonright \ker(\widehat{\Gamma}_0)$ are unitarily equivalent and the corresponding Weyl functions $\widehat{M}_k(\cdot)$ are scalar Nevanlinna functions $\widehat{m}_k(\cdot)$ which coincide with $\widehat{m}(\cdot)$. Setting

$$\widehat{\mathcal{H}} := \bigoplus_{k=1}^{N} \widehat{\mathcal{H}}_k, \quad \widehat{\mathcal{H}}_k = \mathbb{C}, \quad \text{and} \quad \widehat{\Gamma}_i := \bigoplus_{k=1}^{N} \widehat{\Gamma}_i^k, \quad i \in \{0,1\},$$

we define a boundary triple $\widehat{\Pi}$ for $\widehat{S}^*$ where $\widehat{S} = \oplus_{k=1}^{N} \widehat{S}_k$ with Weyl function $\widehat{M}(\cdot)$,

$$\widehat{M}(z) = \bigoplus_{k=1}^{N} \widehat{M}_k(z) = \widehat{m}(z) I_{\widehat{\mathcal{H}}}, \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-.$$

This completes the proof. $\square$

Lemma 4.7 allows us to express the concept of scalar-type Weyl function in geometric terms.

**PROPOSITION 4.8** *Let $S$ be a simple symmetric operator in $\mathfrak{K}$ with equal deficiency indices $n_\pm(S) =: N$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $S^*$. The corresponding Weyl function $M(\cdot)$ is of scalar-type if and only if $S$ and $S_0 := S^* \restriction \ker(\Gamma_0)$ admit the decompositions*

$$S = \bigoplus_{k=1}^{N} S_k \quad \textit{and} \quad S_0 = \bigoplus_{k=1}^{N} S_{k,0} \tag{4.13}$$

*such that*

(i) *$S_k$, $k = 1, 2, \ldots, N$, are closed symmetric operators with deficiency indices $n_\pm(S_k) = 1$ which are unitarily equivalent to each other,*

(ii) *$S_{k,0}$, $k = 1, 2, \ldots, N$, are self-adjoint extensions of $S_k$ which are unitarily equivalent to each other,*

(iii) *there is a boundary triple $\Pi_k = \{\mathcal{H}_k, \Gamma_0^k, \Gamma_1^k\}$ for $S_k^*$ and each $k = 1, 2, \ldots, N$, such that $S_{k,0} = S_k^* \restriction \ker(\Gamma_1^k)$ and the corresponding Weyl function coincides with $m(\cdot)$ for each $k = 1, 2, \ldots, N$.*

*The decomposition* (4.13) *is not unique.*

**PROOF.** The scalar function $m(\cdot)$ is a Nevanlinna function satisfying the assumptions of Corollary 0.2 of [17]. Hence, by this corollary there is a simple closed symmetric operator $A$ with deficiency indices $n_\pm(A) = 1$ on some Hilbert space $\mathfrak{H}_0$ and a boundary triple $\Pi_0 = \{\mathcal{H}_0, \Gamma_0^0, \Gamma_0^0\}$, $\dim(\mathcal{H}_0) = 1$, for $A^*$ such that the corresponding Weyl function coincides with $m(\cdot)$. We set

$$\widehat{\mathfrak{H}} := \bigoplus_{k=1}^{N} \mathfrak{H}_k, \quad \mathfrak{H}_k = \mathfrak{H}_0, \quad \text{and} \quad \widehat{A} := \bigoplus_{k=1}^{N} A_k, \quad A_k = A.$$

With each $A_k$ one associates a boundary triple $\Pi_k = \{\mathcal{H}_k, \Gamma_0^k, \Gamma_1^k\}$, $\mathcal{H}_k = \mathcal{H}_0$, $\Gamma_0^k := \Gamma_0^0$, $\Gamma_1^k := \Gamma_1^0$, with Weyl function $M_k(\cdot) = m_k(\cdot)$, $k = 1, 2, \ldots, N$. Considering the orthogonal sum of all these boundary triples one gets a boundary triple $\widehat{\Pi}$ for $\widehat{A}^*$ with corresponding Weyl function $\widehat{M} = \oplus_{k=1}^N M_k$. Moreover, one has that the extension $\widehat{A}_0 := \widehat{A}^* \upharpoonright \mathrm{dom}(\widehat{\Gamma}_0)$ admits the representation $\widehat{A}_0 := \bigoplus_{k=1}^N A_{k,0}$, $A_{k,0} := A_k^* \upharpoonright \mathrm{dom}(\Gamma_0^k)$. Since the Weyl functions $\widehat{M}(\cdot)$ and $M(\cdot)$ coincide one gets from Corollary 0.1 of [17] that $S$ and $\widehat{A}$ as well as $S_0$ and $\widehat{A}_0$ are unitarily equivalent which proves (i)-(iii). The converse follows from Lemma 4.7. $\square$

Combining Lemma 4.7 with Lemma 4.3 we obtain

**PROPOSITION 4.9** *Let $\{S_k\}_{k=1}^N$ be a sequence of closed symmetric operators $S_k$ defined on the separable Hilbert spaces $\mathfrak{K}_k$ and such that the operators $S_k$ are unitarily equivalent to a given symmetric operator $A$ in $\mathfrak{H}$. Suppose that for some boundary triple $\Pi$ for $A^*$ the corresponding Weyl function $M(\cdot)$ is monotone with respect to $J \subseteq \mathcal{O}_M$. Then there exists a closed symmetric extension $\widehat{S}$ of $S = \oplus_{k=1}^N S_k$ in $\mathfrak{K} = \oplus_{k=1}^N \mathfrak{K}_k$ and a boundary triple for $\widehat{S}^*$ such that the corresponding Weyl function $\widehat{M}(\cdot)$ is of scalar-type, $\mathcal{O}_{\widehat{M}} = \mathcal{O}_M$ and $\widehat{M}(\cdot)$ is monotone with respect to $J \subseteq \mathcal{O}_M$.*

Now we are ready to prove Theorem 1.1:

**PROOF OF THEOREM 1.1.** By Proposition 4.9 it is sufficient to consider a closed symmetric operator admitting a boundary triple with scalar-type Weyl function. Applying Theorem 4.4 we complete the proof. $\square$

# 5 Beyond the gaps

In this section we assume that the simple symmetric operator $A$ admits a boundary triple $\Pi = \{\mathcal{H}, \Pi_0, \Pi_1\}$ such that the corresponding Weyl function $M(\cdot)$ is of scalar-type. We try to complement Theorem 4.4 by results on the spectrum $\sigma(A_B)$ of the operator $A_B$, defined by (4.8), outside the gaps $\mathcal{O}_m^c := \mathbb{R} \setminus \mathcal{O}_m = \mathrm{supp}(m)$. Partially, we try to extend the results to arbitrary extensions. Mainly, we obtain results on the absolutely continuous spectrum, cf. Theorems 5.2,5.6, Corollary 5.4 and Proposition 5.5. However, we are also interested in the singular spectrum, cf. Theorem 5.6 and Proposition 5.8.

We will rely on a Fatou-type theorem (see [4, 5, 19, 20, 28]) which for convenience is repeated here in the form used in [12], Proposition 3.5.

**THEOREM 5.1** *Let $m(\cdot)$ be a scalar Nevanlinna function in $\mathbb{C}_+$ with the integral representation* (3.1) *and the imaginary part $v(z) := \Im\mathrm{m}(m(z))$ which admits the representation*

$$v(x,y) = c_1 y + \int_{\mathbb{R}} \frac{y d\mu(t)}{(t-x)^2 + y^2}, \qquad \int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty$$

*where* $v(x,y) := v(x+iy)$, $z = x+iy \in \mathbb{C}_+$. *Then*

(i) *For any* $x \in \mathbb{R}$ *the limit* $v(x+i0) := \lim_{y\downarrow 0} v(x+iy)$ *exists and is finite if and only if the symmetric derivative* $D\mu(x)$

$$D\mu(x) = \lim_{\epsilon \to 0} \frac{\mu(x+\epsilon) - \mu(x-\epsilon)}{2\epsilon}$$

*exists and is finite. In this case one has* $v(x+i0) = \pi D\mu(x)$.

(ii) *If the symmetric derivative* $D\mu(x)$ *exists and is infinite, then* $v(z) \to +\infty$ *as* $z \to\succ x$.

(iii) *For each* $x \in \mathbb{R}$ *one has* $\Im\mathrm{m}(z-x)v(z) \to \mu(\{x\})$ *as* $z \to\succ x$.

(iv) $v(z)$ *converges to a finite constant as* $z \to\succ x$ *if and only if the derivative* $\mu'(t) := \frac{d\mu(t)}{dt}$ *exists at* $t = x$ *and is finite. Moreover, one has* $v(x_0 + i0) = \pi\mu'(x)$.

The symbol $\to\succ$ means that the limit $\lim_{r\downarrow 0} v(x+re^{i\theta})$, $x \in \mathbb{R}$, exist uniformly in $\theta \in [\epsilon, \pi-\epsilon]$ for each $\epsilon \in (0, \pi/2)$. The main result of this section reads as follows.

**THEOREM 5.2** *Let* $A$ *be a simple symmetric operator in* $\mathfrak{H}$ *with infinite deficiency indices. Further, let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triple for* $A^*$ *with scalar-type Weyl function* $M(\cdot)$, *i.e.* $M(z) = m(z)I_{\mathcal{H}}$, *and let* $B = B^* \in \mathcal{C}(\mathcal{H})$.

(i) *Then* $\sigma_{ac}(A_B) \supset \sigma_{ac}(A_0)$, $A_0 := A^* \upharpoonright \ker(\Gamma_0)$.

(ii) *If the operator* $B$ *is purely absolutely continuous, then the self-adjoint extension* $A_B$ *is purely absolutely continuous, too.*

**PROOF.** (i) By Corollary 2.14 we get that $\sigma_{ac}(A_0) = \mathrm{supp}_{ac}(\mu)$ where $\mu$ is the Radon measure of the representation 3.1. In accordance with [12] we set

$$\Omega_{ac}(m) := \{x \in \mathbb{R} : \exists\, m(x+i0) := \lim_{y\downarrow 0} m(x+iy) \text{ and } 0 < v(x,0) := \Im\mathrm{m}(m(x+i0)) < \infty\}.$$

Notice that the limit $m(x+i0) := \lim_{y\downarrow 0} m(x+iy)$ exists for almost all $x \in \mathbb{R}$. Further, let us introduce the set

$$\mathrm{cl}_{ac}(\mathcal{X}) := \{x \in \mathbb{R} : \mathrm{mes}((x-\epsilon, x+\epsilon) \cap \mathcal{X}) > 0 \quad \text{for all} \quad \epsilon > 0\}.$$

By Lemma 4.1 of [12] we get that $\mathrm{cl}_{ac}(\Omega_{ac}(m)) = \mathrm{supp}_{ac}(\mu)$.

By Remark 2.15 the Weyl function $M_B(\cdot)$ of the extension $A_B$ is given by $M_B(z) := (B - M(z))^{-1} = (B - m(z) \cdot I_{\mathcal{H}})^{-1}$, $z \in \mathbb{C}_+$. Let us introduce the scalar-function

$$M_{B,h}(z) := (M_B(z)h, h) = ((B - m(z)I_{\mathcal{H}})^{-1}h, h) = \int_{\mathbb{R}} \frac{d(E_B(t)h, h)}{t - m(z)}, \qquad z \in \mathbb{C}_+, \quad (5.1)$$

for $h \in \mathcal{H}$. If $z = x + iy$ and $m(z) =: u(x, y) + iv(x, y)$, then we get from (5.1) that

$$F_{B,h}(z) := \Im\mathrm{m}(M_{B,h}(z)) = \int_{\mathbb{R}} \frac{v(x, y)d(E_B(t)h, h)}{(t - u(x, y))^2 + v(x, y)^2}. \tag{5.2}$$

Let $x \in \Omega_{ac}(m)$. Notice that the limits $v(x, 0) := \lim_{y\downarrow 0} v(x, y) > 0$ and $u(x, 0) := \lim_{y\downarrow 0} u(x, y)$ exists if $x \in \Omega_{ac}(m)$. If $y_0 > 0$ is small enough, then

$$\frac{v(x, y)}{(t - u(x, y))^2 + v^2(x, y)} \le \frac{1}{v(x, y)} \le \frac{2}{v(x, 0)}, \qquad y \in [0, y_0), \quad x \in \Omega_{ac}(m). \tag{5.3}$$

Taking into account (5.3) and applying the Lebesgue dominated convergence theorem we obtain from (5.2) that

$$F_{B,h}(x + i0) := \lim_{y\downarrow 0} F_{B,h}(x, y) = v(x, 0) \int_{\mathbb{R}} \frac{d(E_B(t)h, h)}{(t - u(x, 0))^2 + v(x, 0)^2}, \qquad x \in \Omega_{ac}(m). \tag{5.4}$$

Since $v(x, 0) > 0$ for $x \in \Omega_{ac}(m)$ we find

$$0 < F_{B,h}(x + i0) < \infty, \qquad x \in \Omega_{ac}(m).$$

Furthermore, we have

$$G_{B,h}(z) := \Re\mathrm{e}(M_{B,h}(z)) = \int_{R} \frac{(t - u(x, y))d(E_B(t)h, h)}{(t - u(x, y))^2 + v(x, y)^2}$$

Since

$$\frac{|t - u(x, y)|}{(t - u(x, y))^2 + v(x, y)^2} \le \frac{1}{\sqrt{(t - u(x, y))^2 + v(x, y)^2}} \le \frac{\sqrt{2}}{v(x, 0)}$$

for $x \in \Omega_{ac}(m)$ and $y \in (0, y_0)$. Again by the Lebesgue dominated convergence theorem we find

$$G_{B,h}(x + i0) := \lim_{y\downarrow 0} G_{B,h}(x + iy) = \int_{R} \frac{(t - u(x, 0))d(E_B(t)h, h)}{(t - u(x, 0))^2 + v(x, 0)^2}$$

which shows that $x \in \Omega_{ac}(m)$ implies $x \in \Omega_{ac}(M_{B,h})$ for every $h \in \mathcal{H}$ where

$$\Omega_{ac}(M_{B,h}) := \{x \in \mathbb{R} : \exists\, M_{B,h}(x + i0) := \lim_{y\downarrow 0} M_{B,h}(x + iy) \text{ and } 0 < \Im\mathrm{m}(M_{B,h}(x + i0)) < \infty\}.$$

Since $\Omega_{ac}(m) \subseteq \Omega_{ac}(M_{B,h})$ one gets $\sigma_{ac}(A_0) = \mathrm{supp}_{ac}(\mu) = \mathrm{cl}_{ac}(\Omega_{ac}(m)) \subseteq \mathrm{cl}_{ac}(\Omega_{ac}(M_{B,h}))$ for each $h \in \mathcal{H}$. Finally, applying Proposition 4.2 of [12] we verify (i).

(ii) If $B = B^{ac}$, then the measure $\rho_h(\cdot) := (E_B(\cdot)h, h)$ is absolutely continuous for any $h \in \mathcal{H}$, that is, $d\rho_h(t) = \rho'_h(t)dt$, where $\rho'_h(\cdot) \in L^1(\mathbb{R})$ for any $h \in \mathcal{H}$. One rewrites (5.2) as

$$F_{B,h}(z) = \int_{\mathbb{R}} \frac{v(x, y)\rho'_h(t)dt}{(t - u(x, y))^2 + v(x, y)^2} \tag{5.5}$$

From [6] it is well known that the subset $\mathcal{H}_\infty := \{h \in \mathcal{H} : \rho'_h \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})\}$ is dense in $\mathcal{H} = \mathcal{H}^{ac}(B)$. For $h \in \mathcal{H}_\infty$ we obtain from (5.5) that

$$C_\infty(h) := \sup_{0<y<1} \sup_{x\in\mathbb{R}} \Im\mathrm{m} F_{B,h}(x+iy) \le \|\rho'_h\|_{L^\infty} \sup_{v>0} \sup_{u\in\mathbb{R}} \int_{\mathbb{R}} \frac{v\, ds}{(s-u)^2+v^2} \le \pi \|\rho'_h\|_{L^\infty}. \quad (5.6)$$

Using Corollary 4.7 of [12] we complete the proof. □

**REMARK 5.3**

1. The results of Theorem 5.2 are valid if the extensions $\widetilde{A}$ and $A_0$ are disjoint, cf. Remark 2.7. If they are not disjoint and if $A_0$ is not absolutely continuous, then the assertion (ii) of Theorem 5.2 might be false.

2. The inclusion $\sigma_{ac}(A_B) \supset \sigma_{ac}(A_0)$ of Theorem 5.2(i) might be strict. Indeed, if $B = B^{ac}$ and $A_0$ is singular, in particular, pure point, then $A_B = A_B^{ac}$ by Theorem 5.2(ii) but $\sigma_{ac}(A_0) = \emptyset$, i.e. $\emptyset \ne \sigma_{ac}(A_B) \supset \sigma_{ac}(A_0) = \emptyset$.

Assertion(i) of Theorem 5.2 is not only true for extensions which are disjoint with $A_0$ but for any extension.

**COROLLARY 5.4** *Let $A$ be a simple symmetric operator in $\mathfrak{H}$ with infinite deficiency indices. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ a boundary triple for $A^*$ with scalar-type Weyl function $M(\cdot)$. If $\widetilde{A} \in \mathrm{Ext}_A$, then $\sigma_{ac}(A_0) \subseteq \sigma_{ac}(\widetilde{A})$.*

**PROOF.** Taking into account Remark 2.7 the corollary follows from Theorem 5.2 provided the extension $\widetilde{A}$ is disjoint with $A_0$.

If $\widetilde{A}$ is not disjoint with $A_0$ we set $\widehat{A} = A^* \upharpoonright \mathrm{dom}(\widehat{A}) \supseteq A$, $\mathrm{dom}(\widehat{A}) := \mathrm{dom}(A_0) \cap \mathrm{dom}(\widetilde{A})$. The operator $\widehat{A}$ is closed and symmetric. Moreover, one has $A \subseteq \widehat{A} \subseteq A_0$. Notice that $\widetilde{A}$ and $A_0$ are disjoint with respect to $\widehat{A}$. By Lemma 4.3 there is a boundary triple $\widehat{\Pi} = \{\widehat{\mathcal{H}}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ such that the corresponding Weyl function $\widehat{M}(\cdot)$ is a scalar-type one.

If $\widehat{A}$ is simple, then applying Theorem 5.2 one gets $\sigma_{ac}(A_0) \subseteq \sigma_{ac}(\widetilde{A})$. If $\widehat{A}$ is not simple, then $\widehat{A} = \widehat{A}_{self} \oplus \widehat{A}_{sim}$ where $\widehat{A}_{self}$ is self-adjoint and $\widehat{A}_{sim}$ is simple. Notice that $\widehat{A}_{self} \subseteq \widetilde{A}$ and $\widehat{A}_{self} \subseteq A_0$ which yields

$$\widetilde{A} = \widehat{A}_{self} \oplus \widetilde{A}_{sim} \quad \text{and} \quad A_0 = \widehat{A}_{self} \oplus A_{0,sim}.$$

Hence, the self-adjoint operators $\widetilde{A}_{sim}$ and $A_{0,sim}$ are extensions of the symmetric operator $\widehat{A}_{sim}$. We note that

$$\mathrm{dom}(\widehat{A}_{sim}) = \mathrm{dom}(\widetilde{A}_{sim}) \cap \mathrm{dom}(A_{0,sim})$$

which shows that the extensions $\widetilde{A}_{sim}$ and $A_{0,sim}$ are disjoint.

Setting $\widehat{\Gamma}_0^{sim} := \widehat{\Gamma}_0 \upharpoonright \mathrm{dom}(\widehat{A}^*_{sim})$ and $\widehat{\Gamma}_1^{sim} := \widehat{\Gamma}_1 \upharpoonright \mathrm{dom}(\widehat{A}^*_{sim})$ we define a boundary triple $\widehat{\Pi}^{sim} = \{\widehat{\mathcal{H}}, \widehat{\Gamma}_0^{sim}, \widehat{\Gamma}_1^{sim}\}$ for $A^*_{sim}$ such that $A_{0,sim} = \widehat{A}^*_{sim} \upharpoonright \ker(\widehat{\Gamma}_0^{sim})$ and the corresponding Weyl function $\widehat{M}_{sim}(\cdot)$ coincides with $\widehat{M}(\cdot)$. Applying again Theorem 5.2 we find $\sigma_{ac}(A_{0,sim}) \subseteq \sigma_{ac}(\widetilde{A}_{sim})$ which yields $\sigma_{ac}(A_0) \subseteq \sigma_{ac}(\widetilde{A})$. □

Corollary 5.4 shows that under the assumption of a scalar-type Weyl function the absolutely continuous spectrum of any extension always contains $\sigma_{ac}(A_0)$. By Theorem 4.4 the above result implies the following corollary.

**COROLLARY 5.5** *Let $A$ be a simple symmetric operator with infinite deficiency indices on the separable Hilbert space $\mathfrak{H}$. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$ with scalar-type Weyl function $M(\cdot) = m(\cdot)\, I_{\mathcal{H}}$ which is is monotone with respect to the open set $J \subseteq \mathcal{O}_M \subset \rho(A_0)$. Then for any operator $R$ on some separable Hilbert space there is a self-adjoint extension $\widetilde{A}$ such that $\widetilde{A}_J \cong R^{ac}_J$ and $\widetilde{A}$ is absolutely continuous.*

**PROOF.** By Theorem 4.4 there is a self-adjoint extension such that $\widetilde{A}_J \cong R^{ac}_J$. Following the line of reasoning of Theorem 4.4 we find that $\widetilde{A}$ is of the form $\widetilde{A} = A_B$ where $B$ is absolutely continuous. Applying Theorem 5.2(ii) we complete the proof. □

Naturally, the problem arises to find conditions which are sufficient in order that $\sigma_{ac}(\widetilde{A}) = \sigma_{ac}(A_0)$.

**THEOREM 5.6** *Let $A$ be a simple symmetric operator in $\mathfrak{H}$ with infinite deficiency indices. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$ with scalar-type Weyl function $M(\cdot)$, i.e. $M(z) = m(z) I_{\mathcal{H}}$, and let $B = B^* \in \mathcal{C}(\mathcal{H})$.*

- (i) *If $B$ is singular, i.e. $B^s = B$, then the absolutely continuous parts $A^{ac}_B$ and $A^{ac}_0$ are unitarily equivalent, in particular, $\sigma_{ac}(A_B) = \sigma_{ac}(A_0)$.*
- (ii) *If $B$ and $A_0$ are singular, then $A_B$ is singular.*
- (iii) *If $B$ is pure point and the spectrum of $A_0$ consists of isolated eigenvalues, then $A_B$ is pure point.*

**PROOF.** (i) Let $B$ be pure point with the eigenvalues $\{b_k\}_{k=1}^{\infty}$ and eigenprojections $\{Q_k\}_{k=1}^{\infty}$. We set $\mathcal{H}_k := Q_k \mathcal{H}$. Without loss of generality let us assume that $Q_k$ are one dimensional projections. By Proposition 4.8 one gets

$$A = \bigoplus_{k=1}^{\infty} S_k \quad \text{and} \quad A_0 = \bigoplus_{k=1}^{\infty} S_{k,0}$$

where $S_k$ and $S_{k,0}$ obey the properties (i)-(iii) of Proposition 4.8. In particular, by property (iii) of Proposition 4.8 for each $b_k$, $k = 1, 2, \ldots$, there is a boundary triple $\Pi_k = \{\mathbb{C}, \Gamma_0^k, \Gamma_1^k\}$ for $S_k^*$ such that $S_{k,0} = S_k^* \upharpoonright \ker(\Gamma_0^k)$ and the corresponding Weyl function coincides with

$m(\cdot)$. Introducing for each $k$ the boundary triple $\Pi_{b_k} = \{\mathbb{C}, \Gamma_0^{b_k}, \Gamma_1^{b_k}\}$, $\Gamma_0^{b_k} := b_k\Gamma_0^k - \Gamma_1^k$, $\Gamma_1^{b_k} := \Gamma_1^k$, one defines a self-adjoint extension $S_{b_k} := S_k^* \upharpoonright \ker(\Gamma_0^{b_k})$ of $S_k$ with corresponding Weyl function $m_{b_k}(\cdot)$ given by $m_{b_k}(z) := (b_k - m(z))^{-1}$, $z \in \mathbb{C}_+$. Obviously, we have

$$M_B(z) = \bigoplus_{k=1}^{\infty}(b_k - m_k(z))^{-1}, \quad z \in \mathbb{C}_+,$$

which yields $A_B = \bigoplus_{k=1}^{\infty} S_{b_k}$. Since the self-adjoint operators $S_{k,0}$ and $S_{b_k}$ are extensions of the same symmetric operator $S_k$ with deficiency indices $n_\pm(S_k) = 1$ one gets by the Kato-Rosenblum theorem [6] that their absolutely continuous parts $S_{k,0}^{ac}$ and $S_{b_k}^{ac}$ are unitarily equivalent, i.e $S_{k,0}^{ac} \cong S_{b_k}^{ac}$. Hence $A_B^{ac} \cong A_0^{ac}$.

If $B$ is only singular, then by Theorem VI.7 of [29], see also [14], there is a self-adjoint trace class operator $C$ such $B' := B + C$ is pure point. Hence $A_{B'}^{ac} \cong A_0^{ac}$ by the first part. By Theorem 2 of [17] the difference $(A_B - z)^{-1} - (A_{B'} - z)^{-1}$ is a trace class operator if and only if $(B - z)^{-1} - (B' - z)^{-1}$ is a trace class operator. Applying again the Rosenblum-Kato theorem [6] one gets that $A_B^{ac} \cong A_{B'}^{ac} \cong A_0^{ac}$.

(ii) If $\sigma_{ac}(A_0) = \emptyset$, then by (i) we get $\sigma_{ac}(A_B) = \emptyset$ which yields $A_B^{ac} = 0$.

(iii) Following the line of reasoning of (i) one gets that the spectrum of $S_{k,0}$ consists of isolated eigenvalues for each $k = 1, 2, \ldots$, too. Since $S_{k,0}$ and $S_{b_k}$ are self-adjoint extensions of a symmetric operator $S_k$ with $n_\pm(S_k) = 1$ the spectrum of $S_{b_k}$ consists of isolated eigenvalues, too. Hence the spectrum of $A_B = \bigoplus_{k=1}^{\infty} S_{b_k}$ is pure point. □

**REMARK 5.7** The conclusion (iii) of Theorem 5.6 might be false if $A_0$ is only pure point. In this case it can happen that the singular continuous part $A_B^{sc}$ of $A_B$ is not trivial.

Under additional assumptions on the spectral measure $\mu$ of $m(\cdot)$ we can refine the statements of Theorem 5.2.

**PROPOSITION 5.8** *Let $A$ be a simple symmetric operator in $\mathfrak{H}$ with infinite deficiency indices. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for $A^*$ with scalar-type Weyl function $M(\cdot)$, i.e. $M(z) = m(z)I_{\mathcal{H}}$ and*

$$\mathrm{supp}^+(\mu) := \{x \in \mathrm{supp}(\mu) :\ \exists\ D\mu(x) \text{ and } D\mu(x) > 0\}$$

*where $\mu$ is the Radon measure of representation* (3.1). *If $B \in \mathcal{C}(\mathcal{H})$, then*

$$E_{A_B}^{\tau}(\mathrm{supp}^+(\mu)) = 0, \quad \tau \in \{s, pp, sc\}. \tag{5.7}$$

*In particular, it holds*

(i) $\sigma_p(A_B) \cap \mathrm{supp}(\mu) \subseteq \mathrm{supp}(\mu) \setminus \mathrm{supp}^+(\mu)$ *and*

(ii) $E^{sc}_{A_B}(\mathrm{supp}(\mu)) = 0$ *provided* $\mathrm{supp}(\mu) \setminus \mathrm{supp}^+(\mu)$ *is either finite or countable.*

**PROOF.** We set $\mathrm{supp}^+_\infty(\mu) := \{x \in \mathrm{supp}^+(\mu) : D\mu(x) = \infty\}$. By Theorem 5.1(i) we derive that the limit $\lim_{y\downarrow 0} v(x,y)$ exists and is finite for $x \in \mathrm{supp}^+(\mu) \setminus \mathrm{supp}^+_\infty(\mu)$ and

$$v(x,0) := \lim_{y\downarrow 0} v(x,y) = \pi D\mu(x) > 0, \qquad x \in \mathrm{supp}^+(\mu) \setminus \mathrm{supp}^+_\infty(\mu). \tag{5.8}$$

By Proposition 2.5, there exists an operator $B = B^* \in \mathcal{C}(\mathcal{H})$ such that $\widetilde{A} = A_B := A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$. We consider the generalized Weyl function $M_B(z) := (B - M(z))^{-1}$ and define $F_{B,h}(\cdot)$ by (5.2). Following the line of reasoning of Theorem 5.2 we obtain

$$0 < F_{B,h}(x + i0) < \infty, \quad x \in \mathrm{supp}^+(\mu) \setminus \mathrm{supp}^+_\infty(\mu), \qquad h \in \mathcal{H}. \tag{5.9}$$

Further, let $x \in \mathrm{supp}^+_\infty(\mu)$. By Theorem 5.1(ii) and (iii) we find

$$v(x,0) := \lim_{y\downarrow 0} v(x,y) = \infty \quad \text{and} \quad \lim_{y\downarrow 0} y\, v(x,y) = \mu(\{x\}).$$

Therefore for every $y_0 > 0$ there exists $N = N(y_0)$ such that $v(x,y) \geq N$ for $y \in (0, y_0)$. Hence

$$\frac{v(x,y)}{(t - u(x_0,y))^2 + v^2(x,y)} \leq \frac{1}{N}, \qquad y \in (0, y_0).$$

By the Lebesgue dominated theorem we obtain from (5.2) that

$$\lim_{y\downarrow 0} F_{B,h}(x + iy) = 0, \quad x \in \mathrm{supp}^+_\infty(\mu), \qquad h \in \mathcal{H}. \tag{5.10}$$

Let $\Sigma_B(\cdot)$ be the unbounded non-orthogonal spectral measure of the Weyl function $M_B(z) = (B - M(z))^{-1}$, $z \in \mathbb{C}_+$, and $\Sigma_{B,h}(\cdot) = (\Sigma_B(\cdot)h, h)$, $h \in \mathcal{H}$. If

$$S''_s(\Sigma_{B,h}) := \{x \in \mathbb{R} : F_{B,h}(z) \to \infty \quad \text{as} \quad z \to\!\succ x\}, \quad h \in \mathcal{H},$$

then we find from (5.9) and (5.10) that $S''_s(\Sigma_{B,h}) \cap \mathrm{supp}^+(\mu) = \emptyset$. Let $\mathcal{T} = \{h_k\}_{k=1}^\infty$ be a total set in $\mathcal{H}$. Setting

$$S''_s(\Sigma_B; \mathcal{T}) := \bigcup_{k=1}^\infty S''_s(\Sigma_{B,h_k})$$

one gets $S''_s(\Sigma_B; \mathcal{T}) \cap \mathrm{supp}(\mu^+) = \emptyset$. Applying Theorem 3.6 of [12] we find

$$E^s_{A_B}(\mathrm{supp}^+(\mu)) = E_{A_B}(\mathrm{supp}^+(\mu) \cap S''_s(\Sigma_B; \mathcal{T})) = 0$$

which proves (5.7) for $\tau = s$. Similarly, setting

$$S''_{pp}(\Sigma_{B,h}) := \{x \in \mathbb{R} : \lim_{z \to\succ x} (z - x) F_{B,h}(z) > 0\}, \quad h \in \mathcal{H},$$

and

$$S''_{pp}(\Sigma_B;\mathcal{T}) := \bigcup_{k=1}^{\infty} S''_{pp}(\Sigma_{B,h_k})$$

we verify $S''_{pp}(\Sigma_B;\mathcal{T}) \subseteq S''_s(\Sigma_B;\mathcal{T})$. Using Theorem 3.6 of [12] one proves (5.7) for $\tau = pp$. Finally, setting

$$S''_{sc}(\Sigma_{B,h}) := \{x \in \mathbb{R} : F_{B,h}(z) \to \infty \text{ and } (z-x)F_{B,h}(z)) \to 0 \text{ as } z \to\succ x\}, \quad h \in \mathcal{H},$$

and

$$S''_{sc}(\Sigma_B;\mathcal{T}) := \bigcup_{k=1}^{\infty} S''_{sc}(\Sigma_{B,h_k}) \setminus S''_{pp}(\Sigma_B;\mathcal{T})$$

we obtain $S''_{sc}(\Sigma_{B,h}) \subseteq S''_s(\Sigma_{B,h})$ which yields (5.7) for $\tau = sc$ by Theorem 3.6 of [12].

(i) By Theorem 3.6 of [12] we have $\sigma_p(A_B) = S''_{pp}(\Sigma_B;\mathcal{T})$ which yields $\sigma_p(A_B) \cap \operatorname{supp}(\mu) \subset \operatorname{supp}(\mu) \setminus \operatorname{supp}^+(\mu)$.

(ii) We have

$$E^{sc}_{A_B}(\operatorname{supp}(\mu)) = E^{sc}_{A_B}(\operatorname{supp}^+(\mu)) + E^{sc}_{A_B}(\operatorname{supp}(\mu) \setminus \operatorname{supp}^+(\mu)) = E^{sc}_{A_B}(\operatorname{supp}(\mu) \setminus \operatorname{supp}^+(\mu)).$$

Since by assumption $\operatorname{supp}(\mu)\setminus\operatorname{supp}^+(\mu)$ is countable we obtain $E^{sc}_{A_B}(\operatorname{supp}(\mu)\setminus\operatorname{supp}^+(\mu)) = 0$ which shows $E^{sc}_{A_B}(\operatorname{supp}(\mu)) = 0$. $\square$

**REMARK 5.9** We note that if in addition to the assumptions of Proposition 5.8 the condition $\operatorname{supp}(\mu) = \operatorname{supp}^+(\mu)$ is satisfied, then by (i) one has $\sigma_p(A_B) \cap \sigma(A_0) = \emptyset$, cf. [3, 19, 20, 28].

# 6 Examples

In this section we consider several examples in order to illustrate the previous results.

## 6.1 Example

Let $\mathfrak{H} = L^2((0,1))$. By $A$ we denote the closed symmetric operator

$$\begin{aligned} (Af)(x) &:= -i\tfrac{d}{dx}f(x), \quad x \in (0,1), \\ f \in \operatorname{dom}(A) &:= \{f \in W^1_2((0,1)) : f(0) = f(1) = 0\}, \end{aligned}$$

which is simple and has deficiency indices $(1,1)$. We note that $A^*$ is given by $(A^*f)(x) := -i\frac{d}{dx}f(x)$, $f \in \operatorname{dom}(A^*) := W^1_2((0,1))$. A straightforward computation shows that $\Pi = \{\mathcal{H},\Gamma_0,\Gamma_1\}$ where $\mathcal{H} := \mathbb{C}$,

$$\Gamma_0 f := \frac{f(0)-f(1)}{\sqrt{2}}, \quad \Gamma_1 f := i\frac{f(0)+f(1)}{\sqrt{2}}, \quad f \in \operatorname{dom}(A^*) = W^1_2((0,1)), \tag{6.1}$$

forms a boundary triple for $A^*$. The operator $A_0 := A^* \upharpoonright \ker(\Gamma_0)$ is given by

$$(A_0 f)(x) = -i\frac{d}{dx}f(x), \quad x \in (0,1), \quad f \in \mathrm{dom}(A_0) = \{W^1_2((0,1)) : f(0) = f(1)\}.$$

The spectrum of $A_0$ is discrete. It consists of isolated eigenvalues $\sigma(A_0) = \{\lambda_l\}_{l\in\mathbb{Z}}$ with $\lambda_l = 2l\pi$. Obviously, we have $\rho(A_0) = \cup_{l\in\mathbb{Z}}\Delta_l$ where $\Delta_l = (2l\pi, 2(l+1)\pi)$. Trivially, the open intervals $\Delta_l$ are gaps of the operator $A_0 = A_0^*$. Hence they are gaps of the symmetric operator $A$. The extension $A_1 = A^* \upharpoonright \ker(\Gamma_1)$ has the domain $\mathrm{dom}(A_1)$, $\mathrm{dom}(A_1) := \{f \in W^{(1,2)}((0,1)) : f(0) = -f(1)\}$. Its spectrum is discrete and consists of the eigenvalues $\lambda_l = (2l+1)\pi$, $l \in \mathbb{Z}$. Any other extension of $A$ is given by a real constant $\theta \in \mathbb{R}$ and the boundary triple $\Pi_\theta := \{\mathbb{C}, \Gamma^\theta_0, \Gamma^\theta_1\}$, where $\Gamma^\theta_1 = \Gamma_0$ and $\Gamma^\theta_0 = \theta\Gamma_0 - \Gamma_1$. The domain $\mathrm{dom}(A_\theta)$ of the self-adjoint extension $A_\theta = A^* \upharpoonright \ker(\Gamma^\theta_0)$ can be alternatively described by

$$\mathrm{dom}(A_\theta) = \bigl\{f \in \{W^1_2((0,1)) : (\theta - i)(\theta + i)^{-1} f(0) = f(1)\bigr\}$$

Of course, the spectrum of $A_\theta$ is also discrete and consists of the eigenvalues. Setting $\theta = -\cot(\tau/2)$, $\tau \in (0, 2\pi)$, one easily verifies that $\lambda^{(\theta)}_l = \tau + l$, $l \in \mathbb{Z}$. In other words, any extension of $A$, which is different from $A_0$, has an eigenvalue in the gaps $\Delta_l$, $l \in \mathbb{Z}$, i.e., it does not preserve the gaps $\Delta_l$.

It is easily seen that the Weyl function corresponding to the boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of the form (6.1) is

$$m(z) = -\frac{\cos(z/2)}{\sin(z/2)} = -\cot(z/2), \qquad z \in \mathbb{C}_+ \cup \mathbb{C}_-.$$

The open set $\mathcal{O}_m = \mathbb{R} \setminus \mathrm{supp}(m)$ coincides with $\rho(A_0) \cap \mathbb{R}$, i.e. $\mathcal{O}_m = \cup_{l\in\mathbb{Z}}\Delta_l$. The Weyl function admits an extension to $\mathcal{O}_m$ which is given by $m(\lambda) = -\cot(\lambda/2)$, $\lambda \in \mathcal{O}_m$. Obviously, the Weyl function $m(\cdot)$ is increasing on each open interval $\Delta_l$. However, choosing $J = \mathcal{O}_m$ one easily verifies that the Weyl function $m(\cdot)$ is not monotone with respect to $J$. The lack of monotonicity is related to the fact that there does not exist an extension $\widetilde{A}$ of $A$ which has only an eigenvalue in one gap $\Delta_l$ as we have seen above.

Let us consider the closed symmetric operator $S = \oplus^\infty_{k=1} S_k$ on the Hilbert space $\mathfrak{K} = \oplus^\infty_{k=1}\mathfrak{K}_k$ where the operators $S_k$ are unitarily equivalent to $A$ defined above. Obviously, the operator $S$ is unitarily equivalent to the operator $C$ defined on $\mathfrak{H} = L^2((0,\infty))$,

$$(Cf)(x) := -i\frac{d}{dx}f(x), \quad f \in \mathrm{dom}(C) := \{W^1_2(\mathbb{R}_+) : f(k) = 0,\ k \in \{0\} \cup \mathbb{N}\}$$

To apply Theorem 3.3 we note that now $\mathcal{O}_m = \bigcup_{l\in\mathbb{Z}}(2\pi l, 2\pi(l+1))$ and $\varphi_l(t) = -2\mathrm{arccot}(t) + 2\pi(l+1)$, $l \in \mathbb{Z}$. By (3.37) and (3.38) the associated non-orthogonal spectral measures, $\Sigma^0_B(\cdot)$ and $\Sigma_B(\cdot)$ of the Weyl function $M_B(z) := (B - m(z) \cdot I)^{-1}$ are given by

$$\Sigma^0_B(\delta) = \varphi'_l(B)(1 + \varphi_l(B)^2)^{-1} E_B(m(\delta)) = (1 + 2\pi(l+1) - 2\mathrm{arccot}(B)^2)^{-1} E_B(-\cot(\delta/2)), \tag{6.2}$$

and

$$\Sigma_B(\delta) = \varphi_l'(B)E_B(-\cot(\delta/2)) = 2(1+B^2)^{-1}E_B(-\cot(\delta/2)), \tag{6.3}$$

$\delta \in \mathcal{B}(\Delta_l)$. It follows from (6.3) that the measure $\Sigma_B(\cdot)$ is periodic: $\Sigma(\delta + 2\pi l) = \Sigma(\delta)$, $\delta \in \mathcal{B}(\Delta_0)$, $l \in \mathbb{Z}$. Having in mind this fact one obtains that for any $l \in \mathbb{Z}$ the operator $S_B E_{S_B}((2\pi l, 2\pi(l+1))$ is unitarily equivalent to the operator $S_B E_{S_B}((0,2\pi))$.

We note in conclusion that the latter fact is a special case of the following

**PROPOSITION 6.1** *Let $A$ be a simple closed symmetric operator in $\mathfrak{H}$ with $n_+(A) = n_-(A)$ and $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ a boundary triple for $A^*$. If the corresponding Weyl function $M(\cdot)$ is $\tau$-periodic, i.e. $M(z+\tau) = M(z)$, then for any $B = B^* \in \mathcal{C}(\mathcal{H})$ the extension $A_B = A^* \upharpoonright \ker(\Gamma_1 - B\Gamma_0)$ is $\tau$-periodic in the following sense: for every $l \in \mathbb{Z}$ the operator $A_B E_{A_B}(\Delta_0)$ is unitarily equivalent to the operator $(A_B - \tau l \cdot I)E_{A_B}(\Delta_l)$ where $\Delta_l := (l\tau, (l+1)\tau)$.*

**PROOF.** Let $\Sigma_B(\cdot) := \Sigma_{M_B}(\cdot)$ be the unbounded non-orthogonal spectral measure of the Nevanlinna function $M_B(z) := (B - M(z))^{-1}$. It is clear that $M_B(\cdot)$ is $\tau$-periodic, $M_B(z+\tau) = M_B(z)$ since so is $M(\cdot)$. It follows from the Stieltjes inversion formula (2.7) that $\Sigma_B(\cdot)$ is $\tau$-periodic, too, $\Sigma_B(\delta+\tau) = \Sigma_B(\delta)$, $\quad \delta \in \mathcal{B}(\mathbb{R})$.

Next we introduce the operator measures

$$\Sigma^0_{B,l}(\cdot) : \mathcal{B}((0,\tau)) \to [\mathcal{H}], \qquad \Sigma^0_{B,l} : \delta \to \Sigma^0_B(\delta + \tau l), \quad \delta \in \mathcal{B}((0,\tau)).$$

It follows from (2.5) that

$$\Sigma^0_{B,l}(\delta) = \int_{\delta+\tau l} (1+t^2)^{-1} d\Sigma_B(t) = \int_\delta (1+(s+\tau l)^2)^{-1} d\Sigma_B(s+\tau l) = \int_\delta (1+(t+\tau l)^2)^{-1} d\Sigma_B(t) \tag{6.4}$$

for $\delta \in \mathcal{B}((0,\tau))$. Notice that $\Sigma^0_{B,l}(\delta) \neq \Sigma^0_B(\delta))$, $\delta \in \mathcal{B}((0,\tau))$. Thus, the operator measures $\Sigma^0_{B,l}$ and $\Sigma^0_{B,0}$ are not unitarily equivalent for $l \neq 0$. Nevertheless due to (6.4) they are spectrally equivalent (see [26, Proposition 4.18]), that means that they are equivalent in the measure sense and their multiplicity functions are equal. By Proposition 4.9 of [26] the minimal orthogonal dilations of the operator measures $\Sigma^0_{B,l}(\cdot)$ and $\Sigma^0_{B,}(\cdot)$ are unitarily equivalent.

On the other hand, by Proposition 2.1 one gets the representation $\Sigma^0_{B,l}(\delta) = K^* E_{A_B}(\delta + \tau l)K$, $\delta \in \mathcal{B}((0,\tau))$. This identity means that the measure $E_{A_B}(\cdot + \tau l)E_{A_B}(\Delta_l)$ is the orthogonal dilation of $\Sigma^0_{B,l}(\cdot)$ for every $l \in \mathbb{Z}$. Since the operator $A$ is simple, it follows that $E_{A_B}(\cdot + \tau l)E_{A_B}(\Delta_l)$ is the minimal orthogonal dilation of $\Sigma^0_{B,l}(\cdot)$. Therefore the measures $E_{A_B}(\cdot)E_{A_B}(\Delta_0)$ and $E_{A_B}(\cdot + \tau l)E_{A_B}(\Delta_l)$ are unitarily equivalent. By the spectral theorem the operators $A_B E_{A_B}(\Delta_0)$ and $(A_B - \tau l \cdot I)E_{A_B}(\Delta_l)$ are unitarily equivalent. $\square$

Finally, we complement Proposition 6.1 by the following simple result.

**PROPOSITION 6.2** *Let $A$ be a symmetric operator in $\mathfrak{H}$ with two gaps $(\alpha,\beta)$ and $(\alpha+\tau,\beta+\tau)$. Suppose that there exists a boundary triple $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for $A^*$ such that $A_0(=A^*\restriction\ker(\Gamma_0))$ preserves the gaps and the corresponding Weyl function $M(\cdot)$ satisfies $M(x+\tau)=M(x),\quad x\in(\alpha,\beta)$. Then $A$ has infinitely many gaps $(\alpha+k\tau,\beta+k\tau),\ k\in\mathbb{Z}$ and both $M(\cdot)$ and $A_0$ are $\tau$-periodic.*

**PROOF.** Since $M(\cdot)$ is holomorphic in $\mathbb{C}_+\cup\mathbb{C}_-$ we have $M(z+\tau)=M(z), z\in\mathbb{C}_+\cup\mathbb{C}_-$. In particular we have $M(x+iy+k\tau)=M(x+iy)$ for $x\in(\alpha,\beta),\ y\in\mathbb{R}\setminus\{0\}$. It follows that there exist strong limits $s-\lim_{y\downarrow 0}M(x+k\tau\pm iy)$ for $x\in(\alpha,\beta)$ and

$$\begin{aligned}M(x+k\tau+i0) &:= s-\lim_{y\downarrow 0}M(x+k\tau+iy)=s-\lim_{y\downarrow 0}M(x+iy)=M(x)=M(x)^*,\\ M(x+k\tau-i0) &:= s-\lim_{y\downarrow 0}M(x+k\tau-iy)=s-\lim_{y\downarrow 0}M(x-iy)=M(x)^*=M(x)\end{aligned}$$

for $x\in(\alpha,\beta)$ and $k\in\mathbb{Z}$ which yields

$$M(x+k\tau-i0)=M(x+k\tau+i0)^*,\quad k\in\mathbb{Z}.$$

By the Stieltjes inversion formula (2.7) one has $\Sigma_M((\alpha+k\tau,\beta+k\tau))=0$ and the Weyl function $M(\cdot)$ admits a holomorphic continuation through $(\alpha+k\tau,\beta+k\tau)$ for every $k\in\mathbb{Z}$. Hence $(\alpha+k\tau,\beta+k\tau)$ is a gap for both $A$ and $A_0$. By Proposition 6.1 $A_0$ is $\tau$-periodic. □

## 6.2 Example

Let $\mathfrak{H}_1:=L^2(\mathbb{R}_+)$ and let $S_1$ be a closed symmetric operator in $\mathfrak{H}_1$ defined by

$$(S_1f)(x)=-\frac{d^2}{dx^2}f(x),\quad f\in\operatorname{dom}(S_1)=\overset{0}{W}{}_2^2(\mathbb{R}_+):=\{f\in W_2^2(\mathbb{R}_+):f(0)=f'(0)=0\}.\tag{6.5}$$

Obviously $S_1\geq 0$. Setting

$$\Gamma_0^1(\theta)f=f'(0)-\theta f(0),\quad \Gamma_1^1(\theta)f=-f(0),\quad f\in\operatorname{dom}(S_1^*)=W_2^2(\mathbb{R}_+),\quad \theta\in\mathbb{R},$$

we obtain the boundary triple $\Pi_1^\theta=\{\mathbb{C},\Gamma_0^1(\theta),\Gamma_1^1(\theta)\}$ for $S_1^*$. It is clear that the extension $S_1^\theta:=S_1^*\restriction\ker(\Gamma_0^1(\theta))$ is non-negative iff $\theta\geq 0$. The corresponding Weyl function is $m_\theta(\lambda)=(\theta-i\sqrt{\lambda})^{-1}$. It is regular in $\mathbb{C}\setminus\mathbb{R}_+$ if $\theta\geq 0$, where the branch of $\sqrt{\lambda}$ is fixed by the condition $\sqrt{1}=1$. The Weyl function $m_\theta(\cdot)$ admits the following integral representation

$$m_\theta(\lambda)=(\theta-i\sqrt{z})^{-1}=\frac{1}{\pi}\int_0^\infty\frac{\sqrt{t}}{(t-z)(t+\theta^2)}dt,\qquad \theta\geq 0,$$

and the corresponding spectral measure is given by $d\mu_\theta=\pi^{-1}t^{1/2}(t+\theta^2)^{-1}dt$. Clearly, $m_\theta(\cdot)$ is holomorphic within $(-\infty,0)$ such that $m_\theta((-\infty,0))=(0,\theta^{-1})$. The inverse function

$\varphi_\theta(\cdot) : (0, \theta^{-1}) \to (-\infty, 0)$ is given by $\varphi_\theta(\xi) = -(\xi^{-1} - \theta)^2$, $\xi \in (0, \theta^{-1})$. We set $\Delta := (-\infty, 0)$ and $\Delta' := m_\theta(\Delta) = (0, \theta^{-1})$. Notice that $\varphi'_\theta(\xi) = 2(\xi^{-1} - \theta)\xi^{-2}$.

Let $\mathfrak{H} = \oplus_{k=1}^\infty \mathfrak{H}_k$, $A := \oplus_{k=1}^\infty S_k$ and $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\} := \oplus_{k=1}^\infty \Pi_k^\theta$ where $\mathfrak{H}_k := \mathfrak{H}_1$, $S_k := S_1$ and $\Pi_k^\theta = \Pi_1^\theta$ for $k \in \mathbb{N}$. We set $A_0 := A^* \restriction \ker(\Gamma_0)$. The corresponding Weyl function $M(\cdot)$ is of scalar-type, i.e. $M(z) = m_\theta(z) I_\mathcal{H}$. Further, let $B = B^* \in \mathcal{C}(\mathcal{H})$. To the self-adjoint extension $A_B$ it corresponds the Weyl function $M_B(z) := (B - m_\theta(z) I_\mathcal{H})^{-1}$. Let $\Sigma_B(\cdot)$ be the unbounded non-orthogonal spectral measure of the Weyl function $M_B(\cdot)$. It follows from (3.37)

$$\Sigma_B(\delta) = 2(B_{\Delta'}^{-1} - \theta) B_{\Delta'}^{-2} E_B(m_\theta(\delta)), \quad B_{m_\theta(\Delta)} = B_{\Delta'} = B E_B(\Delta'), \quad \delta \in \mathcal{B}(\Delta). \tag{6.6}$$

Let $\delta = (x, 0)$, $x < 0$. Since $m_\theta\big((x,0)\big) = \big((\theta + \sqrt{|x|})^{-1}, \theta^{-1}\big)$ for $x < 0$ we get from (6.6)

$$\Sigma_B((x,0)) = 2\big(B_{\Delta'}^{-1} - \theta\big) B_{\Delta'}^{-2} E_B\big((\theta + \sqrt{|x|})^{-1}, \theta^{-1}\big)\big), \quad x < 0. \tag{6.7}$$

We note that $\Sigma_B(x) \in [\mathcal{H}]$ for every $x < 0$, while $B_{\Delta'}^{-1}$ may be unbounded.

Further, starting with (5.4) we can explicitly calculate the non-orthogonal spectral measure $\Sigma_B(\cdot)$ outside the gap $\Delta = (-\infty, 0)$. Setting $\Sigma_{B,h}(\cdot) := (\Sigma_B(\cdot)h, h)$ and $F_{B,h}(z) = \Im\mathrm{m}(M_B(z)h, h)$ we easily derive from (5.4) and the Fatou theorem that

$$\pi \frac{d\Sigma_{B,h}(x)}{dx} = F_{B,h}(x + i0) = \int_\mathbb{R} \frac{\sqrt{x}\, d(E_B(t)h, h)}{(t\theta - 1)^2 + xt^2}, \qquad x > 0, \quad h \in \mathcal{H}, \tag{6.8}$$

where $\Sigma_{B,h}(x) := \Sigma_{B,h}((0,x))$, $x > 0$. A straightforward computation shows that $\mathrm{supp}^+(\mu_\theta) = (0, \infty)$. By Proposition 5.8 we have $E_{A_B}^\tau((0,\infty)) = 0$, $\tau = s, pp, sc$. Hence $\sigma_\tau(A_B) \subseteq (-\infty, 0]$, $\tau = s, p, sc$. Since $\sigma_{ac}(A_0) = [0, \infty)$ we obtain from Theorem 5.2 that $\sigma_{ac}(A_B) \supseteq [0, \infty)$. Therefore, the orthogonal spectral measure $E_{A_B}(\cdot)$ of $A_B^\theta$ is absolutely continuous on $(0, \infty)$ which yields that $\Sigma_B(\cdot)$ is absolutely continuous on $(0, \infty)$, i.e $\Sigma_B^{ac}(\delta) = \Sigma_B(\delta)$ for $\delta \in \mathcal{B}((0,\infty))$. Hence

$$\Sigma_{B,h}((0,x)) = \frac{1}{\pi} \int_0^x ds \int_\mathbb{R} \frac{\sqrt{s}\, d(E_B(t)h, h)}{(t\theta - 1)^2 + st^2} = \int_\mathbb{R} \Phi_\theta(x, t)\, d(E_B(t)h, h), \quad x > 0, \quad h \in \mathcal{H},$$

where

$$\Phi_\theta(x, t) := \frac{2}{\pi t^2} \left( \sqrt{x} - \frac{|t\theta - 1|}{t} \arctan\left( \frac{t\sqrt{x}}{|t\theta - 1|} \right) \right), \quad x > 0, \tag{6.9}$$

which yields

$$\Sigma_B((0,x))h = \int_\mathbb{R} \Phi_\theta(x,t) dE_B(t)h, \quad x > 0, \quad h \in \mathcal{H}. \tag{6.10}$$

Thus, formulas (6.7) and (6.10) together give the explicit form for the unbounded non-orthogonal spectral measure $\Sigma_B(\cdot)$ of the extension $A_B$.

## 6.3 Example

Let $\mathfrak{H}_1 = L^2(\mathbb{R}_+)$ and let $S_1 \geq 0$ be as in (6.5). Consider a boundary triple $\Pi_1^\infty = \{\mathbb{C}, \Gamma_0^1(\infty), \Gamma_1^1(\infty)\}$ for $S_1^*$ where

$$\Gamma_0^1(\infty)f = f(0), \qquad \Gamma_1^1(\infty)f = -f'(0), \quad f \in \mathrm{dom}(S_1^*) = W_2^2(\mathbb{R}_+). \tag{6.11}$$

It is clear that the extension $S_1^\infty \geq 0$ defined by

$$S_1^\infty := S_1^* \restriction \ker(\Gamma_0^1(\infty)), \quad \ker(\Gamma_0^1(\infty)) = W_{2,0}^2(\mathbb{R}_+) = \{f \in W_2^2(\mathbb{R}_+) : f(0) = 0\}$$

is the Friedrichs extension of $S_1$. The Weyl function corresponding to the triple (6.11) is $m_\infty(\lambda) = i\sqrt{\lambda}$. It admits the integral representation

$$m_\infty(\lambda) = i\sqrt{\lambda} = -\frac{1}{\sqrt{2}} + \frac{1}{\pi}\int_0^\infty \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right) t^{1/2} dt.$$

The associated spectral measure is $d\mu_\infty(t) = \pi^{-1}t^{1/2}dt$. Clearly, $m_\infty(\lambda) = i\sqrt{\lambda}$ is holomorphic and monotone on $(-\infty, 0)$. Its inverse is $\varphi_\infty(\xi) = -\xi^2$. We set $\Delta = (-\infty, 0)$ and $\Delta' = m_\infty(\Delta) = (-\infty, 0)$. Notice that $\varphi_\infty(\xi) = -2\xi$.

As in the previous example let $\mathfrak{H} = \oplus_{k=1}^\infty \mathfrak{H}_k$, $A := \oplus_{k=1}^\infty S_k$ and $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\} := \oplus_{k=1}^\infty \Pi_k^\infty$ where $\mathfrak{H}_k := \mathfrak{H}_1$, $S_k := S_1$ and $\Pi\infty_k = \Pi_1^\infty$ for $k \in \mathbb{N}$. Notice that $\Pi$ forms a boundary triple for $A^*$. The corresponding Weyl function $M(\cdot)$ is of scalar-type, i.e $M(z) = m_\infty(z)I_\mathcal{H}$. The operator $A_0 := A^* \restriction \ker(\Gamma_0)$ is the Friedrichs extension of $A$, is absolutely continuous and $\sigma(A_0) = \sigma_{ac}(A_0) = [0, \infty)$. Let $B = B^* \in \mathcal{C}(\mathcal{H})$. As above to the self-adjoint extension $A_B$ it corresponds the Weyl function $M_B(z) = (B - m_\infty(z)I_\mathcal{H})^{-1}$. By $\Sigma_B(\cdot)$ we denote the unbounded non-orthogonal spectral measure of the Weyl function $M_B(\cdot)$. We obtain from (3.37) that

$$\Sigma_B((x,0)) = -2BE_B((-\infty,0))E_B\big((-|x|^{1/2},0)\big) = -2BE_B\big((-|x|^{1/2},0)\big), \quad x < 0. \tag{6.12}$$

Repeating the reasoning from above we find

$$\Sigma_{B,h}((0,x)) = \frac{1}{\pi}\int_0^x ds \int_\mathbb{R} \frac{\sqrt{s}}{t^2+s}\, d(E_B(t)h,h) = \int_\mathbb{R} \Phi_\infty(x,t)\, d(E_B(t)h,h), \quad x > 0, \tag{6.13}$$

$h \in \mathcal{H}$, where

$$\Phi_\infty(x,t) := 2\pi^{-1}\left(\sqrt{x} - |t| \arctan\left(\sqrt{x}/|t|\right)\right), \qquad x > 0. \tag{6.14}$$

Formula (6.13) leads to the following explicit integral representation for the non-orthogonal spectral measure $\Sigma_B(\cdot)$ outside the gap,

$$\Sigma_B(x)h = \int_\mathbb{R} \Phi_\infty(x,t) dE_B(t)h, \qquad x > 0, \qquad h \in \mathcal{H}, \tag{6.15}$$

with the kernel (6.14).

We note that by Proposition 5.8 one has $\sigma_\tau(A_B) \subseteq (-\infty, 0]$, $\tau \in \{s, pp, sc\}$. Since by Theorem 5.2 the relation $\sigma_{ac}(A_B) \supseteq [0,\infty)$ holds the spectral measure $E_{A_B}(\cdot)$ is absolutely continuous on $(0,\infty)$ independent from the spectral properties of $B$.

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