A PARAMETERIZATION METHOD FOR THE COMPUTATION OF INVARIANT TORI AND THEIR WHISKERS IN QUASI PERIODIC MAPS: RIGOROUS RESULTS.

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Abstract. We prove rigorous results on persistence of invariant tori and their invariant manifolds for quasiperiodically perturbed systems. The proofs are based on the parametrization method of [CFdlL03a, CFdlL03b].

The invariant manifolds results proved here include as particular cases of the usual (strong) stable and (strong) unstable manifolds, but also include other non-resonant manifolds.

The method lends itself to numerical implementations whose analysis and implementation is studied in [HdlL01].

1. Introduction

The goal of this paper is to present a proof of existence of invariant tori and their invariant manifolds in quasi-periodically perturbed systems. The proof is designed to be readily implementable and the implementation is described in the companion paper [HdlL01].

The systems we consider are of the form:

- (discrete time)

$$\begin{aligned} \bar{x} &= F(x,\theta) \ , \\ \bar{\theta} &= \theta + \omega \ , \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ and $\theta \in \mathbb{T}^d$ are variables, and $\omega \in \mathbb{R}^d$ is the *rotation vector*.

- (continuous time)

$$\begin{aligned} \dot{x} &= X(x,\theta) \ , \\ \dot{\theta} &= \omega \ , \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ and $\theta \in \mathbb{T}^d$ are variables, and $\omega \in \mathbb{R}^d$ is the *frequency vector*.

Systems of the form (1), (2) are called skew-products in the mathematical literature. In applications they appear when one forces a system with a quasi-periodic external perturbation.

An important particular case is when the external forcing is small. That is:

- (quasi-autonomous discrete time)

The system (1) has the form

$$F(x,\theta) = F_0(x) + F_1(x,\theta) \ , \tag{3}$$

where F_1 is small. For $F_1 \equiv 0$ the dynamics of x and θ are uncoupled and for a fixed point x_0 of F_0 the torus

$$\mathcal{K}_0 = \{x_0\} \times \mathbb{T}^d$$

is invariant for the whole system (1) given by (3).

- (quasi-autonomous continuous time)
  The system (2) has the form

$$X(x,\theta) = X_0(x) + X_1(x,\theta) \ , \tag{4}$$

  where X_1 is small. For $X_1 \equiv 0$ the dynamics of x and θ are uncoupled and for a fixed point x_0 of X_0 the torus

$$\mathcal{K}_0 = \{x_0\} \times \mathbb{T}^d$$

  is invariant for the whole system (2) given by (4) when $X_1 = 0$.
  Similarly, if $x_0(t)$ is a T periodic orbit for the autonomous system X_0, then

$$\mathcal{K}_0 = \{x_0(t) \mid t \in [0,T]\} \times \mathbb{T}^d$$

  is invariant for X when $X_1 = 0$.

As motivation for the results presented here, we note that in the perturbative case, it is natural to look for invariant manifolds close to $\mathcal{K}_0$ in the perturbed systems. If x_0 had invariant manifolds, we can consider whether there are corresponding objects for the quasiperiodically excited problem.

We will consider two types of problems for quasiperiodic systems (1), (2).

a) Existence of invariant tori (of dimension d) which are normally hyperbolic, and persistence of such tori under perturbations;
b) Existence of asymptotic manifolds attached to an invariant torus (also known as whiskers).

In the case that the fixed point x_0 of F_0 (or X_0) is hyperbolic and that the manifolds we are considering are the stable and unstable manifolds, persistence results follow from the general theory of normally hyperbolic manifolds [Fen72, Fen74, Fen77, HP69, HPS77].

In this paper we present an approach which differs from the classical approach of normal hyperbolicity. The approach is well suited for numerical implementations (which we will discuss in [HdlL01]). For the cases we consider, it also has some mathematical advantages over the general theory of normal hyperbolicity.

- For part a), taking advantage of the special structure of (1),(2) we will show in Section 3 that the invariant tori in a) are as smooth as the system (including analytic) and that they depend smoothly (including analytic) on parameters. Such results are false for more general systems (see e.g. [dlL01] for explicit examples).
- For part b) we obtain in Section 4 asymptotic manifolds (also called *whiskers*) associated to non-resonant parts of the linearization. This included as particular cases, the strong stable and the strong unstable manifolds of [Fen72, Fen74, Fen77, HPS77], but it also includes other cases associated to other invariant bundles of the linearization. In some cases, we can consider invariant manifolds that correspond to the slowest directions. These manifolds have interest in applications since they dominate the asymptotic behavior of the systems.

> Non-resonant invariant manifolds have been considered in [dlL97, ElB01, CFdlL03a, CFdlL03b, dlL03] for fixed points of diffeomorphisms in Banach spaces and, using a device introduced in [HP69], one can extend them to construct non-resonant manifolds for an invariant manifold. The results obtained here take advantage of the special structure of the system and conclude more regularity, including analyticity, which does not follow from the general theory.

A very important motivation for the work undertaken here is that the method of proof can be modified into a numerical algorithm. The method unifies the calculation of invariant tori and of invariant manifolds so that the algorithms for both are based on very similar functional equations, so that the implementations for both cases can share significant parts of the code. In the companion paper [HdlL01], we discuss implementation issues and results that have been obtained applying the method to some test cases that have appeared in the literature.

The results presented here are designed to serve as justification of numerical calculations. We have formulated them as *"a posteriori"* estimates. The existence of invariant objects is equivalent to the existence of a function satisfying a functional equation. We will show that given a function that solves it approximately and that satisfies some non-degeneracy conditions, then there is a true solution. Moreover the distance from the true solution to the approximate one can be bounded by the error in the solution of the functional equation. One can, for example, take as an approximate solution the result of a numerical computation. To verify the reliability of the computed solutions, one does not need to study the algorithm to compute them. It suffices to check that they satisfy the equation approximately and that they satisfy the non-degeneracy conditions.

The method we use has precedents and has been in common use in the BCN group [Sim90, Sim98, CJ00, GLMS01, GSLM01, GJSM01a, GJSM01b]. Other methods are based on graph transforms appear in [BOV97, KO98, DL95].

The proof we present is based on the parameterization method. In general, the parameterization method formulates a functional equation for a parameterization of the invariant manifold as well as for the dynamics on it. With respect to the general parameterization method, the cases considered here have the important advantage that we know that the motion in the angle variables variables is a rotation with the frequency of the external perturbation. This simplifies substantially the functional equations considered and eliminates the main source of difficulties in the analysis considered in [CFdlL03a, CFdlL03b], namely, the existence of unknown functions that appear as composition in the right.

As will be seen in more detail later, the functional equation that we need to deal with is much better behaved for the analysis than those that appear in the graph transform method (they give rise to a differentiable operator in

C^r spaces that can be studied with the regular implicit function theorem in Banach spaces). Since the construction will happen in C^r – even analytic – spaces, the error bounds will happen in very differentiable norms and we will be able to obtain numerical control over high derivatives of the manifolds.

The good analytic properties of the functional equations giving the invariant manifolds seem to translate in numerical stability of the algorithm. We postpone this discussion till [HdlL01]. Also in [HdlL01] we will present variants of the results here which, even if less precise in the number of derivatives used, etc. are designed to validate numerical computations. In particular, in [HdlL01] we will present *a posteriori* bounds that assert that given an approximate solution of the functional equation that satisfy non-degeneracy conditions, there is a true solution close by. In [HdlL01] we will give particular attention to the situation where the non-resonant manifolds have one normal direction. This is the situation that has also been considered in the literature in the stable case e.g. in [OF00].

2. Overview of the paper

In this section we discuss heuristically the main ideas of the paper and explain its organization. For the sake of simplicity, we will discuss the discrete case. Indeed, we will present detailed proofs of the main results only in the discrete case and in Section 5 we will present a short argument which shows that the results for flows can be deduced from the results for maps.

2.1. **Existence of invariant tori.**

The existence and persistence of invariant tori for (1) is developed in Section 3 (see Theorem 3.1).

It is based on the equation

$$F(K(\theta-\omega),\theta-\omega)-K(\theta)=0\ , \tag{5}$$

where $F:\mathbb{R}^n\times\mathbb{T}^d\to\mathbb{R}^n$ and $\omega\in\mathbb{R}^d$ are given and we are supposed to find $K:\mathbb{T}^d\to\mathbb{R}^n$.

We note that (5) is equivalent to

$$F(K(\theta),\theta)=K(\theta+\omega)\ , \tag{6}$$

which makes it clear that the set of points

$$\mathcal{K}=\{(K(\theta),\theta)\mid\theta\in\mathbb{T}^d\} \tag{7}$$

is invariant under the dynamical system (1). Indeed, K is a parameterization of a torus in which the dynamics is a rotation.

In the ergodic theory literature, situations such as (6) are described as saying that the rotation by ω is a subfactor of the skew-product (1), or, somewhat less precisely, that (1) is semiconjugate to a rotation by ω.

Remark 2.1. An interesting phenomenon, widely discussed in applications, is that of *subharmonic* tori. These are invariant tori so that their basic frequencies are a submultiple of those of the perturbation. These can be incorporated in (5) simply by considering the external perturbation as having an internal frequency which is a submultiple of the original one.

We just note that it suffices to consider, instead of $\theta \in \mathbb{T}^d$, $\theta \in (N\mathbb{T})^d$, where $N \in \mathbb{N}$. A more refined version is to change $\theta \in \mathbb{T}^d = \mathbb{T} \times \cdots \times \mathbb{T}$ by $\theta \in N_1\mathbb{T} \times \cdots \times N_d\mathbb{T}$, where $N_1, \ldots, N_d \in \mathbb{N}$ (notice that N is the minimum common multiple of $N_1, \ldots, N_d$).

These coverings transformations are also useful when considering whiskers which are non orientable.

Remark 2.2. One can wonder if it is possible to find some more invariant tori for (1) on which the motion is semiconjugate to a rotation. (In the ergodic theory literature, this is described saying that (1) *"admits the rotation as a factor"*.)

We see that the most general form of an invariant embedding of a torus $\mathbb{T}^d$ in $\mathbb{R}^n \times \mathbb{T}^d$ is given by $\tilde{K} : \mathbb{T}^d \to \mathbb{R}^n \times \mathbb{T}^d$, where $\tilde{K}(\varphi) = (\tilde{K}_x(\varphi), \tilde{K}_\theta(\varphi))$ satisfies

$$\begin{aligned} F(\tilde{K}_x(\varphi), \tilde{K}_\theta(\varphi)) &= \tilde{K}_x(\varphi + \omega) \ , \\ \tilde{K}_\theta(\varphi) + \omega &= \tilde{K}_\theta(\varphi + \omega) \ . \end{aligned} \tag{8}$$

We observe that, when the rotation ω is ergodic, the only measurable solutions of the second equation in (8) are $\tilde{K}_\theta(\varphi) = \varphi + a$, where $a \in \mathbb{R}^d$.

Noticing that $\tilde{K}(\varphi + a)$ is a solution of (8) if $\tilde{K}(\varphi)$ is, we obtain that, when ω is ergodic, the solutions of (5) are in one-to-one correspondence with the solutions of (8), i.e. the existence of rotations as subfactors of (1).

The study we will undertake is based in the study of the equation (5).

It is quite important to notice that provided $F \in C^{r+1}(\mathbb{R}^n \times \mathbb{T}^d, \mathbb{R}^n)$, the operator $\mathcal{T} : C^r(\mathbb{T}^d, \mathbb{R}^n) \to C^r(\mathbb{T}^d, \mathbb{R}^n)$ defined by

$$\mathcal{T}(K)(\theta) \equiv F(K(\theta - \omega), \theta - \omega) - K(\theta) \tag{9}$$

is a differentiable operator when $C^r(\mathbb{T}^d, \mathbb{R}^n)$ is given the C^r topology. See [dlLO99]. Hence, as we will see, we can study (5) using standard implicit function theorems in C^r spaces in the case that the invariant manifold is normally hyperbolic.

Note that a formal calculation – which is justified in [dlLO99] – gives

$$\mathrm{D}\mathcal{T}(K)\Delta(\theta) = \mathrm{D}_x F(K(\theta - \omega), \theta - \omega)\Delta(\theta - \omega) - \Delta(\theta) \ . \tag{10}$$

Hence, once we show that $\mathrm{D}\mathcal{T}$ is invertible, it is clear by the Implicit Function Theorem that the existence of approximate solutions implies existence of true solutions (see Section 3).

In particular, if we have a true solution for a certain F, for which $\mathrm{D}\mathcal{T}$ is invertible, it will be an approximate solution if we modify F slightly and, hence, we have a true solution for the modified F.

For an invariant torus, the invertibility of $\mathrm{D}\mathcal{T}$ is closely related to the fact that the manifold is normally hyperbolic. This is an extension of the theory of the characterization of Anosov diffeomorphisms in [Mat68], that has been studied in many places [Mn78, HPS77, Swa83, HdlL03c]. An elaboration of this theory for rotations on tori can be found in [HdlL03b].

In particular, it is very important for us that the invertibility of $\mathrm{D}\mathcal{T}$ comes from properties of the spectrum of the operator $\mathcal{M}_\omega$ defined by

$$\mathcal{M}_\omega \Delta(\theta) = \mathrm{D}_x F(K(\theta-\omega), \theta-\omega)\Delta(\theta-\omega) \ , \tag{11}$$

which is a shift multiplied by the monodromy matrix

$$M(\theta) = \mathrm{D}_x F(K(\theta), \theta) \ .$$

The operator $\mathcal{M}_\omega$ is referred to as the transfer operator associated to M_ω, which is a vector bundle map over a rotation ω. The spectral theory of transfer operators, not necessarily over rotations, has been studied in [Mat68, LS90, LS91, CL99, HdlL03a]. The spectral theory of transfer operators over rotations has been studied in [HdlL03b].

One of the main developments in [HdlL03b] is that the equivalence between normal hyperbolicity and invertibility of $\mathrm{D}\mathcal{T}$ in C^0 spaces that holds in general, for systems of the form (1) becomes an equivalence between normal hyperbolicity and invertibility of $\mathrm{D}\mathcal{T}$ in C^r spaces, $r \in \mathbb{N} \cup \{\infty, a\}$ (where a means analytic). This is not true in systems which are not of the form (1).

Remark 2.3. Equation (5) can also be used to find invariant tori in some cases where $\mathcal{K}$ is not normally hyperbolic.

Notably, in the case of Hamiltonian systems, equations very similar to (5) has been used to compute KAM tori or lower dimensional tori, including also their existence under quasiperiodic perturbations.

A version of KAM theory related to the theory developed here is found in [Rüs76, CC97, JdlLZ99, dlL01, GJdlLV00] (see also [JS92] for perturbative results in the context of lower dimensional tori).

For the systems (2), the analogous of (5) is

$$X(K(\theta), \theta) = \mathrm{D}K(\theta)\omega \ . \tag{12}$$

Due to the appearance of a derivative, (12) is apparently much worse behaved that (5). Nevertheless, by passing to integral forms equation (12) can be dealt with also using implicit function theorems. Also, it can be reduced to taking time-one map or Poincaré sections with transversals. We discuss this in Section 5.

2.2. **Existence of asymptotic invariant manifolds (whiskers).** The study of the invariant manifolds attached to an invariant torus $\mathcal{K}$ as in Section 2.1 is developed in Section 4.

We note that the perturbations in the dynamic variables propagate by the variational equations of (1) on the torus $\mathcal{K}$:

$$\begin{aligned} \Delta\bar{x} &= \mathrm{D}_x F(K(\theta),\theta)\Delta x \ , \\ \bar{\theta} &= \theta + \omega \ , \end{aligned} \tag{13}$$

where $\Delta x \in \mathbb{R}^n$ and $\theta \in \mathbb{T}^d$. To study the variational equations (13) it is natural to consider them as acting in $\mathbb{R}^n \times \mathcal{K} \simeq \mathbb{R}^n \times \mathbb{T}^d$. The points in $\mathcal{K}$ move according to the rotation ω.

In the general theory of normal hyperbolic manifolds one studies the action of the variational equations on $T_{\mathcal{K}}(\mathbb{R}^n \times \mathbb{T}^d)$, the tangent bundle of the (extended) phase space $\mathbb{R}^n \times \mathbb{T}^d$ restricted to the invariant object $\mathcal{K}$. In this theory one uses a splitting

$$T_{\mathcal{K}}(\mathbb{R}^n \times \mathbb{T}^d) = T\mathcal{K} \oplus N\mathcal{K} \ , \tag{14}$$

where $T\mathcal{K}$ is the tangent bundle of $\mathcal{K}$ and $N\mathcal{K}$ is a bundle transversal to $T\mathcal{K}$. In the general theory of normal hyperbolic manifolds, the splitting (14) is generally not invariant under the action of the variational equations. In fact, $T\mathcal{K}$ is invariant while $N\mathcal{K}$ generally is not (the variational equations have a block triangular structure). However, in our case we can take $N\mathcal{K}$ to correspond to the directions along the $\mathbb{R}^n$ factor of the (extended) phase space. With this choice, the splitting (14) is invariant and the variational equations are block diagonal. Since the block corresponding to $T\mathcal{K}$ is just the identity, to study dynamical properties it suffices to study the cocycle corresponding to (13). In the language of global differential geometry, the variational equations can be considered as a linear vector bundle map on a bundle $N\mathcal{K}$ whose fibbers are $\mathbb{R}^n$ and whose base points are the points in $\mathcal{K} \simeq \mathbb{T}^d$.

We show in Theorem 4.1 that given a subbundle E_1 of $N\mathcal{K}$ invariant under the variational equations and such that the spectrum of the linearization restricted to it satisfies certain non-resonance conditions, then, there is an invariant manifold tangent to this subbundle which is invariant under the map.

This result includes, as a particular case, the usual strong stable and strong unstable invariant manifold theorems, but it also includes some more exotic manifolds. In particular, sometimes we can find invariant manifolds corresponding to the less contracting part of the spectrum. These are the "slow" manifolds, which dominate the approach to the manifold.

Our study of whiskers is based on the study of the equation

$$F(W(\eta,\theta),\theta) = W(\Lambda(\eta,\theta),\theta+\omega) \ , \tag{15}$$

where, as before, $F : \mathbb{R}^n \times \mathbb{T}^d \to \mathbb{R}^n$ and $\omega \in \mathbb{R}^d$ are given and we are supposed to find $W : \mathbb{R}^{n_1} \times \mathbb{T}^d \to \mathbb{R}^n$ and $\Lambda : \mathbb{R}^{n_1} \times \mathbb{T}^d \to \mathbb{R}^{n_1}$, where $n_1 \le n$. Moreover, $W(0,\theta) = K(\theta)$ is the parameterization obtained in (1) of the invariant torus and $\Lambda(0,\theta) = 0$. The equation (15) implies that

$$\mathcal{W} = \{(W(\eta,\theta),\theta) \mid \theta \in \mathbb{T}^d \ , \eta \in \mathbb{R}^{n_1}\} \ , \tag{16}$$

is invariant under F, and Λ is the induced dynamics on the manifold. The conditions

$$W(0,\theta) = K(\theta) \ ,$$
$$\Lambda(0,\theta) = 0 \ ,$$

express that $\mathcal{W}$ extends the invariant manifold found in (1). Notice also that

$$\mathrm{D}_x F(K(\theta),\theta) W^1(\theta) = W^1(\theta+\omega)\Lambda^1(\theta) \ ,$$

where $W^1(\theta) = \mathrm{D}_\eta W(0,\theta)$, $\Lambda^1(\theta) = \mathrm{D}_\eta \Lambda(0,\theta)$. This says that the bundle spanned by the columns of W^1 is invariant under the variational equations (13). Notice that in this formulation such a bundle is trivial. This is the case which often appears in practice but, even if the bundles could be non-trivial, using a device in [HP69] one can augment the bundles so that they become trivial.

Notice that W and Λ are not uniquely defined, Nevertheless, as we will see, it is possible to chose normalizations that make them unique. We will try to find simple expressions for Λ, in particular, polynomial expressions. Notice that the case $n_1 = n$ amounts to find normal forms for the dynamics around the torus (these are the nonstationary normal forms).

When

$$|\mathrm{D}_\eta \Lambda(0,\theta)| \le \lambda < 1$$

or, more generally, that for some $n \in \mathbb{N}$,

$$|\mathrm{D}_\eta \Lambda(0,\theta+n\omega) \cdots \mathrm{D}_\eta \Lambda(0,\theta)| \le \lambda < 1$$

we obtain that the points of $\mathcal{W}$ close to $\mathcal{K}$ converge to $\mathcal{K}$ upon iteration of the map. In other words, when $\Lambda(\cdot,\theta)$ is a contraction for all θ, all η sufficiently small, the manifolds that we obtain are submanifolds of the usual stable manifold.

The solution of the equation (15) is more complicated than that of (5). In general, it involves non-resonance conditions on the spectrum of $\mathrm{D}\mathcal{T}(K)$. Such non-resonance conditions are automatic when one studies strong stable manifolds or the classical stable manifolds. Hence, we obtain the classical theorems as particular cases, but we can obtain invariant manifolds associated to other non-resonant subbundles.

We also note that (15) gives the invariant manifold $\mathcal{W}$ in a parametric form. In some cases – notably the rotating Henon map, which has been extensively used in the literature – this parameterization is global (it is an entire function). From the numerical point of view, the fact that the parameterization is global has the advantage that the algorithm presented here – in principle – does not require the step of globalization. The parameterization (15) does not have any difficulty following the twists and turns typical of invariant manifolds in its domain of definition. Of course, these turns interfere severely with the possibility of studying the manifolds as graphs. In [HdlL01], we present a more detailed discussion of numerical issues. Even if the numerical parameterization can follow several folds of the manifold, it sometimes reduces the error to evaluate it in a small domain – where the error is very small – and then, propagate it.

2.3. **One dimensional asymptotic manifolds.** In this section, we will discuss the simplest case of the method, which happens when the invariant subbundle is one dimensional ($n_1 = 1$) and trivial (we can get this using covering transformations), ω is Diophantine, and F is smooth enough. This case has been considered in the literature (e.g [OF00]), and it is also given special attention in [HdlL01]. Given the special interest of the case, it is worth presenting it in detail. The presentation includes all the essential ideas of the problem but avoids some of the technical complications that will be incorporated later. The discussion will be informal and we will not keep track of what are the differentiability assumptions, etc.

Let us consider the invariance equation

$$F(W(\eta, \theta), \theta) = W(\lambda\eta, \theta + \omega) \ , \tag{17}$$

where $\lambda \in \mathbb{R}$ is a number to be determined. Notice that we are fixing the dynamics on the manifold as the simplest one: it is linear in the normal direction, and the expansion rate is constant.

Remark 2.4. The fact that the expansion rate is constant is not too restrictive when the subbundle is one dimensional and the rotation ω is Diophantine. In such a case, one can make the expansion constant by multiplying by a factor chosen conveniently. This factor satisfies a cohomological equation involving small divisors, so we loose some regularity in this formulation.

We emphasize, however, that the transformation to constant rates is only done in this pedagogical section to simplify the notation so that the ideas come across better. Later, we will also present results when λ is a function of θ. This leads to optimal results on regularity of the invariant manifolds, and does not requires arithmetical assumptions on ω.

We write

$$W(\eta, \theta) = W^{\leq}(\eta, \theta) + W^{>}(\eta, \theta) \ ,$$

where

$$W^{\leq}(\eta, \theta) = \sum_{i=0}^{L} W^i(\theta)\eta^i$$

is a polynomial in the variable η whose degree L will be made explicit later, and the high order part of the function W satisfies

$$\frac{\partial^i W^{>}}{\partial \eta^i}(0, \theta) = 0 \text{ for } i = 0, \dots, L.$$

We seek the coefficients $W^0, \dots, W^L$ of $W^{\leq}$ and the remainder $W^{>}$ from the equation (17), which leads to a hierarchy

$$
\begin{aligned}
&F(W^0(\theta),\theta) = W^0(\theta+\omega)\\
&\mathrm{D}_x F(W^0(\theta),\theta) W^1(\theta) = \lambda W^1(\theta+\omega)\\
&\mathrm{D}_x F(W^0(\theta),\theta) W^2(\theta) + P^2(\theta) = \lambda^2 W^2(\theta+\omega)\\
&\vdots\\
&\mathrm{D}_x F(W^0(\theta),\theta) W^L(\theta) + P^L(\theta) = \lambda^L W^L(\theta+\omega)\ ,
\end{aligned}
\tag{18}
$$

where P^i stands for a polynomial expression in $W^1, \dots, W^{i-1}$ for $i = 2, \dots L$ whose coefficients are derivatives of F of order up to i evaluated at $(W^0(\theta), \theta)$. (For example, $P^2 = \frac{1}{2}\mathrm{D}_x^2 F(W^0(\theta),\theta)(W^1(\theta))^{\otimes 2}$.)

The high order part $W^{>}$ satisfies

$$
\mathrm{D}_x F(W^0(\theta),\theta) W^{>}(\eta,\theta) + P^{>}(\eta,\theta) = W^{>}(\lambda\eta, \theta+\omega)\ , \tag{19}
$$

where $P^{>}$ contains terms which vanish to order higher than L.

The hierarchy of equations (18) can be solved by recursion in the degree of the polynomials matched, provided that some non-resonance conditions, that we will discuss now are satisfied.

The equation for W^0 is an equation of the type we have studied in the theory of existence of invariant tori. Hence, we take

$$
W^0(\theta) = K(\theta)
$$

to be the solution of the equation for the invariant torus.

The equation for W^1 states that W^1 is an eigenfunction for the transfer operator $\mathcal{M}_\omega$ defined in (11), whose eigenvalue is λ. Note that the equation $\mathcal{M}_\omega W^1 = \lambda W^1$ has a very clear geometric interpretation, namely that the bundle spanned by W^1 is invariant under M, and also that the expansion rate in appropriate coordinates is constant. Hence, the geometric interpretation of the second equation in (18) is that the bundle spanned by W^1 is invariant for the linearization of F. As it is apparent from the theory developed in [Joh80, JS81, HdlL03b], in the case that ω is Diophantine and the bundle is one dimensional (the case considered in (17)) *the geometric and the analytical characterization are equivalent.*

The other equations of the hierarchy (18) have the form

$$
\mathcal{M}_\omega W^i(\theta) - \lambda^i W^i(\theta) = -P^i(\theta)\ ,
$$

where $i = 2, \dots L$. Hence, under the assumption that λ^i is not in the spectrum of $\mathcal{M}_\omega$ for $i = 2, \dots L$, we can recursively solve the equations of the hierarchy.

We also note that the equation for $W^{>}$ can be solved automatically if L is large enough (depending on the spectrum of $\mathcal{M}_\omega$, see the condition (20) below). Indeed, since $W^{>}(\eta,\theta)$ vanishes to high order in η, we have that the norm defined by

$$
\|W^{>}\|_{C^0} = \|W^{>}\|_{C^0([-\rho,\rho]\times\mathbb{T}^d,\mathbb{R}^n)} = \sup_{|\eta|\leq\rho,\theta\in\mathbb{T}^d} |W^{>}(\eta,\theta)|
$$

satisfies, for small enough $\rho > 0$,

$$\|W^{>}(\lambda\eta, \theta+\omega)\|_{C^0} \leq \lambda^{L+1}\|W^{>}(\eta,\theta)\|_{C^0} \ .$$

Therefore, if

$$\|M\|_{C^0}\lambda^{L+1} < 1 \ , \tag{20}$$

we see that the equation of the remainder (19) is solvable in C^0. As it turns out, one can develop a theory of solutions in higher regularity that goes along the same lines.

In summary, provided that λ is an eigenvalue of $\mathcal{M}_\omega$, $|\lambda| < 1$, and such that λ^i is not in the spectrum of $\mathcal{M}_\omega$, for $i = 2, \dots L$, one can obtain that there is a solution for the hierarchy (18).

In the case that the functions are analytic, it can be shown (e.g. using the majorant method) that the series for W converge and, hence, there is an analytical solution for (17).

In the finitely differentiable case, we will show that the equations (18) can be solved to a finite order, and there is a solution for (17) which is obtained applying a fixed point argument starting with the solutions of (18) to a suitable finite order.

The existence of a solution to (17) shows that there is a parameterization of the asymptotic manifold associated to λ in such a way the restriction to the manifold is an exponential contraction toward the invariant torus.

As we will see, if λ^i is in the spectrum of $\mathcal{M}_\omega$ for some $i = 2, \dots L$, we can modify slightly the procedure indicated by constructing a dynamics on the manifold that is polynomial in η.

The hierarchy of equations (18) can be numerically solved to a finite order. The fixed point argument alluded above shows that, given one such numerical solution, there is a true solution nearby. Hence, the fixed point argument ca be though of as “a posteriori” validating estimates in the sense of numerical analysis. The numerical analysis issues are discussed in [HdlL01].

2.4. **Notation.**

2.4.1. *Differential geometry.* The appropriate language to express the results of this paper is that of Differential Geometry, in particular vector bundles, Finsler metrics, bundle maps, etc. Related to these geometric objects, it is natural to define spaces of functions adapted to them and study the functional equations in terms of operators acting on these spaces. See [MS89] for more details on bundles in a context very similar to the one considered here.

For instance, the quasiperiodic skew product

$$\begin{aligned} \bar{x} &= F(x,\theta) \ , \\ \bar{\theta} &= \theta + \omega \ , \end{aligned}$$

is a bundle map in $E = \mathbb{R}^n \times \mathbb{T}^d$, a trivial bundle over $\mathcal{P} = \mathbb{T}^d$. An invariant torus $x = K(\theta)$ is given by a section $K : \mathbb{T}^d \to \mathbb{R}^n$ on such a bundle that satisfies the invariance equation $F(K(\theta),\theta) = K(\theta+\omega)$. A whisker $x = W(\eta,\theta)$ of the torus is given by a bundle map over the identity $W : E_1 \to E = \mathbb{R}^n \times \mathbb{T}^d$ where E_1 is a linear subbundle of E, and its dynamics is a bundle map $\Lambda : E_1 \to E_1$ over the rotation ω in E_1: $F(W(\eta,\theta),\theta) = W(\Lambda(\eta,\theta),\theta+\omega)$. The whiskers we will obtain can be topologically non-trivial. Notice also that we will obtain polynomial approximations of the whiskers, and the dynamics on the whisker will be polynomial.

The more general set up would be the following. Given two C^r vector bundles E_1 and E_2 over the same base manifold $\mathcal{P}$, we consider bundle maps $F_f : B_1 \to E_2$, where B_1 is a tubular neighborhood of E_1. The subindex f denotes the motion on the base manifold, that is $f : \mathcal{P} \to \mathcal{P}$, the map such that $\Pi_2 \circ F = f \circ \Pi_1$. We denote the elements of E_1 as $x_\theta = (x,\theta)$ and of E_2 as $y_\theta = (y,\theta)$, and pictorially, we write

$$\begin{aligned} \bar{y} &= F(x,\theta) \ , \\ \bar{\theta} &= f(\theta) \ . \end{aligned}$$

Notice that the geometric objects are C^r in the horizontal variables and C^∞ (in fact, analytic) in the vertical variables (using trivialization charts). Hence, the regularity of F in θ is at most C^r.

Along this paper, the motion on the base manifold $\mathcal{P} = \mathbb{T}^d$ is a rotation $f(\theta) = \theta + \omega$. In particular, the system (1) is a bundle map over the trivial bundle $E = \mathbb{R}^n \times \mathbb{T}^d$. This is not a restriction, since one could uses the trick in [HP69] to trivialize the bundle. On the other hand, we will not use such a trick in order to trivialize the subbundles of E.

2.4.2. *Spaces of differentiable functions with anisotropic differentiability.* In order to obtain sharp results on regularity of the invariant manifolds, it will be very important for us to distinguish the regularities of the functions with respect to the horizontal variables (θ) and the vertical variables (x, η), because the angle variables parameterizing the torus and the real variables used to parameterize the stable directions enter very differently in the functional equations that we need to solve. Hence, when one is interested in optimal regularity it is natural to introduce spaces in which the regularity along these two variables is not the same.

The following spaces are an adaptation of the definitions used in [CFdlL03a, CFdlL03b]. They are designed to make easy induction arguments for the functional equations.

Definition 2.5. *We consider subsets $\Sigma \subset \mathbb{N}^2$ is such that $(i,j) \in \Sigma$ and $\tilde{\imath} \le i, \tilde{\jmath} \le j$ implies $(\tilde{\imath},\tilde{\jmath}) \in \Sigma$.*

We denote $C^\Sigma = C^\Sigma_\cdot(B_1,E_2)$ the set of maps F for which $D^i_\theta D^j_x F$ exists, is continuous and bounded for every $(i,j) \in \Sigma$. We consider C^Σ endowed with the norm

$$||F||_{C^\Sigma(B_1,E_2)} = \sup_{(i,j)\in\Sigma,(x,\theta)\in B_1} |D^i_\theta D^j_x F(x,\theta)| \ ,$$

which makes it a Banach space.

We denote by $C^{\Sigma}_{\cdot} = C^{\Sigma}_{\cdot}(B_1, E_2)$ the Banach subspace of bundle maps with such regularities. If we fix the base dynamics $f : \mathcal{P} \to \mathcal{P}$, we consider the Banach subspace $C^{\Sigma}_{f}(B_1, E_2)$ of bundle maps over f.

Special cases are:

- *F is $C^{r,s}$ when $D^i_\theta D^j_x F(x, \theta)$ exists, is continuous and bounded for $0 \le i \le r$, $0 \le j \le s$;*
- *F is jointly C^r when it is a C^r mapping with bounded derivatives up to order r. This is equivalent to the existence, continuity and boundedness of $D^i_\theta D^j_x F(x, \theta)$ for $0 \le i + j \le r$.*
- *F is $C^{\Sigma_{r,s}}$ when $D^i_\theta D^j_x F(x, \theta)$ exists, is continuous and bounded for $(i, j) \in \Sigma_{r,s} = \{(i, j) \in \mathbb{N}^2 \mid \ i \le r \ , i + j \le r + s\}$.*

We will see that the classes $C^{\Sigma_{r,s}}$ are well adapted to the study of optimal regularity with respect to the horizontal and vertical directions. The results of this paper also work if one changes $C^{\Sigma_{r,s}}$ by C^{r+s}, because $C^{r+s} \subset C^{\Sigma_{r,s}} \subset C^{r,s}$.

Definition 2.6. *We say that the derivative $D^k_\theta D^l_x$ is of lower order than the derivative $D^i_\theta D^j_x$ if $(k, l) \in \Sigma_{i,j}$.*

2.4.3. *Spaces of analytic functions.* Some of the results obtained in the present paper have simpler proofs when working in the analytic category. For the sake of simplicity, we will consider functions that are analytic in both horizontal and vertical variables.

A real-analytic vector bundle is a smooth bundle that has a vector atlas for which the transition maps between vector charts are real-analytic (and linear in the vertical components).

In the following, E_1, E_2 are two real-analytic vector bundles.

Definition 2.7. *A real-analytic bundle map $F : B_1 \to E_2$ defined in a tubular neighborhood B_1 of E_1 is real-analytic if its local representations (in real-analytic charts) are real-analytic.*

In order to introduce topology in the space of real-analytic bundle maps, we fix finite vector atlases in both E_1 and E_2. Then, we define the complex neighborhood of B_1 of size ξ as

$$B^\xi_1 = \{(x + \mathbf{i}y, \theta + \mathbf{i}\varphi) \mid (x, \theta) \in B_1 \ , |y| < \xi, |\varphi| < \xi\} \ ,$$

where the expressions above are understood once one has taken local charts. Similarly, we define E^η_2.

Once we have fixed a complex neighborhood E^η_2, we can define a scale of spaces of real-analytic bundle maps from B_1 to E_2.

Definition 2.8. *A real-analytic bundle map $F : B_1 \to E_2$ is $C^{a,\xi}$ if it has an holomorphic extension $\hat{F} : B^\xi_1 \to E^\eta_2$, and $\hat{F}$ is continuous in the closure. We equip $C^{a,\xi}$ with the norm*

$$\|F\|_{C^{a,\xi}(B_1,E_2)} = \sup_{(\hat{x},\hat{\theta}) \in B^\xi_1} |\hat{F}(\hat{x}, \hat{\theta})| = \|\hat{F}\|_{C^0(B^\xi_1, E^\eta_2)} \ ,$$

where the norms in E_1 and E_2 have also been complexified.

It is standard that, with the indicated norm, $C^{a,\xi}$ is a Banach space.

2.4.4. *Polynomial bundle maps.* We introduce now the polynomial bundle maps, which are the objects that we will use to represent the dynamics on an invariant manifold.

Definition 2.9. *A bundle map $P_f : E_1 \to E_2$ is said to be a C^r polynomial bundle map of degree k if f is C^r and P is of the form*

$$P(x, \theta) = P^0(\theta) + P^1(\theta)x + P^2(\theta)x^{\otimes 2} + \cdots + P^k(\theta)x^{\otimes k} \ ,$$

where for all $i = 0, \ldots, k$, P^i is a i-multilineal bundle map over f from $E_1 \times \overset{i}{\ldots} \times E_1$ to E_2, and of class C^r. Each P^i can be chosen symmetric. We will say that $P^i_\theta(x) = P^i(x, \theta) = P^i(\theta)x^{\otimes i}$ is homogeneous of degree i. Notice that $P \in C^{r,\infty}_f$.

Obviously, for $r = a$ this is a real-analytic polynomial bundle map, and $P \in C^a$.

We observe that a C^r i-multilineal bundle map over the identity is identified with a C^r section on the i-multilineal symmetric bundle $L^i_s(E_1; E_2)$. In particular, a polynomial map over the identity is equivalent to a section of the bundle $\bigoplus_{i=0}^k L^i_s(E_1; E_2)$.

The definition and comments in the analytic case corresponds to $r = a$.

2.4.5. *Transfer operators, cocycles.* The variational equations (13) over an invariant torus $\mathcal{K}$ for (1) define a linear skew-product M given by

$$\begin{aligned} \bar{v} &= M(\theta)v \ , \\ \bar{\theta} &= \theta + \omega \ , \end{aligned}$$

where $v \in \mathbb{R}^n$ and $\theta \in \mathbb{T}^d$, and $M(\theta) = \mathrm{D}_x F(K(\theta), \theta)$.

An useful notation is that of cocycles. We will write

$$\begin{aligned} M^0_\theta &= M(\theta, 0) = \mathrm{Id} \ , \\ M^m_\theta &= M(\theta, m) = M(\theta + (m-1)\omega) \cdot \cdots \cdot M(\theta) \ \text{ if } m > 0 \ , \\ M^m_\theta &= M(\theta, m) = M(\theta + m\omega)^{-1} \cdot \cdots \cdot M(\theta - \omega)^{-1} \ \text{ if } m < 0 \ . \end{aligned} \tag{21}$$

We will also write $M(v, \theta, m) = M^m_\theta v = M(\theta, m)v$.

We associate to the linear skew-product M a *transfer operator* over the rotation ω. This transfer operator is the map $\mathcal{M}_\omega : \Gamma \to \Gamma$ on the space of sections $\Gamma = \{v : \mathbb{T}^d \to \mathbb{R}^n\}$ defined by

$$(\mathcal{M}_\omega v)(\theta) = M(\theta - \omega)v(\theta - \omega) \ . \tag{22}$$

Clearly, the linearization of the map near an invariant torus, (11) is a particular case of (11). Other examples that will play a role in our discussion are the tensor products of the linearization, which will play a role in the study of higher derivatives.

Remark 2.10. Notice that we identify a section in the trivial bundle $E = \mathbb{R}^n \times \mathbb{T}^d$ over $\mathbb{T}^d$ with a function from $\mathbb{T}^d$ to $\mathbb{R}^n$.

Remark 2.11. The previous definitions and notations (22) and (21) can be introduced for general bundle maps $F_f : E \to E$, and in such a case, we will made explicit the non linear character of the objects produced, saying that $\mathcal{F}_F$ is a non-linear transfer operators, or the family $F(x, \theta, m) = F_\theta^m(x)$ is a non-linear cocycle.

The transfer operator of an analytic M can be considered as acting on spaces of sections with different regularities. For example, it can be considered as acting on bounded sections (Γ_b), continuous sections (Γ_{C^0}), C^r sections (Γ_{C^r}), analytic sections (Γ_{C^a}), etc. In particular, the spectral theory of these operators (in fact, the complexification of these operators, acting on complex sections in $E_{\mathbb{C}} = E \oplus \mathbf{i}E \simeq \mathbb{C}^d \times \mathbb{T}^d$) will be very important for us.

In general, it can happen that the spectrum of the operator depends on the spaces it is considered as acting on. Nevertheless, for the case that the motion on the base is a rotation, the spectrum does not depend on the space.

The following result is established in [HdlL03b]. Note that it depends crucially on the fact that the motion on the base is a rotation. It could be false if the motion in the base is a general map.

Theorem 2.12. *Let $M_\omega : E \to E$ be a C^r $r \in \mathbb{N}$ vector bundle automorphism over a rotation. Then:*

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_b(E)) = \mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{C^r}(E)) \ . \tag{23}$$

For the proof of Theorem 2.12 we refer to [HdlL03b].

The Theorem 2.12 does not apply to the case that the spaces are analytic. The best result for analytic spaces that can be found in [HdlL03b] is:

Theorem 2.13. *Let $M_\omega : E \to E$ be a C^{a,ξ^*}, vector bundle automorphism over a rotation. Then for all $\xi < \xi^*$ we have*

$$\mathrm{dist}(\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_b(E)), \mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{C^{a,\xi}}(E))) \leq O(\xi) \tag{24}$$

where by dist *we mean the Hausdorff distance among sets.*

3. Invariant tori

3.1. **Existence and persistence of invariant tori.**

In this section, we formulate the result on existence and persistence of invariant tori. The main result is the following.

Theorem 3.1. *Let $U \subset \mathbb{R}^n$ be an open set. Let $F : U \times \mathbb{T}^d \subset \mathbb{R}^n \times \mathbb{T}^d \to \mathbb{R}^n$ be a map of class $C^{\Sigma_{r,1}}$, with $r \geq 0$ – including $C^{\Sigma_{r,1}} = C^a$ in the analytic case $r = a$ –, such that for all $\theta \in \mathbb{T}^d$ the map $F(\cdot, \theta) : U \to \mathbb{R}^n$ is a local diffeomorphism. Let $\omega \in \mathbb{R}^d$ be a rotation.*

We consider the skew product

$$\begin{aligned} \bar{x} &= F(x, \theta) \ , \\ \bar{\theta} &= \theta + \omega \ , \end{aligned}$$

that is a vector bundle map on the bundle $E = \mathbb{R}^n \times \mathbb{T}^d$.

Let $K : \mathbb{T}^d \to U \subset \mathbb{R}^n$ be a C^r map such that:

- *$\mathcal{K}$ is an approximate invariant torus, that is*

$$\|F(K(\theta), \theta) - K(\theta + \omega)\|_{C^r} < \varepsilon \ . \tag{25}$$

- *For the C^r matrix valued map $M : \mathbb{T}^d \to \mathrm{GL}_n(\mathbb{R})$, defined by*

$$M(\theta) = \mathrm{D}_x F(K(\theta), \theta) \ ,$$

  the corresponding transfer operator $\mathcal{M}_\omega$ satisfies the spectral gap condition

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_b(E)) \cap \{z \in \mathbb{C} \mid |z| = 1\} = \emptyset \ . \tag{26}$$

Then:

- *If ε is small enough, there exists a C^r map $K_F : \mathbb{T}^d \to U \subset \mathbb{R}^n$ such that*

$$F(K_F(\theta), \theta) = K_F(\theta + \omega) \ , \tag{27}$$

  and $\|K_F - K\|_{C^r} = \mathrm{O}(\varepsilon)$.
- *The solution K_F above is the only C^0 solution of (27) in a C^0 neighborhood of K.*
- *The torus K_F is normally hyperbolic.*

Moreover, the map $F \to K_F$ is C^1 when F is given the $C^{\Sigma_{r,1}}$ topology and K_F the C^r topology.

Remark 3.2. Notice that the spectral gap assumption (26) for an invariant torus is equivalent to normal hyperbolicity ([Mn78, HPS77, Swa83]).

Remark 3.3. Notice that the spectral gap assumption (26) is formulated in the space of bounded sections – not necessarily continuous –. Using the results in [HdlL03b] about the spectrum of transfer operators over rotations, we obtain that the spectrum over bounded sections is the same as that over C^r sections. This is what allows to obtain C^r regularity in the conclusions.

Of course, these results depend very heavily on the fact that the motion on the torus is a rotation.

Remark 3.4. The linear operator corresponding to $\hat{K} = K_F$ is

$$\hat{M}(\theta) = \mathrm{D}_x F(\hat{K}(\theta), \theta) \ ,$$

and by the mean value theorem, if F is $C^{\Sigma_{r,2}}$, then

$$\|M - \hat{M}\|_{C^0} \le \|F\|_{C^{\Sigma_{0,2}}} \|K - \hat{K}\|_{C^0} \ .$$

Hence, for the transfer operators $\mathcal{M}_\omega, \hat{\mathcal{M}}_\omega$ corresponding to $M, \hat{M}$, respectively, we have

$$\|\mathcal{M}_\omega - \hat{\mathcal{M}}_\omega\|_{C^0} \le \|F\|_{C^{\Sigma_{0,2}}} C\varepsilon \ ,$$

Under the hypothesis of Theorem 3.1, F is $C^{\Sigma_{r,1}}$, one has

$$\|M - \hat{M}\|_{C^0} \le \eta \left(\|K - \hat{K}\|_{C^0} \right) \ ,$$

where η is the modulus of continuity of $\mathrm{D}_x F$, and one obtains the estimate

$$\|\mathcal{M}_\omega - \hat{\mathcal{M}}_\omega\|_{C^0} \leq \eta(C\varepsilon) \ .$$

Therefore, by the usual properties of the stability of the spectrum [Kat76], we can ensure that $\hat{\mathcal{M}}_\omega$ is also hyperbolic. Indeed, if we can know that

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_b) \cap \{z \in \mathbb{C} \mid \lambda_- \leq |z| \leq \lambda_+\} = \emptyset \ ,$$

with $\lambda_- < 1 < \lambda_+$, then we can ensure that

$$\mathrm{Spec}(\hat{\mathcal{M}}_\omega, \Gamma_b) \cap \{z \in \mathbb{C} \mid \lambda_- + C\varepsilon \leq |z| \leq \lambda_+ - C\varepsilon\} = \emptyset \ ,$$

and we can obtain bounds for the norms of the spectral projections.

Hence, the hyperbolicity properties of the exact torus are similar to those of the approximate one and their difference is bounded by the error in the approximation.

Remark 3.5. The formulation we have presented of Theorem 3.1, implies the usual formulation of the Theorem on persistence of normally hyperbolic invariant tori. If K_F is a parameterization of a torus invariant under a map F, it will be smooth and it will satisfy (25) for all the maps G close to F. Furthermore, if the torus is normally hyperbolic for F, then, the operator $M(\theta) = \mathrm{D}_x F(K(\theta), \theta)$ is hyperbolic. By the stability of the spectrum under perturbations, we will obtain that (26) will be satisfied for G close to F.

Hence, we have verified that, given a normally hyperbolic invariant torus, if we perturb the map slightly, we have all the assumptions of Theorem 3.1 for the perturbed map and the original invariant torus. The conclusions of Theorem 3.1 give the persistence of the invariant torus.

Remark 3.6. We call attention to the fact that the proof works for ω resonant or non-resonant (ergodic). For ω irrational, the spectral gap condition is equivalent to $1 \notin \mathrm{Spec}(\mathcal{M}_\omega, \Gamma_b(E))$, since in such a case the spectrum is rotationally invariant. See [Mat68, HdlL03a].

Remark 3.7. The results we have formulated here immediately imply smooth dependence on parameters. If $F_\gamma : U \times \mathbb{T}^d \subset \mathbb{R}^n \times \mathbb{T}^d \to \mathbb{R}^n$ depends on a possibly multidimensional parameter γ we can, without loss of generality assume that the range of the parameter is $\mathbb{T}^\ell$. The extended map $\tilde{F} : U \times \mathbb{T}^d \times \mathbb{T}^\ell \to \mathbb{R}^n$ given by $F(x, \theta, \gamma) = F_\gamma(x, \theta)$ defines an extended skew product in $\mathbb{R}^n \times \mathbb{T}^d \times \mathbb{T}^\ell$ by

$$\bar{x} = F_\gamma(x, \theta) \ ,$$
$$(\bar{\theta}, \bar{\gamma}) = (\theta, \gamma) + (\omega, 0) \ .$$

We see that verifying the assumptions of Theorem 3.1 for F_γ uniformly on the parameters γ is the same as verifying the assumptions of Theorem 3.1 for the extended system $\tilde{F}$. The existence of smooth invariant tori for the extended system is the same as the smooth dependence on parameters for the family F_γ. In this formulation, we obtain that the regularity of in θ and the parameters is joint. There are examples that show that this is optimal.

Remark 3.8. The formulation of Theorem 3.1 is very similar to the a-posteriori estimates of numerical analysis. A numerical method can produce an approximate solution that satisfies (25) up to a few units of round–off error. It is also possible to verify the other hypothesis of Theorem 3.1 on the computed solution. These issues of numerical analysis, as well and results of implementations are discussed in more detail in [HdlL01].

Remark 3.9. The good analytical properties of the operators appearing in the invariance equation of the torus seems to translate in stability of numerical methods. This analysis is completed in the companion paper [HdlL01].

Proof: We will prove first Theorem 3.1 for finite differentiable maps.

We have to solve the equation (5)

$$F(K(\theta-\omega),\theta-\omega)-K(\theta)=0$$

in C^r. In this case, the map $\mathcal{T}_F : C^r(\mathbb{T}^d, U) \to C^r(\mathbb{T}^d,\mathbb{R}^n)$ defined by

$$\mathcal{T}_F(K)(\theta) = F(K(\theta-\omega),\theta-\omega)-K(\theta) \tag{28}$$

is a C^1 operator [dlLO99]. Moreover, the derivative of $\mathcal{T}_F$ is given by

$$\begin{aligned} \mathrm{D}\mathcal{T}_F(K)\Delta(\theta) &= \mathrm{D}_x F(K(\theta-\omega),\theta-\omega)\Delta(\theta-\omega)-\Delta(\theta)\\ &= \mathcal{M}_\omega\Delta(\theta)-\Delta(\theta)\ , \end{aligned}$$

i.e., $\mathrm{D}\mathcal{T}_F(K) = \mathcal{M}_\omega - \mathrm{Id}$.

By Theorem 2.12 we have that the spectral gap does not depend on the spaces considered. Therefore, the spectral gap assumption (26) on the bounded sections, implies that $\mathrm{D}\mathcal{T}_F(K)$ is invertible as a linear operator acting on C^r sections Δ. The existence and uniqueness of $\hat{K} = K_F$ in C^r spaces follows from the Inverse Function Theorem. The torus is normally hyperbolic since the transfer operator $\hat{\mathcal{M}}_\omega$ associated to $\hat{M}(\theta) = \mathrm{D}_x F(\hat{K}(\theta),\theta)$ is hyperbolic (see Remark 3.4).

The uniqueness in the C^0 space follows from the fact that $\hat{K}$ is obviously a C^0 invariant tori, and that the transfer operator $\hat{\mathcal{M}}_\omega$ is hyperbolic in C^0. ¿From the Inverse Function Theorem in C^0 spaces, $\hat{K}$ is the unique C^0 invariant torus in a C^0 neighborhood of the C^r torus $\hat{K}$.

The persistence of the torus under perturbations and the C^1 dependence on F follows by applying the Implicit Function Theorem on the C^1 operator $\mathcal{T} : C^{\Sigma_{r,1}}_\omega(U\times\mathbb{T}^d,\mathbb{R}^n)\times C^r(\mathbb{T}^d,U)\to C^r(\mathbb{T}^d,\mathbb{R}^n)$ defined by

$$\mathcal{T}(F,K)(\theta) = F(K(\theta-\omega),\theta-\omega)-K(\theta)\ .$$

The proof of Theorem 3.1 in the analytic case follows the same lines, but it is actually much simpler.

The C^1 regularity of the composition operator in (28) for analytic cases follows from the results for C^0 in the complex extension. For more details, we refer to [Mey75].

In the analytic case, by Theorem 2.13, we have that if there is a spectral gap in $\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_b(E))$ then, for ξ small enough, $\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{C^{a,\xi}}(E))$ also has a spectral gap. The proof, as before, is an application of the Implicit Function Theorem. □

3.2. **Bootstrap on the regularity.** Theorem 3.1 produces C^r invariant tori from a C^r approximate invariant tori, and gives estimates in the C^r norms of the difference between the approximate solutions and the true ones in terms of the C^r norm of $\mathcal{T}(K)$.

The following theorem says that a C^0 invariant torus of a $C^{\Sigma_{r,1}}$ quasiperiodic skew product is necessarily C^r. Henceforth, from the existence of a C^0 approximate invariant torus, we can deduce the existence of a C^r invariant torus. Notice, however, that Theorem 3.10 does not produce C^r estimates of the distance between the approximation and the true solution from C^0 estimates on $\mathcal{T}$.

As mentioned before, we will find it convenient to use spaces of functions with anisotropic regularity introduced in Definition 2.5.

Theorem 3.10. *Let $U \subset \mathbb{R}^n$ be an open set. Let $F : U \times \mathbb{T}^d \subset \mathbb{R}^n \times \mathbb{T}^d \to \mathbb{R}^n$ be a map of class $C^{\Sigma_{r,1}}$, with $r \geq 0$ – including $C^{\Sigma_{r,1}} = C^a$ in the analytic case $r = a$ –, such that for all $\theta \in \mathbb{T}^d$ the map $F(\cdot, \theta) : U \to \mathbb{R}^n$ is a local diffeomorphism. Let $\omega \in \mathbb{R}^d$ be a rotation. Let $K : \mathbb{T}^d \to U \subset \mathbb{R}^n$ be a C^0 parameterization of a normally hyperbolic invariant torus $\mathcal{K}$.*

Then, the parameterization K is C^r.

Proof: For $r = 0$ we has nothing to do.

We will prove first the result in the analytic case. By smoothing K, we can consider tori $\tilde{K}_\eta$ in $C^{a,\eta}$ with $\|K - \tilde{K}_\eta\|_{C^0}$ small enough and η small enough. Since

$$\|F(K(\theta - \omega), \theta - \omega) - K(\theta)\|_{C^0} = 0 \ ,$$

by choosing η sufficiently small we have

$$\|F(\tilde{K}_\eta(\theta - \omega), \theta - \omega) - \tilde{K}_\eta(\theta)\|_{C^{a,\eta}} < \varepsilon \ .$$

Since K is normally hyperbolic, by choosing η small enough we can get that the transfer operator associated to $M_\eta(\theta) = \mathrm{D}_x F(K_\eta(\theta), \theta)$ is hyperbolic in $\Gamma_{C^{a,\eta}}$ [HdlL03b]. So, by the Inverse Function Theorem, there is an analytic invariant torus K_η near $\tilde{K}_\eta$. By uniqueness, $K_\eta = K$.

The technique of the proof in the finite differentiable case ($r \geq 1$) is similar to the proof of the regularity in [dlLW95]. First, we show that the formal equations for derivatives have unique solutions, which are continuous. Then, under the assumptions of regularity of F, we show that the "Taylor" expansions obtained with these derivatives satisfy the equations with a smallness condition which is a power of the displacement. Then, using the quantitative estimates for the Theorem 3.1, we conclude that the C^0 solution K differs from its Taylor approximation less than a power and, by the converse of Taylor's theorem [AMR88, Nel69] we conclude that K is indeed C^r.

We will work in detail the case $r = 1$. We will show that the case for $r > 1$ can be deduced from this by induction. In Remark 3.11 we will also sketch the relatively easy modifications that are needed for a direct proof.

If K were C^1, taking derivatives in (5) we would obtain

$$\mathrm{D}_x F(K(\theta-\omega), \theta-\omega)\mathrm{D}_\theta K(\theta-\omega) + \mathrm{D}_\theta F(K(\theta-\omega), \theta-\omega) - \mathrm{D}_\theta K(\theta) = 0 \ ,$$

so $\mathrm{D}_\theta K(\theta)$ would solve the equation

$$\mathrm{D}_x F(K(\theta-\omega), \theta-\omega)K'(\theta-\omega) - K'(\theta) = -\mathrm{D}_\theta F(K(\theta-\omega), \theta-\omega) \ , \tag{29}$$

in $C^0(\mathbb{T}^d, L(\mathbb{R}^d, \mathbb{R}^n))$. We note that the right hand side of (29) is a continuous function.

By the hyperbolicity assumptions, there exists one and only one C^0 solution K' of this equation (29). We will see that in fact K is differentiable and $\mathrm{D}_\theta K = K'$.

To do so, given $\eta \in \mathbb{R}^d$ sufficiently small, we consider

$$\tilde{K}_\eta(\theta) = K(\theta-\eta) + K'(\theta-\eta)\eta \ , \tag{30}$$

and we will see that $\|K - \tilde{K}_\eta\|_{C^0} \leq |\eta|\gamma(|\eta|)$, with γ converging to zero as $|\eta|$ converges to zero. Since K' is continuous, this will prove that K is C^1 and $\mathrm{D}_\theta K = K'$.

We note that, by the uniform continuity of K and the fact that K' is bounded,

$$\|\tilde{K}_\eta - K\|_{C^0} \leq \gamma(|\eta|) \ .$$

We now compute $\mathcal{T}(\tilde{K}_\eta)$ using the first order Taylor expansion of F:

$$\begin{aligned}
\mathcal{T}_F(\tilde{K}_\eta)(\theta) &= F(\tilde{K}_\eta(\theta-\omega), \theta-\omega) - \tilde{K}_\eta(\theta) \\
&= F(K(\theta-\eta-\omega) + K'(\theta-\eta-\omega)\eta, (\theta-\eta-\omega)+\eta) \\
&\quad - K(\theta-\eta) - K'(\theta-\eta)\eta \\
&= F(K(\theta-\eta-\omega), \theta-\eta-\omega) \\
&\quad + \mathrm{D}_x F(K(\theta-\eta-\omega), \theta-\eta-\omega)K'(\theta-\eta-\omega)\eta \\
&\quad + \mathrm{D}_\theta F(K(\theta-\eta-\omega), \theta-\eta-\omega)\eta + R(\eta, \theta) \\
&\quad - K(\theta-\eta) - K'(\theta-\eta)\eta \ ,
\end{aligned} \tag{31}$$

where R is the remainder of the Taylor expansion, and $|R(\eta, \theta)| \leq |\eta|\gamma(|\eta|)$. We also see that, using the fact that K is invariant and K' satisfies the equation (29), we obtain that all the terms in (31) except R cancel. Hence, we obtain

$$\|\mathcal{T}_F(\tilde{K}_\eta)\|_{C^0} \leq |\eta|\gamma(|\eta|) \ .$$

Since $\tilde{K}_\eta$ is C^0 close to K, the hyperbolicity property of the cocycle remain uniform, and as $\tilde{K}_\eta$ is a C^0 approximate invariant torus, applying Theorem 3.1 we conclude that there is a torus K_η solving $\mathcal{T}_F(K_\eta) = 0$ and

$$\|K_\eta - \tilde{K}_\eta\|_{C^0} \leq |\eta|\gamma(|\eta|) \ .$$

On the other hand, by the uniqueness statements of Theorem 3.1 we conclude that $K_\eta = K$ for η small. Hence, for η small

$$\|\tilde{K}_\eta - K\|_{C^0} \leq |\eta|\gamma(|\eta|) \ .$$

This shows that indeed K' is the derivative of K.

The case of higher regularity is obtained by the *"tangent functor trick"*. Later in Remark 3.11 we will sketch an alternative proof which avoids the induction.

Let $r \geq 2$. Assume we have proved that if F is $C^{\Sigma_{r-1,1}}$ then a normally hyperbolic invariant torus K is C^{r-1}. We will prove now that if F is $C^{\Sigma_{r,1}}$ then K is C^r. We extend the map F to

$$\hat{F}(x, Y, \theta) = \begin{pmatrix} F(x,\theta) \\ \mathrm{D}_x F(K(\theta),\theta)Y + \mathrm{D}_\theta F(x,\theta) \end{pmatrix} ,$$

where $x \in \mathbb{R}^n$, $Y \in L(\mathbb{R}^d, \mathbb{R}^n)$, $\theta \in \mathbb{T}^d$. Notice that $\hat{F}$ is in $C^{\Sigma_{r-1,1}}$ and that $(K(\theta), \mathrm{D}_\theta K(\theta))$ is a C^{r-2} normally hyperbolic invariant torus of the skew product associated to $\hat{F}$. By induction, it is C^{r-1}, and, in particular $\mathrm{D}_\theta K(\theta)$ is C^{r-1}. So K is C^r, and we are done with the proof of the bootstrap. □

Remark 3.11. A direct proof follows the following lines. We proceed again by induction in the order of derivatives. Assume that $r-1$ derivatives of K exist and are continuous. Taking derivatives of (5) formally up to order r we obtain that $K^r = \mathrm{D}^r K$ satisfies the equation

$$\mathrm{D}_x F(K(\theta-\omega), \theta-\omega) K^r(\theta-\omega) - K^r(\theta) = R_r(\theta) \ , \tag{32}$$

where

$$\begin{aligned} R_r(\theta) = &- \sum_{j=0}^{r-2} \begin{pmatrix} r-1 \\ j \end{pmatrix} \mathrm{D}^{r-1-j} \left(\mathrm{D}_x F(K(\theta-\omega), \theta-\omega)\right) \mathrm{D}^{j+1} K(\theta-\omega) \\ &- \mathrm{D}^{r-1} \left(\mathrm{D}_\theta F(K(\theta-\omega), \theta-\omega)\right) \ . \end{aligned}$$

Notice that, since F is $C^{\Sigma_{r,1}}$ and K is C^{r-1}, the right hand side R_r of (32) is continuous. Again, by the hyperbolicity of the cocycle, we can find a continuous solution K^r of (32), and we have to see that in fact K is C^r and $\mathrm{D}^r K = K^r$.

It is obvious that for $\eta \in \mathbb{R}^d$, the torus K_η given by $K_\eta(\theta) = K(\theta+\eta)$ is invariant under the skew product F_η given by $F_\eta(x,\theta) = F(x, \theta+\eta)$.

Let us consider the expansion

$$K_\eta^{\leq r}(\theta) = \sum_{j=0}^{r-1} \frac{1}{j!} \mathrm{D}^j K(\theta)\eta^j + \frac{1}{r!} K^r(\theta)\eta^r \ .$$

Notice that this is a polynomial in η, and $\mathrm{D}_\eta^i K_\eta^{\leq r}(\theta)_{|\eta=0} = \mathrm{D}^i K(\theta)$ for $i < r$, and $\mathrm{D}_\eta^r K_\eta^{\leq r}(\theta)_{|\eta=0} = K^r(\theta)$.

Notice also that, at this point, $\|K_\eta^{\leq r} - K_\eta\|_{C^0} \leq |\eta|^{r-1}\gamma(|\eta|)$. Since F is $C^{\Sigma_{r,1}}$ we can take the first r derivatives with respect to η of

$$\begin{aligned}\mathcal{T}_{F_\eta} K_\eta^{\leq r} =& F_\eta(K_\eta^{\leq r}(\theta-\omega), \theta-\omega) - K_\eta^{\leq r}(\theta)\\ =& F(K_\eta^{\leq r}(\theta-\omega), \theta+\eta-\omega) - K_\eta^{\leq r}(\theta)\ .\end{aligned}$$

We claim that $\mathrm{D}_\eta^i \left(\mathcal{T}_{F_\eta} K_\eta^{\leq r}\right)_{|\eta=0} = 0$ for $i \leq r$. For $i < r$ we have the result because $K_\eta^{\leq r} = K_\eta + R_\eta$ with $R_\eta \simeq |\eta|^{r-1}\gamma(|\eta|)$, and $\mathcal{T}_{F_\eta} K_\eta = 0$. Moreover, $\mathrm{D}_\eta^r \left(\mathcal{T}_{F_\eta} K_\eta^{\leq r}\right)_{|\eta=0} = 0$ because of the equation satisfied by K^r.

Hence, by uniform continuity, we have

$$\|\mathcal{T}_{F_\eta} K_\eta^{\leq r}\|_{C^0} \leq |\eta|^r \gamma(|\eta|)\ .$$

This is equivalent to say that $\tilde{K}_\eta$ defined by $\tilde{K}_\eta(\theta) = K_\eta^{\leq r}(\theta - \eta)$ satisfies

$$\|\mathcal{T}_F \tilde{K}_\eta\|_{C^0} \leq |\eta|^r \gamma(|\eta|)\ .$$

Arguing as before we conclude that, if η is sufficiently small, the invariant torus K satisfies

$$\|\tilde{K}_\eta - K\|_{C^0} \leq |\eta|^r \gamma(|\eta|)\ ,$$

from where we obtain, applying the converse of Taylor theorem, that K is C^r and $K^r(\theta) = \mathrm{D}^r K(\theta)$.

4. Asymptotic invariant manifolds

In this section, given an invariant torus, we consider the existence of other invariant manifolds so that the motion on them converges to the torus. Well known examples in the literature are the stable or strong stable invariant manifolds. The main result of this section is Theorem 4.1 which generalizes the classical strong stable manifold theorem.

The main geometric requirement is that there exists and invariant bundle around the torus such that the spectrum of the transfer operator restricted to this bundle is contractive and satisfies some finite non-resonance assumptions with respect to the transfer operator on the whole tangent bundle. Then, we can find an invariant manifold tangent to this bundle. Note that we do not need that the invariant bundle that we consider has an invariant complement. In the case that the bundle is an spectral bundle associated to the most contractive sectors – in such a case the non-resonance assumptions are satisfied automatically, we recover the classical strong stable manifold theorem.

One reason to be interested in non-resonant manifolds is that, if one wants to study the asymptotic convergence to the torus, the motion along the strong stable manifolds converges the fastest so that the dominant effect for the long time behavior is the directions for which the contraction is the weakest. One therefore observes a convergence to the slow manifolds.

As indicated by the theory developed here, a smooth slow manifold may exist or not depending on whether the resonance conditions are met.

Nonresonant invariant manifolds in a neighborhood of a point were introduced in [dlL97] and studied further in [ElB01] and – more closely to the approach presented here – in [CFdlL03a, CFdlL03b]. Using the method of lifting diffeomorphism to actions on sections [HP69], one obtains results for asymptotic manifolds out of the results in [dlL97, CFdlL03a]. Nevertheless, using the special structure of the maps considered in this paper we obtain sharper results that those obtained just invoking the general theory.

4.1. **Statement of results.** The main theorem of this section is the following.

Theorem 4.1. *Let $U \subset \mathbb{R}^n$ be an open set. Let $F : U \times \mathbb{T}^d \subset \mathbb{R}^n \times \mathbb{T}^d \to \mathbb{R}^n$ be a map of class $C^{\Sigma_{r,s}}$, with $r \geq 0$ and $s \geq 2$ – including $C^{\Sigma_{r,s}} = C^a$ in the analytic case $r = a$ –, such that for all $\theta \in \mathbb{T}^d$ the map $F(\cdot, \theta) : U \to \mathbb{R}^n$ is a local diffeomorphism. Let $\omega \in \mathbb{R}^d$ be a rotation. Let $\mathcal{K}$ be an invariant torus whose parameterization is given by $K \in C^r(\mathbb{T}^d, U)$.*

Let $\mathcal{M}_\omega$ be the transfer operator defined from $M(\theta) = \mathrm{D}_x F(K(\theta), \theta)$.

Assume that there is a decomposition

$$N\mathcal{K} = E_1 \oplus E_2 \tag{33}$$

into C^r subbundles such that E_1 is invariant under M. Equivalently, if we take a representation of the transfer operator in a frame associated to the decomposition (33) *we have*

$$M(\theta) = \begin{pmatrix} M_1(\theta) & B(\theta) \\ 0 & M_2(\theta) \end{pmatrix} . \tag{34}$$

We denote by $\mathcal{M}_{1,\omega}$, $\mathcal{M}_{2,\omega}$ the transfer operators acting on sections of E_1, E_2, respectively associated to M_1, M_2. The annular hull of the spectrum

$$\mathcal{A} = \mathcal{A}\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_b) = \{ze^{i\alpha} \mid z \in \mathrm{Spec}(\mathcal{M}_\omega, \Gamma_b), \alpha \in \mathbb{R}\} ,$$

is then the union of the annular hulls of each $\mathcal{M}^1_\omega$, $\mathcal{M}^2_\omega$:

$$\mathcal{A}_1 = \mathcal{A}\mathrm{Spec}(\mathcal{M}_{1,\omega}, \Gamma_b(E_1)) \ , \ \mathcal{A}_2 = \mathcal{A}\mathrm{Spec}(\mathcal{M}_{2,\omega}, \Gamma_b(E_2)) \ ,$$

and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$.

Assume that:

H.1 $\mathcal{A}_1 \subset \{z \in \mathbb{C} \mid |z| < 1\}$*;*
H.2 $\mathcal{A}_1^{L+1}\mathcal{A}^{-1} \subset \{z \in \mathbb{C} \mid |z| < 1\}$*, for a certain $L \geq 1$.*
H.3 $\mathcal{A}_1^i \cap \mathcal{A}_2 = \emptyset$ *for every i with $2 \leq i \leq L$ (in case that $L \geq 2$);*
H.4 $L + 1 \leq s$*.*

Then:

a) *We can find a polynomial bundle map $\Lambda : E_1 \to E_1$ over the rotation ω, of degree not larger than L and of class $C^{r,\infty}$ with*

$$\Lambda(0, \theta) = 0 \ , \quad \mathrm{D}_\eta \Lambda(0, \theta) = M_1 \ , \tag{35}$$

and a $C^{\Sigma_{r,s}}$ *bundle map* $W : U_1 \subset E_1 \to X$ *over the identity, where* U_1 *is an open tubular neighborhood of the zero section of* E_1*, such that*

$$F(W(\eta,\theta),\theta) = W(\Lambda(\eta,\theta),\theta+\omega) \tag{36}$$

holds in U_1*, and*

$$\begin{aligned} &W(0,\theta) = K(\theta)\ , \\ &\Pi_1 \mathrm{D}_\eta W(0,\theta) = \mathrm{Id}_{E_1}\ , \quad \Pi_2 \mathrm{D}_\eta W(0,\theta) = 0\ , \end{aligned} \tag{37}$$

for all $\theta \in \mathbb{T}^d$*, where* Π_1, Π_2 *are the projections on* E_1, E_2*, respectively.*

b) *In case that we further assume for* $M \geq 2$ *that*

$$\mathcal{A}_1^i \cap \mathcal{A}_1 = \emptyset \qquad \text{for every integer } i \text{ with } M \leq i \leq L\ , \tag{38}$$

then we can choose Λ *in a) above to be a polynomial of degree not larger than* $M-1$*.*

In particular, if (38) *happens for* $M = 2$*, then*

$$\mathcal{A}_1^i \cap \mathcal{A}_1 = \emptyset \qquad \text{for every integer } i \text{ with } 2 \leq i \leq L\ , \tag{39}$$

and we can choose Λ *in a) above to be linear.*

c) *The* $C^{\Sigma_{r,s}}$ *manifold produced in a) is unique among the* $C^{\Sigma_{r,L+1}}$ *locally invariant manifolds tangent to* E_1 *at* $\mathcal{K}$*. That is, every two* $C^{\Sigma_{r,L+1}}$ *locally invariant manifolds will coincide in a neighborhood of* K *in* E*.*

Remark 4.2. $\mathcal{A}_1, \mathcal{A}_2$ are both union of annuli centered in the origin. If ω is irrational, the spectrum is invariant under rotations centered in the origin, so the spectrum coincides with its annular hull.

Remark 4.3. Note that we do not assume that $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$ since the assumption [H.3] only requires that the intersection is empty for powers bigger or equal that 2.

Also, we do not need that E_2 is invariant.

One example of this situation occurs when the linearization is a Jordan block. The bundle E_1 corresponds to the eigenspace and the bundle E_2 corresponds to the generalized eigenvalues. Note that E_2 is not invariant, indeed, in this example, there are no invariant complementary bundles.

Remark 4.4. Note that a consequence of (36) is that $\mathcal{W} = W(U_1)$ is a $C^{\Sigma_{r,s}}$ manifold invariant under F and tangent to E_1 at $\mathcal{K}$.

About the uniqueness statement c) of Theorem (4.1), note that the parameterization W and the map Λ need not be unique; it is the manifold $\mathcal{W} = W(U_1)$ which is unique.

Remark 4.5. We do not need that either E_1 or E_2 are trivial bundles. As we will see in [HdlL01], the fact that the bundles are non trivial happens very often near resonant situations.

Remark 4.6. Also, we do not assume that the invariant torus is normally hyperbolic, that is $\mathcal{A}_2$ can contain the unit circle. Compared with Theorem 3.1, notice also that we assume 1 degree more of differentiability in the vertical variables.

Remark 4.7. It follows from the results in [HdlL03a] that the spectrum of a transfer operator changes by an small amount if the transfer operator changes by an small amount.

Therefore, the non-resonance conditions H.1 – H.4 in Theorem 4.1 hold for open sets of transfer operators.

In particular, in case that the torus is normally hyperbolic, applying Theorem 3.1 we obtain that the torus persists and that the linearization is close to the original one.

Hence, if the original torus has spectral spaces which satisfy the hypothesis H.1 – H.4, then, the perturbed tori will also have spectral subspaces satisfying these hypothesis and, hence, by Theorem 4.1, it will also have invariant manifolds associated to these spectral subspaces.

Remark 4.8. Theorem 4.1 can be considered as a generalization of a the Poincaré-Dulac theorem for fixed points.

We recall that the Poincaré-Dulac theorem (see [Arn88]) has as a corollary that the dynamics in the stable manifold of a point can be conjugated to a polynomial which only has terms which are resonant for the spectrum of the linearization. The observation of [dlL97] was that the appropriate non-resonance conditions imply that the normal form has an invariant manifold.

In Theorem 4.1 the role of the spectrum of the derivative is taken by the spectrum of the transfer operator related to the linearization.

4.2. **Examples.** As mentioned above, Theorem 3.1 works for small quasi-periodic perturbations of a hyperbolic fixed point for a time independent map. We obtain persistent invariant tori.

If the hyperbolic fixed point has invariant subspaces satisfying the hypothesis of Theorem 4.1, then, to the invariant tori produced above, we can attach invariant manifolds which are analogues to the invariant spaces of the linearization at the fixed section.

Some of the examples considered in [CFdlL03a] are hyperbolic fixed points whose linearization is

$$A = \begin{pmatrix} 1/2 & 1 & & & \\ 0 & 1/2 & & & \\ & & 1/3 & & \\ & & & 1/5 & 1 \\ & & & & 1/5 \end{pmatrix}.$$

Then, denoting by E_i the i^{th} coordinate axis, there are invariant manifolds tangent to $E_1, E_3, E_4,\ E_1 \oplus E_2,\ E_4 \oplus E_5$, or to sums of these spaces, e.g., $E_1 \oplus E_4,\ E_1 \oplus E_2 \oplus E_4$.

These invariant manifolds have analogues under quasi-periodic perturbations.

4.3. **Proof of the theorem on invariant manifolds.** In this section we present the proof of Theorem 4.1. The method of proof is reminiscent of the proof of the corresponding result for invariant manifolds in the neighborhood of fixed points in [CFdlL03a].

After some useful normalizations and scalings, we show that there is a polynomial approximation that solves the desired equation to a very high order in the distance to the invariant torus. Then, we establish a result that allows us to conclude that the approximate solution developed in this way is close to a true solution of the full system of equations.

4.3.1. *Preliminaries.* By translating the torus $\mathcal{K}$, we can assume that it is the zero-section of the bundle $E = \mathbb{R}^n \times \mathbb{T}^d$. Notice that the vertical translation T given by

$$\begin{aligned} \bar{x} &= K(\theta) + x \ , \\ \bar{\theta} &= \theta \end{aligned}$$

is $C^{\Sigma_{r,\infty}}$ and then $\bar{F} = T^{-1}\circ F\circ T$ is $C^{\Sigma_{r,s}}$ in a neighborhood of the zero-section, which is invariant. So, from now on we assume that $K(\theta) = 0$.

We will write

$$F(x,\theta) = M(\theta)x + N(x,\theta) \ ,$$

where $M : E \to E$ is a C^r vector bundle map over the rotation ω (the monodromy of the torus), and N is a $C^{\Sigma_{r,s}}$ bundle map over the rotation ω, defined in a tubular neighborhood of the zero-section $E_0 = \{0\} \times \mathbb{T}^d$, and such that $N(0,\theta) = 0$, $\mathrm{D}_x N(0,\theta) = 0$ for every $\theta \in \mathbb{T}^d$.

For $i = 1, 2$, we denote by Π_i the projection of E onto E_i, by I_i the inclusion of E_i in E. With this notation: $M_1 = \Pi_1\circ M\circ I_1$, $M_2 = \Pi_2\circ M\circ I_2$, $B = \Pi_1\circ M\circ I_2$ (notice that $\Pi_2\circ M\circ I_1 = 0$ because E_1 is invariant under M).

¿From the hypothesis of the theorem we can find a Finsler metric on $E = \mathbb{R}^n \times \mathbb{T}^d$, adapted to the splitting $E = E_1 \oplus E_2$, so that

$$\|M^{-1}\|_{C^0} \left(\|M_1\|_{C^0} + \varepsilon\right)^{L+1} < 1 \ , \tag{40}$$

for $\varepsilon > 0$ small enough.

Rather than considering small tubular neighborhoods where the objects (bundle maps, invariant manifolds, ...) are defined, we will scale the maps involved in the equations to work in the unit tubular neighborhood of the zero section $B_E(1)$ (notice that $B_{E_1}(1) \subset B_E(1)$). More concretely, if we have a bundle map $H : E \to E$ we define $H^\delta(x,\theta) = \frac{1}{\delta} H(\delta x, \theta)$, for a given $\delta > 0$. Notice then that the invariance equation (36)

$$F_\theta\circ W_\theta = W_{\theta+\omega}\circ\Delta_\theta$$

holds in $B_E(\delta)$ if and only if

$$F^\delta_\theta\circ W^\delta_\theta = W^\delta_{\theta+\omega}\circ\Delta^\delta_\theta$$

holds in $B_E(1)$. Moreover,

$$F^\delta_\theta = M_\theta + N^\delta_\theta \ ,$$

where N^δ satisfies $N^\delta_\theta(0) = 0, \mathrm{D}_x N^\delta_\theta(0) = 0$ for all $\theta \in \mathbb{T}^d$ and that $\|N^\delta\|_{C^{\Sigma_{r,s}}}$ is small in $B_E(3)$ by taking a small δ.

4.3.2. *Finding the dynamics on the manifold.* In this section we show that, under the non-resonance hypotheses of the theorem, we can solve the invariance equation (36) up to order L, that is there exists a polynomial bundle map $W^{\leq} : E_1 \to E$ over the identity and a polynomial bundle map $\Lambda : E_1 \to E_1$ over ω, both of them of degree L and C^r in θ, such that

$$F(W^{\leq}(\eta,\theta),\theta) = W^{\leq}(\Lambda(\eta,\theta),\theta+\omega) + o(|\eta|^L) \ .$$

The discussion follows along the lines of Section 2.2.

We write $W^{\leq}(\eta,\theta) = \sum_{k=0}^{L} W^k(\eta,\theta)$, and $\Lambda(\eta,\theta) = \sum_{k=1}^{L} \Lambda^k(\eta,\theta)$. This broken up the invariance equation (36) into a hierarchy of equations, and we can study them recursively.

The zero-order equation is

$$F(W^0(\theta),\theta) = W^0(\theta+\omega) \ ,$$

that amounts to the torus parameterized by W^0 is invariant under F. So, we take $W^0 = 0$ (after the election of suitable coordinates made in Section 4.3.1, the torus is the zero-section).

The first-order equation is

$$M(\theta)W^1(\theta)\eta = W^1(\theta+\omega)\Lambda^1(\theta)\eta \ .$$

We point out that is just the equation of invariance of the bundle generated by W^1, and Λ^1 is the linearized dynamics on such a bundle. Hence, we take $W^1 = I_1$ (the immersion of E_1 into E) and $\Lambda^1(\theta) = M_1(\theta)$. In contrast to the Diophantine one-dimensional case discussed briefly in Section 2.2, there is no simple uniqueness for $\Lambda^1(\theta)$. We will assume that some choice is made.

The subsequent equations matching terms of order $k = 2, \ldots L$ are to be considered equations for W^k,Λ^k, assuming that $W^1, \ldots, W^{k-1}$, $\Lambda^1, \ldots, \Lambda^{k-1}$ are known. More concretely, the equation for the order k is

$$M(\theta)W^k(\eta,\theta) = W^1(\theta+\omega)\Lambda^k(\eta,\theta) + W^k(\Lambda^1(\theta)\eta,\theta+\omega) + R^k(\eta,\theta) \ , \tag{41}$$

where R^k is a homogeneous polynomial of degree k over the rotation ω, depending polynomially on $W^1, \ldots, W^{k-1}$ and $\Lambda^1, \ldots, \Lambda^{k-1}$, and so R^k is C^r in θ. We rewrite equation (41) as:

$$M(\theta-\omega)W^k(\Lambda^1(\theta-\omega)^{-1}\eta,\theta-\omega) - W^1(\theta)\hat{\Lambda}^k(\eta,\theta) - W^k(\eta,\theta) = \hat{R}^k(\eta,\theta) \ , \tag{42}$$

where $\hat{R}^k \in \Gamma_{C^r}(L^k_s(E_1;E))$ is known, and it is defined by

$$\hat{R}^k(\eta,\theta) = R^k(\Lambda^1(\theta-\omega)^{-1}\eta,\theta-\omega) \ ,$$

and the unknown terms are $W^k \in \Gamma_{C^r}(L^k_s(E_1;E))$ and $\hat{\Lambda}^k \in \Gamma_{C^r}(L^k_s(E_1;E_1))$, where $\hat{\Lambda}^k(\eta,\theta) = \Lambda^k(\Lambda^1(\theta-\omega)^{-1}\eta,\theta-\omega)$.

Taking projections over E_1, E_2 in equation (42), and taking into account that $W^1 = I_1$, we obtain:

A: Habiamos perdido la "B"

$$
\begin{aligned}
(43)\quad \hat{R}_1^k(\eta,\theta) &= M_1(\theta-\omega)W_1^k(M_1(\theta-\omega)^{-1}\eta,\theta-\omega) - W_1^k(\eta,\theta) \\
&\quad + B(\theta-\omega)W_2^k(M_1(\theta-\omega)^{-1}\eta,\theta-\omega) - \hat{\Lambda}^k(\eta,\theta)\ , \\
(44)\quad \hat{R}_2^k(\eta,\theta) &= M_2(\theta-\omega)W_2^k(M_1(\theta-\omega)^{-1}\eta,\theta-\omega) - W_2^k(\eta,\theta)\ .
\end{aligned}
$$

(43) is an equation for W_1^k and $\hat{\Lambda}_1^k$ in $\Gamma_{C^r}(L_s^k(E_1;E_1))$ and (44) is an equation for W_2^k in $\Gamma_{C^r}(L_s^k(E_1;E_2))$. Notice we have to solve first (44), and the solve (43). The operators that appear in both equation are a generalization of the Sylvester operators in [BK98, dlLW95, CFdlL03a], and have been studied in great detail in [HdlL03a].

A: He añadido esta frase

In general, given vector bundle maps E, F over the same base manifold $\mathcal{P}$, and two vector bundle maps $M_f : E \to E$ and $N_f : F \to F$ over the same homeomorphism $f : \mathcal{P} \to \mathcal{P}$, we construct a vector bundle map over f on the bundle of k-multilinear maps $S_f^k = S_{f,M,N}^k : L^k(F;E) \to L^k(F;E)$, by

$$(S^k(\theta)w_\theta)(v_1,\ldots,v_k) = M(\theta)w_\theta(N(\theta)^{-1}v_1,\ldots,N(\theta)^{-1}v_k)\ ,$$

where $w_\theta \in L^k(F_\theta;E_\theta)$, and $v_1,\ldots,v_k \in F_{f(\theta)}$. We will refer to this action as the Sylvester vector bundle map associated to M_f and N_f. The spectrum of the corresponding transfer operator is clarified in the following proposition (see [HdlL03a] for the proof, a similar argument happens in [CFdlL03a]).

Proposition 4.9. *Let $M_f : E \to E$, $N_f : F \to F$ be two vector bundle maps over the same homeomorphism $f : \mathcal{P} \to \mathcal{P}$, and let $S_f^k = S_{f,M,N}^k$ be the corresponding Sylvester vector bundle map on $L^k(F;E)$, where $k \geq 1$. Then:*

$$
\begin{aligned}
\mathrm{Spec}(\mathcal{S}_f^k, \Gamma_b(L_s^k(F;E))) &\subset \mathrm{Spec}(\mathcal{S}_f^k, \Gamma_b(L^k(F;E))) \\
&\subset \mathrm{Spec}(\mathcal{M}_f, \Gamma_b(E)) \cdot (\mathrm{Spec}(\mathcal{N}_f, \Gamma_b(F)))^{-k}\ .
\end{aligned}
$$

The idea of the proof is that one can factor the multilinear operators into linear operators in each of the factors. The spectrum of these elementary operators can be readily be related to the operators of the one variable operator. On the other hand, the action on each of the coordinates commutes with the others. Hence, we can apply a well known result in Banach algebra theory that ensures that the spectrum of the product of two communting operators is contained in the product of the spectra. We refer to the references above for complete details.

Another result of [HdlL03b] is that for C^r vector bundle maps over rotations the spectra of transfer operators acting on bounded sections and on continuous sections coincide with that on acting on C^r sections.

Introducing the Sylvester vector bundle maps $S_1^k = S_{\omega,M_1,M_1}^k$, $S_2^k = S_{\omega,M_2,M_1}^k$ and $S_B^k = S_{\omega,B,M_1}^k$, (43) and (44) can be rewritten as

$$
\begin{aligned}
(45)\quad \mathcal{S}_1^k W_1^k - W_1^k - \hat{\Lambda}^k &= \hat{R}_1^k - S_B^k W_2^k\ , \\
(46)\quad \mathcal{S}_2^k W_2^k - W_2^k &= \hat{R}_2^k\ .
\end{aligned}
$$

A: He añadido el termino en “B”, antes estaba implicito en la “R”

Since

$$\mathrm{Spec}(\mathcal{S}_2^k, \Gamma_b(L_s^k(E_2;E_1))) \subset \mathcal{A}_2 \cdot \mathcal{A}_1^{-k}$$

by Proposition 4.9 and by assumption H.3 $\mathcal{A}_1^{-k} \cap \mathcal{A}_2 = \emptyset$ for $k = 0, \ldots, L$, we conclude that

$$1 \notin \mathrm{Spec}(\mathcal{S}_2^k, \Gamma_{C^r}(L_s^k(E_2;E_1))) \ .$$

Therefore, (46) admits a unique solution $W_2^k \in \Gamma_{C^r}(L_s^k(E_2;E_1)))$.

We solve (45) as follows. If $\mathcal{A}_1 \cap \mathcal{A}_1^k = \emptyset$ we conclude that

$$1 \notin \mathrm{Spec}(\mathcal{S}_1^k, \Gamma_{C^r}(L_s^k(E_1;E_1))) \ ,$$

and we take $\Lambda^k = 0$ and W_1^k solving $\mathcal{S}_1^k W_1^k - W_1^k = \hat{R}_1^k - S_B^k W_2^k$. Otherwise we will choose $W_1^k = 0$ and $\hat{\Lambda}^k = -\hat{R}_1^k + S_B^k W_2^k$. This proves claim b) of Theorem 4.1.

Remark 4.10. Notice that solutions of (45) are not unique. We could choose $W_1^k = 0$ and $\hat{\Lambda}^k = -\hat{R}_1^k + S_B^k W_2^k$ (and then we compute Λ^k), for which we do not need non-resonance condition such as $\mathcal{A}_1 \cap \mathcal{A}_1^k = \emptyset$. Notice that this election corresponds to find the invariant manifold as a graph (over E_1).

4.3.3. *Standing hypotheses.* Since the coefficients of $W^{\leq}$ and Λ are computed recursively from N, the smallness condition on N state at the end of Section 4.3.1 implies that $W^{\leq}$ is close to the immersion $I_1 : E_1 \to E$ and Λ is close to $M_{1,\omega} : E_1 \to E_1$. In summary:

We assume that

$$\|N\|_{C^{\Sigma_{r,s}}(B_E(3),E)} \ , \ \|W^{\leq} - I_1\|_{C^{r,s}(B_{E_1}(1),E)} \ , \ \|\Lambda - M_1\|_{C^{r,s}(B_{E_1}(1),E_1)}$$

are as small as we need.

Using scaling arguments similar to those in Section 4.3.1, by taking the scaling parameter δ small enough we may assume that Λ is approximately linear and it is a contraction which maps $B_{E_1}(1)$ in $B_{E_1}(\lambda)$ with $\lambda < 1$, and

$$\|\mathrm{D}_\eta \Lambda\|_{C^0(B_{E_1}(1),E_1)} \leq \|M_1\|_{C^0} + \varepsilon < 1$$

and then

$$\|M^{-1}\|_{C^0} \|\mathrm{D}_\eta \Lambda\|^{L+1}_{C^0(B_{E_1}(1),E_1)} \leq \|M^{-1}\|_{C^0} (\|M_1\|_{C^0} + \varepsilon)^{L+1} < 1 \ .$$

4.3.4. *The equation for the higher order terms.* Once we have obtained the polynomial vector bundle map Λ over the rotation ω and the L-order approximation $W^{\leq}$ of the invariant manifold W, we have to find the higher order terms of the parameterization of the invariant manifold, $W^{>}$. We will write

$$W = W^{\leq} + W^{>},$$

where $W^{>} : E_1 \to E$ is a $C^{\Sigma_{r,s}}$ bundle map over the identity such that $\mathrm{D}_\eta^j W^{>}(0,\theta) = 0$ for every $j \leq L$.

The invariance equation (36) is reformulated in terms of $W^{>}$ as

$$\begin{aligned} & M(\theta)\left[W^{\leq}(\eta,\theta)+W^{>}(\eta,\theta)\right]+N\left(W^{\leq}(\eta,\theta)+W^{>}(\eta,\theta),\theta\right) \\ = \ & W^{\leq}(\Lambda(\eta,\theta),\theta+\omega)+W^{>}(\Lambda(\eta,\theta),\theta+\omega) \end{aligned}$$

or, with more compact notation,

$$\begin{aligned} W_\theta^{>} - M_\theta^{-1}\cdot W_{\theta+\omega}^{>}\circ\Lambda_\theta = & -\left(W_\theta^{\leq} - M_\theta^{-1}\cdot W_{\theta+\omega}^{\leq}\circ\Lambda_\theta\right) \\ & - M_\theta^{-1}\cdot N_\theta\circ(W_\theta^{\leq}+W_\theta^{>})\ . \end{aligned} \tag{47}$$

Equation 47 is an equation for $W^{>}$ to be solved in the space of $C^{\Sigma_{r,s}}$ bundle maps from E_1 to E, over the identity, whose L first vertical derivatives vanish on the zero section of E_1.

Notice that the way we have constructed $W^{\leq}$ and Λ ensures that, if $W^{>}$ satisfies the conditions above, then the right hand side of (47) satisfies also the same conditions.

If we define the operator $\mathcal{S}$ by

$$(\mathcal{S}H)_\theta = H_\theta - M_\theta^{-1}H_{\theta+\omega}\circ\Lambda_\theta\ , \tag{48}$$

then (47) reduces to the fixed point equation

$$W_\theta^{>} = -\mathcal{S}^{-1}(W_\theta^{\leq} - M_\theta^{-1}\cdot W_{\theta+\omega}^{\leq}\circ\Lambda_\theta + M_\theta^{-1}\cdot N_\theta\circ(W_\theta^{\leq}+W_\theta^{>}))\ , \tag{49}$$

provided that $\mathcal{S}^{-1}$ exists and it is continuous in suitable spaces.

The existence of $\mathcal{S}^{-1}$ is equivalent to solve the linearized equation

$$(\mathcal{S}H)_\theta = H_\theta - M_\theta^{-1}\cdot H_{\theta+\omega}\circ\Lambda_\theta = R_\theta\ . \tag{50}$$

Formally, the solution of (50) is

$$H_\theta = \sum_{k=0}^{\infty} M_{\theta+k\omega}^{-k} R_{\theta+k\omega}\circ\Lambda_\theta^k\ . \tag{51}$$

To prove the existence of $\mathcal{S}^{-1}$, we will analyze the convergence of (51) is suitable spaces. Then we will show that (49) defines a solution $W^{>}$.

The spaces will be introduced in Section 4.3.5. The analysis of the linearized equation (50) will be undertaken in Section 4.3.6. The fixed point equation (49) will be partially solved in Section 4.3.7, since we will loose one derivative. This derivative will be recovered in Section 4.3.8.

4.3.5. *Functional spaces and lemmas on derivatives of highly iterated functions.* With this motivation (cf. [CFdlL03a]), given two C^r vector bundles E, F over the same manifold $\mathcal{P}$ we define the Banach space of l-flat $C^{\Sigma_{r,s}}$ bundle maps over the identity

$$\Gamma_{r,s,l}(E;F) = \{H \in C_{\mathrm{id}}^{\Sigma_{r,s}}(B_E(1),F) \mid \mathrm{D}_x^j H(0,\theta)=0 \text{ for } 0\le j\le l,\ \|H\|_{\Gamma_{r,s,l}}<\infty\}$$

where the norm $\|\cdot\|_{\Gamma_{r,s,l}}$ is given by

$$\|H\|_{\Gamma_{r,s,l}} = \max\left\{ \|H\|_{C^{\Sigma_{r,s}}(B_E(1),F)}\ ,\ \max_{i\le r}\ \sup_{(x,\theta)\in B_E(1)\setminus E_0} \frac{|\mathrm{D}_\theta^i \mathrm{D}_x^l H(x,\theta)|}{|x|} \right\}\ . \tag{52}$$

The first part in this definition controls the derivatives in $\Sigma_{r,s}$, and the second part controls the derivatives of order l in x and any order of θ. We impose that the derivatives in the highest order are controlled by x.

We do not require that the functions have $l+1$ derivatives with respect to the vertical directions, but we require that the derivatives of order l are estimated by linear functions on x. Of course, the functions whose derivatives up to order $l+1$ in x exist and such that the derivatives up to order l in x vanish at the zero section $x = 0$ are in our space. For such functions, the norm $\|\mathrm{D}_\theta^i \mathrm{D}_x^{l+1} H\|_{C^0(B_E(1),F)}$ estimates the last term in (52). Indeed, the arguments will work with this norm in place of the "conical" norm in (52).

In the real-analytic case $r = a$, and once we have fixed a complexification given by ξ, we define

$$\Gamma_{a,l}(E;F) = \{H \in C^{a,\xi}_{\mathrm{id}}(B_E(1), F) \mid \mathrm{D}_x^j H(0,\theta) = 0 \text{ for } 0 \le j \le l,\ \|H\|_{\Gamma_{a,l}} < \infty\}$$

where the norm $\|\cdot\|_{\Gamma_{a,l}}$ is given by

$$\|H\|_{\Gamma_{a,l}} = \|\mathrm{D}_x^{l+1} H\|_{C^{a,\xi}}\ . \tag{53}$$

The following lemmas will be very useful. Although the notation we use is for the finite differentiable case, they also work for the analytic case. In fact, in the analytic case the proofs are easier, because we do not take derivatives with respect the horizontal directions and the norm (53) do not involves "conical" terms as in (52).

The following lemma follows immediately from Taylor's theorem. We use the notation $t_+ = \max(t, 0)$ for $t \in \mathbb{R}$.

Lemma 4.11. *Given E, F two C^r vector bundles over the same manifold $\mathcal{P}$, let $H \in \Gamma_{r,s,l}(E;F)$ and $(i,j) \in \Sigma_{r,s}$. Then, for every $(x,\theta) \in B_E(1)$:*

$$|\mathrm{D}_\theta^i \mathrm{D}_x^j H(x,\theta)| \le \frac{1}{(l-j)_+!} \|H\|_{\Gamma_{r,s,l}} |x|^{(l-j+1)_+}$$

If we multiply $H \in \Gamma_{r,s,l}$ by a matrix, we obtain again a function in $\Gamma_{r,s,l}$. More concretely,

Lemma 4.12. *Given E, F, G three C^r vector bundles over the same manifold $\mathcal{P}$, let $H \in \Gamma_{r,s,l}(E;F)$ and $P \in C^{\Sigma_{r,s}}_{\mathrm{id}}(B_E(1), L(F,G))$. Then $P \cdot H \in \Gamma_{r,s,l}(E;G)$, where we define $(P \cdot H)_\theta(x) = P_\theta(x) \cdot H_\theta(x)$, and*

$$\|P \cdot H\|_{\Gamma_{r,s,l}} \le C \|P\|_{C^{\Sigma_{r,s}}} \|H\|_{\Gamma_{r,s,l}}\ ,$$

where C is a constant.

Proof: Applying Leibnitz's rule to compute $\mathrm{D}_\theta^i \mathrm{D}_\eta^j (P \cdot H)$ for $(i,j) \in \Sigma_{r,s}$ we obtain the estimates

$$
\begin{aligned}
|\mathrm{D}_\theta^i \mathrm{D}_x^j (P_\theta(x) \cdot H_\theta(x))| &\le \sum_{m=0}^{i} \sum_{n=0}^{j} \begin{pmatrix} i \\ m \end{pmatrix} \begin{pmatrix} j \\ n \end{pmatrix} |\mathrm{D}_\theta^{i-m} \mathrm{D}_x^{j-n} P_\theta(x)| |\mathrm{D}_\theta^m \mathrm{D}_\eta^n H_\theta(x)| \\
&\le \sum_{m=0}^{i} \sum_{n=0}^{j} \frac{\begin{pmatrix} i \\ m \end{pmatrix} \begin{pmatrix} j \\ n \end{pmatrix}}{(l-n)_+!} \|P\|_{C^{\Sigma_{r,s}}} \|H\|_{\Gamma_{r,s,l}} |x|^{(l-n+1)_+} \\
&\le C \|P\|_{C^{\Sigma_{r,s}}} \|H\|_{\Gamma_{r,s,l}} |x|^{(l-j+1)_+} \\
&\le C \|P\|_{C^{\Sigma_{r,s}}} \|H\|_{\Gamma_{r,s,l}} \ .
\end{aligned}
$$

Notice also that, for $j = l$,

$$
\tfrac{1}{|x|} |\mathrm{D}_\theta^i \mathrm{D}_x^l (P_\theta(x) \cdot H_\theta(x))| \ \le \ C \|P\|_{C^{\Sigma_{r,s}}} \|H\|_{\Gamma_{r,s,l}} \ ,
$$

and we are done with the proof of Lemma 4.12. □

Remark 4.13. In particular, if the matrix P only depends on θ, then we have the estimate $\|P \cdot H\|_{\Sigma_{r,s,l}} \le C \|P\|_{C^r} \|H\|_{\Gamma_{r,s,l}}$.

In the analysis of the convergence of the expansion (51) in $\Gamma_{r,s,l}$ spaces, we have to estimate the norms of its terms, that involve derivatives in the horizontal and vertical directions. This kind of problems will also appear in further arguments.

Following [CFdlL03a, CFdlL03b], we use the following sets of indices, closely related to $\Sigma_{i,j}$, to describe in some detail the structure of the expression of the derivatives of the composition in terms of the derivatives of the bundle maps, with respect to horizontal and vertical directions:

(54)
$$
\begin{aligned}
\Sigma_{i,0}^* &= \{(a,b) \in \mathbb{N}^2 \mid \ a+b \le i,\ b \ge 1\} \cup \{(i,0)\} \subset \Sigma_{i,0} \ , \\
\Sigma_{i,j}^* &= \{(a,b) \in \mathbb{N}^2 \mid \ a+b \le i+j,\ a \le i,\ b \ge 1\} \subset \Sigma_{i,j} \qquad \text{if } j \ge 1 \ ,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Sigma}_{i,0} &= \{(a,b) \in \mathbb{N}^2 \mid \ a+b \le i\} \ , \\
\tilde{\Sigma}_{i,j} &= \Sigma_{i,j}^* \qquad\qquad \text{if } j \ge 1 \ .
\end{aligned}
\tag{55}
$$

We will also use the notation

$$
\sigma(t,m) = \sum_{j=0}^{m} t^j \le (1+t)^m \tag{56}
$$

for $t \ge 0$.

We have the following estimates of the norms of composition of bundle maps.

Lemma 4.14. *For $M \in C^{r,\infty}_\omega(E,E)$ (linear), $R \in C^{\Sigma_{r,s}}_{\mathrm{id}}(E_1,E)$, and $\Lambda \in C^{\Sigma_{r,s}}_\omega(E_1,E_1)$, then $M_\theta^{-1}\cdot R_{\theta+\omega}\circ\Lambda_\theta \in C^{\Sigma_{r,s}}_{\mathrm{id}}(E_1,E)$ and for all $(i,j)\in\Sigma_{r,s}$*

$$\mathrm{D}^i_\theta \mathrm{D}^j_\eta (M_\theta^{-1}\cdot R_{\theta+\omega}\circ\Lambda_\theta) =$$

$$\sum_{m=0}^{i}\sum_{(a,b)\in\Sigma^*_{m,j}}\sum_{I,J} C_{m,j,a,b,I,J}\ \mathrm{D}^{i-m}_\theta(M_\theta^{-1})\ \mathrm{D}^a_\theta\mathrm{D}^b_\eta R_{\theta+\omega}\circ\Lambda_\theta\ \mathrm{D}^{i_1}_\theta\mathrm{D}^{j_1}_\eta\Lambda_\theta\ldots\mathrm{D}^{i_b}_\theta\mathrm{D}^{j_b}_\eta\Lambda_\theta\ .$$

where $I=(i_1,\ldots,i_b)$, $J=(j_1,\ldots,j_b)$ are multi-indices with $|I|_1 = m-a$, $|J|_1 = j$, $i_l + j_l \geq 1$ for $l=1,\ldots b$, and $C_{m,j,a,b,I,J}$ is a combinational coefficient depending on the indices.

Moreover, we have a bound

$$\|M_\theta^{-1}\cdot R_{\theta+\omega}\circ\Lambda_\theta\|_{C^{\Sigma_{r,s}}} \leq C\|M^{-1}\|_{C^r}\|R\|_{C^{\Sigma_{r,s}}}\sigma(\|\Lambda\|_{C^{\Sigma_{r,s}}}, r+s)\ .$$

Proof: First compute $\mathrm{D}^m_\theta\mathrm{D}^j_\eta(R_{\theta+\omega}\circ\Lambda_\theta)$ by using induction arguments, and then compute $\mathrm{D}^i_\theta\mathrm{D}^j_\eta(M_\theta^{-1}\cdot R_{\theta+\omega}\circ\Lambda_\theta)$. A key point for obtaining this formulas is that the derivatives of $\theta\to\theta+\omega$ are bounded uniformly independently of the order.

Notice that the resulting expression contains derivatives $D^a_\theta D^b_\eta R_{\theta+\omega}$ of orders $(a,b)\in\Sigma^*_{0,j}\cup\Sigma^*_{1,j}\cup\cdots\cup\Sigma^*_{i,j} = \tilde\Sigma_{i,j}$ if $j>0$, or $\{(a,b)\mid a+b\leq i\} = \tilde\Sigma_{i,0}$ if $j=0$. □

Lemma 4.15. *For $M \in C^{r,\infty}_\omega(E,E)$ (linear), $F \in C^{\Sigma_{r,s}}_\omega(E,E)$, and $W \in C^{\Sigma_{r,s}}_{\mathrm{id}}(E_1,E)$ we have for all $(i,j)\in\Sigma_{r,s}$*

$$\mathrm{D}^i_\theta \mathrm{D}^j_\eta (M_\theta^{-1}\cdot F_\theta\circ W_\theta) =$$

$$\sum_{m=0}^{i}\sum_{(a,b)\in\Sigma^*_{m,j}}\sum_{I,J} C_{m,j,a,b,I,J}\ \mathrm{D}^{i-m}_\theta(M_\theta^{-1})\ \mathrm{D}^a_\theta\mathrm{D}^b_x F_\theta\circ W_\theta\ \mathrm{D}^{i_1}_\theta\mathrm{D}^{j_1}_\eta W_\theta\ldots\mathrm{D}^{i_b}_\theta\mathrm{D}^{j_b}_\eta W_\theta\ .$$

where $I=(i_1,\ldots,i_b)$, $J=(j_1,\ldots,j_b)$ are multi-indices with $|I|_1 = m-a$, $|J|_1 = j$, $i_l + j_l \geq 1$ for $l=1,\ldots b$, and $C_{m,j,a,b,I,J}$ is a combinational coefficient depending on the indices.

Moreover, we have a bound

$$\|M_\theta^{-1}\cdot F_\theta\circ W_\theta\|_{C^{\Sigma_{r,s}}} \leq C\|M^{-1}\|_{C^r}\|F\|_{C^{\Sigma_{r,s}}}\sigma(\|W\|_{C^{\Sigma_{r,s}}}, r+s)\ .$$

4.3.6. *Solving the linearized equation.* In this section we prove the invertibility of the linear operator $\mathcal{S}$ introduced in (48), which is equivalent to solve the linear equation (50). We start with two lemmas.

Lemma 4.16. *Let $M : E\to E$ be a C^r vector bundle map over a rotation ω. Then, for all $0\leq m\leq r$, and $k\geq 0$,*

$$\|\mathrm{D}^m_\theta M^{-k}_{\theta+k\omega}\|_{C^0} \leq C_r k^m \|M_\theta^{-1}\|^k_{C^0}\ ,$$

where C_r is a constant that does not depend on k.

Proof: Applying Leibnitz rule, we have that

$$\mathrm{D}_\theta^m(M(\theta)^{-1}M(\theta+\omega)^{-1}\dots M(\theta+(k-1)\omega)^{-1})$$

contains a sum of $C_{m,k} \le k^m$ terms of the form

$$M(\theta)^{-1}\cdots \mathrm{D}^{r_1}M(\theta+k_1\omega)^{-1}\cdots \mathrm{D}^{r_j}M(\theta+k_j\omega)^{-1}\cdots M(\theta+(k-1)\omega)^{-1} \ ,$$

where $j \in \{1,\dots,m\}$, $r_1+\cdots+r_j = m$, $r_1,\dots,r_j > 0$ and $0 \le k_1 < \cdots < k_j < k$. So, each term can be bounded by

$$\begin{aligned}\|M^{-1}\|_{C^0}^{k-j}\|M^{-1}\|_{C^m}^{j} &\le \|M^{-1}\|_{C^0}^{k}\left(\frac{\|M^{-1}\|_{C^m}}{\|M^{-1}\|_{C^0}}\right)^j \\ &\le \|M^{-1}\|_{C^0}^{k}\left(\frac{\|M^{-1}\|_{C^m}}{\|M^{-1}\|_{C^0}}\right)^m \ ,\end{aligned}$$

and we obtain that

$$\|\mathrm{D}_\theta^m M_{\theta+k\omega}^{-k}\|_{C^0} \le C_m k^m \|M^{-1}\|_{C^0}^k \ ,$$

where $C_m = \left(\dfrac{\|M^{-1}\|_{C^m}}{\|M^{-1}\|_{C^0}}\right)^m$. Notice that $C_m \le C_r$, hence we can always use C_r. □

The fact that the dynamics on the torus is a rotation is crucial for obtaining the estimates in Lemma 4.16. We are now to compute estimates of the derivatives of the nonlinear cocycle associated to Λ_ω.

Lemma 4.17. *For Λ constructed in Section 4.3.2, for all $i \le r$, for all j and for all $k \ge 0$*

$$|\mathrm{D}_\theta^i\mathrm{D}_\eta^j(\Lambda(\eta,\theta,k)| \le C_{i,j}(\|M_1\|+\varepsilon)^k|\eta|^{(1-j)_+}$$

where $|\eta| < 1$.

Proof: Follows from the fact that Λ is polynomial in η, and it is arbitrarily close to M_1. □

Now, we will prove the convergence of the expansion (51) in the space $\Gamma_{r,s,L}$.

Proposition 4.18. *The operator*

$$\mathcal{S}H_\theta = H_\theta - M_\theta^{-1}H_{\theta+\omega}\circ\Lambda_\theta$$

is a bounded linear operator in $\Gamma_{r,s,L} = \Gamma_{r,s,L}(E_1,E)$, whose inverse is also bounded. Obviously, if $r = a$ then $\Gamma_{r,s,L} = \Gamma_{a,L}$.

Proof: We have to bound the general term

$$H_\theta^k = M_{\theta+k\omega}^{-k}R_{\theta+k\omega}\circ\Lambda_\theta^k$$

of (51). For all $(i,j) \in \Sigma_{r,s}$, and for all $(\eta,\theta) \in B_{E_1}(1)$:

$$
\begin{aligned}
&|\mathrm{D}_\theta^i \mathrm{D}_\eta^j H_\theta^k(\eta)| \\
&\leq \sum_{m=0}^{i} \sum_{(a,b)\in\Sigma^*_{m,j}} \sum_{I,J} C \, \|\mathrm{D}_\theta^{i-m}(M^{-k}_{\theta+k\omega})\|_{C^0} \, \|\mathrm{D}_\theta^a \mathrm{D}_\eta^b R_{\theta+k\omega} \circ \Lambda_\theta^k\|_{C^0} \prod_{l=1}^{b} |\mathrm{D}_\theta^{i_l} \mathrm{D}_\eta^{j_l} \Lambda_\theta^k(\eta)| \ ,
\end{aligned}
$$

by Lemma 4.14. For each term,

$$
\|\mathrm{D}_\theta^{i-m}(M^{-k}_{\theta+k\omega})\|_{C^0} \leq C_r k^{i-m} \|M^{-1}\|^k \ ,
$$

by Lemma 4.16,

$$
\begin{aligned}
\|\mathrm{D}_\theta^a \mathrm{D}_\eta^b R_{\theta+k\omega} \circ \Lambda_\theta^k\|_{C^0} &\leq \frac{1}{(L-b)_+!} \|R\|_{\Gamma_{r,s,L}} \|\Lambda_\theta^k\|_{C^0}^{(L-b+1)_+} \\
&\leq \frac{1}{(L-b)_+!} \|R\|_{\Gamma_{r,s,L}} (\|M_1\| + \varepsilon)^{k(L-b+1)_+} \ ,
\end{aligned}
$$

by Lemma 4.11 and Lemma 4.17 for $i = j = 0$, and

$$
\begin{aligned}
\prod_{l=1}^{b} |\mathrm{D}_\theta^{i_l} \mathrm{D}_\eta^{j_l} \Lambda_\theta^k(\eta)| &\leq \prod_{l=1}^{b} C_{i_l,j_l} (\|M_1\| + \varepsilon)^k |\eta|^{(1-j_l)_+} \\
&\leq C (\|M_1\| + \varepsilon)^{kb} |\eta|^{(b-j)_+}
\end{aligned}
$$

by Lemma 4.17, where the last inequality follows from $(\alpha+\beta)_+ \leq \alpha_+ + \beta_+$, and from $|\eta| < 1$. We therefore obtain the estimate

$$
|\mathrm{D}_\theta^i \mathrm{D}_\eta^j H_\theta^k(\eta)| \leq \tilde{C} \sigma(k,i) \left(\|M^{-1}\| (\|M_1\| + \varepsilon)^{L+1} \right)^k \cdot \|R\|_{\Gamma_{r,s,L}} \cdot |\eta|^{(L-j+1)_+} \ ,
$$

where $\tilde{C}$ does not depend on k, and σ is defined in (56).

Hence, for all $(\eta,\theta) \in B_{E_1}(1)$, for all $(i,j) \in \Sigma_{r,s}$, we have

$$
|\mathrm{D}_\theta^i \mathrm{D}_\eta^j H_\theta^k(\eta)| \leq \tilde{C} \sigma(k,i) \left(\|M^{-1}\| (\|M_1\| + \varepsilon)^{L+1} \right)^k \cdot \|R\|_{\Gamma_{r,s,L}} \ ,
$$

and, since for $j = L$ we have $|\eta|^{(L-j+1)_+} = |\eta|$, then if $|\eta| \neq 0$

$$
\frac{|\mathrm{D}_\theta^i \mathrm{D}_\eta^L H_\theta^k(\eta)|}{|\eta|} \leq \tilde{C} \sigma(k,i) \left(\|M^{-1}\| (\|M_1\| + \varepsilon)^{L+1} \right)^k \cdot \|R\|_{\Gamma_{r,s,L}} \ .
$$

In summary, we obtain that the general term of the the sum (51) has a $\Gamma_{r,s,L}$-norm

$$
\|H_k\|_{\Gamma_{r,s,L}} \leq \tilde{C} \sigma(k,r) \left(\|M^{-1}\| (\|M_1\| + \varepsilon)^{L+1} \right)^k \cdot \|R\|_{\Gamma_{r,s,L}}
$$

and the series (51) is absolutely convergent in $\Gamma_{r,s,L}$ because

$$
\|M^{-1}\| (\|M_1\| + \varepsilon)^{L+1} < 1
$$

and $\sigma(k,r) \leq (k+1)^r$. □

4.3.7. *Solving the equation with one less derivative.* We will write now (49) as a fixed point problem for the operator

$$\mathcal{T}(W^{>}) = \mathcal{S}^{-1}\left(-(W_\theta^{\le} - M_\theta^{-1}W_{\theta+\omega}^{\le}\circ\Lambda_\theta) - M_\theta^{-1}N_\theta\circ(W_\theta^{\le} + W_\theta^{>})\right) , \tag{57}$$

in $\Gamma_{r,s,L}$. First, we will prove that $\mathcal{T}$ is a contraction, but in a closed ball of $\Gamma_{r,s-1,L}$ (recall that $L \le s-1$, by hypothesis). The derivative that we loose in the following result will be recovered later.

Lemma 4.19. *Under the hypotheses of Theorem 4.1, and the standing hypotheses of Section 4.3.3, we have that the operator* $\mathcal{T} : \Gamma_{r,s-1,L} \to \Gamma_{r,s-1,L}$ *sends the closed unit ball* $\bar{B}_{\Gamma_{r,s-1,L}}(1)$ *into itself, and it is a contraction there. Obviously, if* $r = a$ *then* $\Gamma_{r,s-1,L} = \Gamma_{a,L}$.

Proof: If $W^{>} \in \bar{B}_{\Gamma_{r,s-1,L}}(1)$, $W^{>}$ is defined in $B_{E_1}(1)$ and $\|W^{>}\|_{C^0} \le 1$, then for all $\eta_\theta \in B_{E_1}(1)$,

$$\begin{aligned} |W_\theta^{\le}(\eta) + W_\theta^{>}(\eta)| &\le |(W_\theta^{\le} - I_\theta^1)(\eta)| + |\eta| + |W_\theta^{>}(\eta)| \\ &< 3 , \end{aligned}$$

and then $N_\theta\circ(W_\theta^{\le} + W_\theta^{>})$ is well defined and of class $C^{\Sigma_{r,s-1}}$.

Let H be the bundle map over the identity defined by

$$H_\theta = -(W_\theta^{\le} - M_\theta^{-1}W_{\theta+\omega}^{\le}\circ\Lambda_\theta) - M_\theta^{-1}N_\theta\circ(W_\theta^{\le} + W_\theta^{>}) .$$

Notice that $H \in \Gamma_{r,s-1,L}$. We are going to bound $\|H\|_{\Gamma_{r,s-1,L}}$. To do so, we bound $\mathrm{D}_\theta^i\mathrm{D}_\eta^j H_\theta(\eta)$ for $i, j \in \Sigma_{r,s-1}$ and $\eta_\theta \in B_{E_1}(1)$. We write $H_\theta = H_\theta^1 + H_\theta^2$, where

$$\begin{aligned} H_\theta^1 &= -W_\theta^{\le} + M_\theta^{-1}W_{\theta+\omega}^{\le}\circ\Lambda_\theta - M_\theta^{-1}N_\theta\circ W^{\le} \\ &= -W_\theta^{\le} + M_\theta^{-1}I_{\theta+\omega}^1\circ\Lambda_\theta + \\ &\quad M_\theta^{-1}(W_{\theta+\omega}^{\le} - I_{\theta+\omega}^1)\circ\Lambda_\theta - M_\theta^{-1}N_\theta\circ W^{\le} \end{aligned} \tag{58}$$

and

$$\begin{aligned} H_\theta^2 &= M_\theta^{-1}N_\theta\circ W_\theta^{\le} - M_\theta^{-1}N_\theta\circ(W_\theta^{\le} + W_\theta^{>}) \\ &= -\int_0^1 M_\theta^{-1}\mathrm{D}_x N_\theta\circ(W_\theta^{\le} + tW_\theta^{>})\, W_\theta^{>}\, dt . \end{aligned} \tag{59}$$

Since $W^{\le}$ and Λ are polynomial of degree L (with $L+1 \le s$), then $H^1 \in C^{\Sigma_{r,s}}$. If $j \le L$, we apply Taylor's theorem to obtain the estimate

$$|\mathrm{D}_\theta^i\mathrm{D}_\eta^j H_\theta^1(\eta)| \le \|\mathrm{D}_\theta^i\mathrm{D}_\eta^{L+1} H_\theta^1\|_{C^0} \cdot |\eta|^{L+1-j} , \tag{60}$$

so we have to estimate $|\mathrm{D}_\theta^i\mathrm{D}_\eta^j H_\theta^1(\eta)|$ for $L < j \le s-1$. Since

$$\begin{aligned} |\mathrm{D}_\theta^i\mathrm{D}_\eta^j H_\theta^1(\eta)| \;\le\; & C\|M^{-1}\|_{C^r}\big(\|W^{\le} - I^1\|_{C^{\Sigma_{r,s}}}\sigma(\|\Lambda\|_{C^{r,L}}, r+s) \\ & +\|N\|_{C^{\Sigma_{r,s}}}\sigma(\|W^{\le}\|_{C^{r,L}}, r+s)\big) \end{aligned}$$

where σ was defined in (56), we obtain the bound

$$\begin{aligned} \|H^1\|_{\Gamma_{r,s-1,L}} \quad \le \quad & C\|M^{-1}\|_{C^r}\left(\|W^{\le}-I^1\|_{C^{\Sigma_{r,s}}}\sigma(\|\Lambda\|_{C^{r,L}}, r+s)\right. \\ & \left.+\|N\|_{C^{\Sigma_{r,s}}}\sigma(\|W^{\le}\|_{C^{r,L}}, r+s)\right) \ . \end{aligned} \tag{61}$$

For H^2, we argue that

$$\begin{aligned} & |\mathrm{D}^i_\theta \mathrm{D}^j_\eta H^2_\theta(\eta)| \\ & \le \int_0^1 \left|\mathrm{D}^i_\theta \mathrm{D}^j_\eta \left(M^{-1}_\theta \mathrm{D}_x N_\theta \circ (W^{\le}_\theta + tW^{>}_\theta)\ W^{>}_\theta\ \right)\right| dt \\ & \le C\|M^{-1}\|_{C^r}\|N\|_{C^{\Sigma_{r,s}}}\sigma(\|W^{\le}\|_{C^{r,L}}+1, r+s)\|W^{>}\|_{\Gamma_{r,s-1,L}}|\eta|^{(L-j+1)_+} \ , \end{aligned}$$

where we use the bound

$$\sup_{t\in[0,1]} \|W^{\le}+tW^{>}\|_{C^{\Sigma_{r,s}}} \le \|W^{\le}\|_{C^{r,L}}+1 \ ,$$

and then

$$\|H^2\|_{\Gamma_{r,s-1,L}} \le C\|M^{-1}\|_{C^r}\|N\|_{C^{\Sigma_{r,s}}}\sigma_{r+s}(\|W^{\le}\|_{C^{r,L}}+1)\|W^{>}\|_{\Gamma_{r,s-1,L}} \ . \tag{62}$$

Remark 4.20. This is the point in which we loose one derivative with respect to the vertical direction and we are forced to work in $\Gamma_{r,s-1,L}$ instead of $\Gamma_{r,s,L}$. Obviously, this drawback has not to do in the analytic case.

The standing hypotheses of Section 4.3.3 about the smallness of N and $W^{\le}-I^1$ give that (61),(62) are small, so

$$\|\mathcal{T}W^{>}\|_{\Gamma_{r,s-1,L}} = \|\mathcal{S}^{-1}H\|_{\Gamma_{r,s-1,L}} < 1 \ .$$

We have to prove now that $\mathcal{T}$ is a contraction in the closed unit ball of $\Gamma_{r,s-1,L}$. To do so, let $W^{>}$ and Δ such that $W^{>}$ and $W^{>}+\Delta \in \bar{B}_{\Gamma_{r,s-1,L}}(1)$. Notice that

$$\mathcal{T}(W^{>}+\Delta)_\theta - \mathcal{T}(W^{>})_\theta = -\int_0^1 \mathcal{S}^{-1}M^{-1}_\theta \mathrm{D}_x N_\theta \circ (W^{\le}_\theta + W^{>}_\theta + t\Delta_\theta)\Delta_\theta\ dt \ ,$$

and using the same arguments are those leading to (62) we obtain that

$$\begin{aligned} & \|\mathcal{T}(W^{>}+\Delta)-\mathcal{T}(W^{>})\|_{\Gamma_{r,s-1,L}} \\ & \le \|\mathcal{S}\|_{\Gamma_{r,s-1,L}}\|M^{-1}\|_{C^r}\|N\|_{C^{\Sigma_{r,s}}}\sigma_{r+s}(1+\|W^{\le}\|_{C^{r,L}})\|\Delta\|_{\Gamma_{r,s-1,L}} \ . \end{aligned} \tag{63}$$

Under the smallness conditions on N the operator $\mathcal{T}$ is a contraction in $\bar{B}_{\Gamma_{r,s-1,L}}(1)$. □

Remark 4.21. The previous lemma close the proof of statements a) and b) of Theorem 4.1 in the analytic case.

4.3.8. *The last derivative.* Lemma 4.19 proves that the operator $\mathcal{T}$ defined in (57) has a fixed point. It is the result claimed except for the fact that in the finite differentiable case we obtain that W is $C^{\Sigma_{r,s-1}}$ instead of $C^{\Sigma_{r,s}}$.

In this section we will see that $\mathrm{D}_\eta W^> \in \Gamma_{r,s-1,L-1}$, and as a result we will obtain $W^> \in \Gamma_{r,s,L}$, ending the proof of statements a) and b) of Theorem 4.1.

Recall that $W^>$ solves the equation (47)

$$\begin{aligned} W_\theta^> - M_\theta^{-1} \cdot W_{\theta+\omega}^> \circ \Lambda_\theta = &- (W_\theta^\leq - M_\theta^{-1} \cdot W_{\theta+\omega}^\leq \circ \Lambda_\theta) \\ &- M_\theta^{-1} \cdot N_\theta \circ (W_\theta^\leq + W_\theta^>) \ . \end{aligned} \tag{64}$$

So, $\mathrm{D}_\eta W^>$ solves the equation

$$\begin{aligned} \mathrm{D}_\eta W_\theta^> - M_\theta^{-1} \cdot \mathrm{D}_\eta W_{\theta+\omega}^> \circ \Lambda_\theta \cdot \mathrm{D}_\eta \Lambda_\theta = &- \mathrm{D}_\eta \left(W_\theta^\leq - M_\theta^{-1} \cdot W_{\theta+\omega}^\leq \circ \Lambda_\theta \right) \\ &- M_\theta^{-1} \cdot \mathrm{D}_x N_\theta \circ W_\theta \cdot (\mathrm{D}_\eta W_\theta^\leq + \mathrm{D}_\eta W_\theta^>) \ . \end{aligned} \tag{65}$$

Let U be the bundle map

$$\begin{aligned} U_\theta = &- \mathrm{D}_\eta \left(W_\theta^\leq - M_\theta^{-1} \cdot W_{\theta+\omega}^\leq \circ \Lambda_\theta \right) \\ &- M_\theta^{-1} \cdot \mathrm{D}_x N_\theta \circ W_\theta \cdot \mathrm{D}_\eta W_\theta^\leq \ . \end{aligned}$$

Notice that $U \in \Gamma_{r,s-1,L-1}(E_1, L(E_1, E))$.

We consider now the operators $\tilde{\mathcal{S}}, \tilde{\mathcal{T}}$ defined by

$$\begin{aligned} (\tilde{\mathcal{S}}\tilde{H})_\theta = &\tilde{H}_\theta - M_\theta^{-1} \cdot \tilde{H}_{\theta+\omega} \circ \Lambda_\theta \cdot \mathrm{D}_\eta \Lambda_\theta \\ (\tilde{\mathcal{T}}\tilde{H})_\theta = &M_\theta^{-1} \cdot \mathrm{D}_x N_\theta \circ W_\theta \cdot \tilde{H}_\theta \ . \end{aligned}$$

Both operators act on bundle maps $\tilde{H} \in \Gamma_{r,s-1,L-1}(E_1, L(E_1, E))$.

Lemma 4.22. *Under the hypotheses of Theorem 4.1, and the standing hypotheses of Section 4.3.3, we have that the operators* $\tilde{\mathcal{S}}, \tilde{\mathcal{T}} : \Gamma_{r,\bar{s},L-1} \to \Gamma_{r,\bar{s},L-1}$ *are bounded for* $L-1 \leq \bar{s} \leq s-1$. *Moreover, taking* $\|N\|_{C^{\Sigma_{r,s}}}$ *small enough,* $\tilde{\mathcal{S}}$ *is invertible and* $\|\tilde{\mathcal{S}}\| \|\tilde{\mathcal{T}}\| < 1$.

Proof: For $\tilde{H} \in \Gamma_{r,\bar{s},L-1}$:

$$\begin{aligned} \|M_\theta^{-1} \mathrm{D}_x N \circ W_\theta \cdot \tilde{H}\|_{\Gamma_{r,\bar{s},L-1}} \leq &C \|M_\theta^{-1} \mathrm{D}_x N \circ W_\theta\|_{C^{\Sigma_{r,s-1}}} \|\tilde{H}\|_{\Gamma_{r,\bar{s},L-1}} \\ \leq &C \|M_\theta^{-1}\|_{C^r} \|N\|_{C^{\Sigma_{r,s}}} \|\tilde{H}\|_{\Gamma_{r,\bar{s},L-1}} \sigma(\|W\|_{C^{\Sigma_{r,\bar{s}}}}, r+\bar{s}) \ . \end{aligned}$$

where we have applied Lemma 4.12 and Lemma 4.15. This proves that $\tilde{\mathcal{T}}$ is bounded and as small as necessary.

The operator $\tilde{\mathcal{S}}$ is obviously bounded in $\Gamma_{r,\bar{s},L-1}$ (see Proposition 4.18 for the arguments). Given $\tilde{G}$ in $\Gamma_{r,\bar{s},L-1}$, the series

$$\tilde{H}_\theta = \sum_{k=0}^{\infty} M_{\theta+k\omega}^{-k} \tilde{G}_{\theta+k\omega} \circ \Lambda_\theta^k \cdot \mathrm{D}_\eta \Lambda_\theta^k \tag{66}$$

provides a formal solution of $\tilde{\mathcal{S}}\tilde{H} = \tilde{G}$. By repeating again the arguments of Proposition 4.18, we can bound each term in (66) as follows:

$$\begin{aligned}\|M_{\theta+k\omega}^{-k}\tilde{G}_{\theta+k\omega}\circ\Lambda_\theta^k \cdot \mathrm{D}_\eta\Lambda_\theta^k\|_{\Gamma_{r,\bar{s},L-1}} \leq & C\|M_{\theta+k\omega}^{-k}\|_{C^r}\|\tilde{G}_{\theta+k\omega}\circ\Lambda_\theta^k\|_{\Gamma_{r,\bar{s},L-1}}\|\Lambda_\theta^k\|_{C^{\Sigma_{r,\bar{s}}}} \\ \leq & C(k+1)^r\|M^{-1}\|_{C^0}^k(\|M_1\|_{C^0}+\varepsilon)^{(L+1)k}\|\tilde{G}\|_{\Gamma_{r,\bar{s},L-1}}\end{aligned}$$

Since $\|M^{-1}\|_{C^0}(\|M_1\|_{C^0}+\varepsilon)^{L+1} < 1$ (see Section 4.3.3), we are done with the proof of Lemma 4.22. □

The following lines are the final arguments of the proof of statements a) and b) of Theorem 4.1.

At this point, $W^> \in \Gamma_{r,s-1,L}$, so $\mathrm{D}_\eta W^> \in \Gamma_{r,s-2,L-1}$. Moreover, $\mathrm{D}_\eta W^>$ is a solution of (65), that reads

$$\tilde{\mathcal{S}}\mathrm{D}_\eta W^> = U - \tilde{\mathcal{T}}\mathrm{D}_\eta W^> \ .$$

Applying Lemma 4.22 with $\bar{s} = s-2$, we conclude that

$$\mathrm{D}_\eta W^> = (\mathrm{Id} + \tilde{\mathcal{S}}\tilde{\mathcal{T}})^{-1}\tilde{\mathcal{S}}^{-1}U \ , \tag{67}$$

is the only solution in $\Gamma_{r,s-2,L-1}$. Notice that $U \in \Gamma_{r,s-1,L-1}$, and the operators $\tilde{\mathcal{S}},\tilde{\mathcal{T}}$ are also well defined in $\Gamma_{r,s-1,L-1}$, by Lemma 4.22 with $\bar{s} = s-1$. Much more, since $\|\tilde{\mathcal{S}}\tilde{\mathcal{T}})\| < 1$ in $\Gamma_{r,s-1,L-1}$, again by Lemma 4.22, we conclude that (67) is also well defined in $\Gamma_{r,s-1,L-1}$.

In summary, $\mathrm{D}_\eta W^> \in \Gamma_{r,s-1,L-1}$, so $W^> \in \Gamma_{r,s,L}$ and $W \in C^{\Sigma_{r,s}}$.

4.3.9. *Proof of the uniqueness of the invariant manifold.* In this section we will prove the uniqueness of the invariant manifold mentioned in c) of Theorem 4.1 (see [CFdlL03a]).

Notice that if $W = W(\eta,\theta)$ is a $C^{\Sigma_{r,L+1}}$-parameterization of an invariant manifold $\mathcal{W}$ attached to the torus and tangent to E_1 then, after the election of coordinates given in Section 4.3.1,

$$W(0,\theta) = 0 \ , \ \mathrm{D}_\eta W(0,\theta) = \begin{pmatrix} \mathrm{Id}_{E_1} \\ O \end{pmatrix} \ .$$

So, we can write locally $\eta = W_{1,\theta}^{-1}(x_1)$ and $G_\theta(x_1) = W_{2,\theta}\circ W_{1,\theta}^{-1}$. So, $G : B_1 \subset E_1 \to E_2$ is $C^{\Sigma_{r,L+1}}$, where B_1 is a tubular neighborhood of the zero section in E_1. Notice that, locally, the manifold $\mathcal{W}$ is a graph $\{x_2 = G_\theta(x_1)\}$. Moreover, $G_\theta(0) = 0$ and $\mathrm{D}_{x_1}G_\theta(0) = 0$. Notice that this graph representation is independent of the former parameterization of the manifold. If we see that there is one and only one invariant $C^{\Sigma_{r,L+1}}$ graph, tangent to E_1, we will be done with the proof of the uniqueness.

The invariance equation of the graph $x_2 = G_\theta(x_1)$ is

(68)

$$\begin{aligned}G_\theta(x_1) =& A_{2,\theta}^{-1}\left(G_{\theta+\omega}(F_{1,\theta}(x_1,G_\theta(x_1))) - N_{2,\theta}(x_1,G_\theta(x_1))\right) \\ =& A_{2,\theta}^{-1}\left(G_{\theta+\omega}(A_{1,\theta}x_1 + B_\theta G_\theta(x_1) + N_{1,\theta}(x_1,G_\theta(x_1))) - N_{2,\theta}(x_1,G_\theta(x_1))\right) \\ =& \mathcal{U}(G)_\theta(x_1) \ .\end{aligned}$$

In Remark 4.10 we showed how to solve this equation up to order L. We found a polynomial $G^{\leq}_\theta$ of degree L and coefficients of class C^r such that

$$G^{\leq}_\theta(x_1) = \mathcal{U}(G^{\leq})_\theta(x_1) + o(|x_1|^L) \ .$$

This polynomial is unique.

So, we obtain a fixed point equation for the higher order terms of the graph $x_2 = G^{\leq}_\theta(x_1) + G^{>}_\theta(x_1)$,

$$\begin{aligned} G^{>}_\theta(x_1) &= -G^{\leq}_\theta(x_1) + \mathcal{U}(G^{\leq} + G^{>})_\theta(x_1) \\ &= \mathcal{V}(G^{>})_\theta(x_1) \ . \end{aligned} \tag{69}$$

We will see that this equation has at most one solution $G^{>} : B_1 \subset E_1 \to E_2$ such that

$$\begin{aligned} [G^{>}]_{L+1} = &\sup_{(x_1,\theta)\in B_1} \frac{|G^{>}_\theta(x_1)|}{|x_1|^{L+1}} \\ &< \infty \ . \end{aligned}$$

In fact, we will fix $B_1 = B_{E_1}(1)$ and some smallness conditions on B, N, etc. using the scaling arguments in Section 4.3.1 and Section 4.3.3.

Assume that there are two solutions $G^1 = G^{\leq} + G^{1,>}$ and $G^2 = G^{\leq} + G^{2,>}$ of (68) (or $G^{1,>}$ and $G^{2,>}$ of (69)). Then, for $(x_1,\theta) \in B_1$,

$$\begin{aligned} &|(\mathcal{U}(G^2))_\theta(x_1) - (\mathcal{U}(G^1))_\theta(x_1)| \\ &\quad \leq \|A^{-1}_{2,\theta}\| \Big[|G^{\leq}_{\theta+\omega}(F_{1,\theta}(G^2_\theta(x_1))) - G^{\leq}_{\theta+\omega}(F_{1,\theta}(G^1_\theta(x_1)))| \\ &\qquad + |G^{2,>}_{\theta+\omega}(F_{1,\theta}(G^2_\theta(x_1))) - G^{1,>}_{\theta+\omega}(F_{1,\theta}(G^2_\theta(x_1)))| \\ &\qquad + |G^{1,>}_{\theta+\omega}(F_{1,\theta}(G^2_\theta(x_1))) - G^{1,>}_{\theta+\omega}(F_{1,\theta}(G^1_\theta(x_1)))| \\ &\qquad + |N_{2,\theta}(x_1, G^2_\theta(x_1)) - N_{1,\theta}(x_1, G^1_\theta(x_1))| \Big] \\ &\quad \leq \|A^{-1}_{2,\theta}\| \Big[\mathrm{Lip} G^{\leq}_{\theta+\omega}(\|B_\theta\| + \mathrm{Lip}_{x_2} N_{1,\theta}) |G^{2,>}_\theta(x_1) - G^{1,>}_\theta(x_1)| \\ &\qquad + [G^{2,>} - G^{1,>}]_{L+1} |F_{1,\theta}(G^2_\theta(x_1))|^{L+1} \\ &\qquad + \mathrm{Lip} G^{1,>}_{\theta+\omega}(\|B_\theta\| + \mathrm{Lip}_{x_2} N_{1,\theta}) |G^{2,>}_\theta(x_1) - G^{1,>}_\theta(x_1)| \\ &\qquad + \mathrm{Lip}_{x_2} N_{2,\theta} |G^{2,>}_\theta(x_1) - G^{1,>}_\theta(x_1)| \Big] \ . \end{aligned}$$

Notice that we can get

$$\begin{aligned} |F_{1,\theta}(G^2_\theta(x_1))| &= |A_{1,\theta} x_1 + B_\theta G^2_\theta(x_1) + N_{1,\theta}(x_1, G^2_\theta(x_1))| \\ &\leq (\|A_1\|_{C^0} + \varepsilon)|x_1| \end{aligned}$$

by using smallness assumptions on B and N. The bound depends also on $G^{2,>}$. So, again using the smallness assumptions that will depend also on

$G^{1,>}$, we have

$$
\begin{aligned}
[\mathcal{V}(G^{2,>}) - \mathcal{V}(G^{1,>})]_{L+1} = &\sup_{(x_1,\theta)\in B_1} \frac{|\mathcal{U}(G^2)_\theta(x_1) - \mathcal{U}(G^1)_\theta(x_1)|}{|x_1|^{L+1}} \\
&\leq \|A_2^{-1}\|_{C^0}(\varepsilon + (\|A_1\|_{C^0} + \varepsilon)^{L+1})[G^{2,>} - G^{1,>}]_{L+1} \\
\leq \nu[G^{2,>} - G^{1,>}]_{L+1}
\end{aligned}
$$

for some $\nu < 1$. So $G^{2,>} = G^{1,>}$, and the proof of Theorem 4.1 is finished.

5. Results for flows

5.1. **Reduction of the results for flows to results for maps.** The results proved for discrete time maps imply results for the discrete time problem.

If $F_{t,\theta} = F_t(\cdot,\theta)$ is the time t flow of the the vector field X given in (2) that is,

$$
\begin{aligned}
&\frac{d}{dt}F_t(x,\theta) = X(F_t(x,\theta), \theta + t\omega) \\
&F_0(x,\theta) = x
\end{aligned}
\tag{70}
$$

we see that if a torus is invariant for the vector field, it is invariant for the time-one map, F_1.

If F_1 has hyperbolicity properties, then, it is possible to use Theorem 3.1 to study tori invariant under F_1 and Theorem 4.1 to study their invariant manifolds. This provides with candidates for invariant tori and invariant manifolds for the vector field.

We want to argue that, given the uniqueness properties that we have found for invariant tori, the tori and manifolds which are invariant under F_1 have to be invariant for the whole flow.

Recall that

$$
\begin{aligned}
F_{t+s,\theta}(x) = &F_{t,\theta+s\omega} \circ F_{s,\theta}(x) \\
= &F_{s,\theta+t\omega} \circ F_{t,\theta}(x)\ .
\end{aligned}
\tag{71}
$$

If for a fixed $s \in \mathbb{R}\setminus\{0\}$, $K_s : \mathbb{T}^d \to \mathbb{R}^n$ is a solution of (5) for F_s, that is

$$
F_s(K_s(\theta),\theta) = K_s(\theta + s\omega),
\tag{72}
$$

then the torus $\mathcal{K}_s = \{(K_s(\theta),\theta) \mid \theta \in \mathbb{T}^d\}$ is invariant under the s-time map F_s: $F_{s,\theta}(\mathcal{K}_{s,\theta}) = \mathcal{K}_{s,\theta+s\omega}$. Then, for any $t \in \mathbb{R}$ $K_{s,t} : \mathbb{T}^d \to \mathbb{R}^n$ defined by

$$
K_{s,t}(\theta) = F_t(K_s(\theta - t\omega), \theta - t\omega)
$$

parameterizes a torus $\mathcal{K}_{s,t} = F_{t,\theta-t\omega}(\mathcal{K}_{s,\theta-t\omega})$ such that

$$
\begin{aligned}
F_{s,\theta}(\mathcal{K}_{s,t,\theta}) &= F_{s,\theta} \circ F_{t,\theta-t\omega}(\mathcal{K}_{s,\theta-t\omega}) \\
&= F_{t,\theta+(s-t)\omega} \circ F_{s,\theta-t\omega}(\mathcal{K}_{s,\theta-t\omega}) \\
&= F_{t,\theta+(s-t)\omega}(\mathcal{K}_{s,\theta+(s-t)\omega}) \\
&= \mathcal{K}_{s,t,\theta+s\omega}\ .
\end{aligned}
\tag{73}
$$

We see that $K_{s,t}(\theta) = F_t(K_s(\theta - t\omega), \theta - t\omega)$ satisfies the same equation (72). Therefore, given the uniqueness properties of K obtained in Theorem 3.1 we obtain that for all $|t|$ sufficiently small, $F_t(K_s(\theta - t\omega), \theta - t\omega) = K_s(\theta)$, equivalently,

$$F_t(K_s(\theta), \theta) = K_s(\theta + t\omega)$$

Repeating the application of the above equation, we obtain that for any integer n

$$F_{nt}(K_s(\theta), \theta) = K_s(\theta + nt\omega)$$

Hence, using the uniqueness statements in Theorem 3.1, we have shown that the solutions for one time of (72) are invariant under the flow.

We can treat analogously the solutions for a fixed s of

$$F_s(W_s(\eta, \theta), \theta) = W_s(\Lambda_s(\eta, \theta), \theta + s\omega). \tag{74}$$

The key point is using uniqueness of the invariant manifold (is the parameterization which is not unique). So, assume we have an invariant manifold $\mathcal{W}_s = \{(W_s(\eta, \theta) \mid (\eta, \theta) \in U_1\}$ of F_s, where U_1 is a tubular neighborhood of the zero section of E_1, that is $F_{s,\theta}(\mathcal{W}_{s,\theta}) \subset \mathcal{W}_{s,\theta+s\omega}$, and the tangent bundle of $\mathcal{W}_s$ over $\mathcal{K}$ is $T_{\mathcal{K}}\mathcal{W}_s = E_1$. We assume also that the subbundle E_1 is invariant under the whole flow, not just F_s, which means that $\mathrm{D}_x F_{t,\theta}(\mathcal{K}_\theta) E_{1,\theta} = E_{1,\theta+t\omega}$ for all time t, and that E_1 satisfies the hypothesis of Theorem (4.1) for F_s.

Then, for $\mathcal{W}_{s,t}$ defined by $\mathcal{W}_{s,t,\theta} = F_{t,\theta-t\omega}(\mathcal{W}_{s,\theta-t\omega})$ we have

$$F_{s,\theta}(\mathcal{W}_{s,t,\theta}) \subset \mathcal{W}_{s,t,\theta+s\omega} \ . \tag{75}$$

So, $\mathcal{W}_{s,t}$ is invariant under F_s. Moreover,

$$\begin{aligned} T_{\mathcal{K}_\theta}\mathcal{W}_{s,t,\theta} &= \mathrm{D}_x F_t(\mathcal{K}_{\theta-t\omega}) T_{\mathcal{K}_{\theta-t\omega}} \mathcal{W}_{s,\theta-t\omega} \\ &= \mathrm{D}_x F_t(\mathcal{K}_{\theta-t\omega}) E_{1,\theta-t\omega} \\ &= E_{1,\theta} \ , \end{aligned}$$

and the uniqueness of the invariant manifold established in Theorem 4.1 gives that $\mathcal{W}_{s,t} = \mathcal{W}_s$. Repeating the previous arguments given for $\mathcal{K}_s$ we see that $\mathcal{W}_s$ is invariant under the flow.

5.2. **The Poincaré trick.** We can also derive the results for flows from the results for maps using the Poincaré trick. To do so, we split the angle variables as $\theta = (\varphi, \theta_d) \in \mathbb{T}^{d-1} \times \mathbb{T}$. We also write the frequency vector as $\omega = \omega_d(\alpha, 1)$. Hence, the Poincaré map, with respect to the angle variable θ_d is

$$\begin{aligned} \bar{x} &= f(x, \varphi) = F_{\frac{1}{\omega_d}}(x, (\varphi, 0)) \ , \\ \bar{\varphi} &= \varphi + \alpha \ . \end{aligned} \tag{76}$$

This in a skew product in $\mathbb{R}^n \times \mathbb{T}^{d-1}$, over the rotation $\alpha \in \mathbb{R}^{d-1}$.

If $k(\varphi)$ is an invariant torus for (76), that is $f(k(\varphi), \varphi) = k(\varphi + \alpha)$, then the torus $K(\theta)$ defined by

$$K(\varphi, \theta_d) = F_{\frac{\theta_d}{\omega_d}}(k(\varphi - \theta_d\alpha), (\varphi - \theta_d\alpha, 0))$$

is invariant under the whole system (2). Notice that the torus is well defined (the definition does not depend on the representative of $\theta_d \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$).

If $w(\eta, \varphi)$ is a whisker of $k(\varphi)$ for (76), and the dynamics is given by $\lambda(\eta, \varphi)$, that is $f(w(\eta, \varphi)) = w(\lambda(\eta, \varphi), \varphi + \alpha)$, then the parameterization

$$\hat{W}(\eta, \varphi, s) = F_{\frac{s}{\omega_d}}(w(\eta, \theta), (\varphi, 0))$$

covers a whisker of the torus K.

5.3. **A direct treatment of the differential equations case.** In spite of the fact that we have shown that the rigorous results for flows can be deduced from the results for maps, it is instructive to sketch a direct treatment. We can obtain the invariance equations either in integral form or differential form.

A torus $K(\theta)$ is invariant under (2) if

$$F_t(K(\theta), \theta) = K(\theta + t\omega) \ , \tag{77}$$

for all $t \in \mathbb{R}$. Notice that, for ω ergodic, it is enough solving the equation for a given $t \neq 0$. The differential form of (77) is

$$X(K(\theta), \theta) = \mathrm{D}K(\theta)\omega \ . \tag{78}$$

The integral form of (77) is

$$K(\theta + t\omega) = K(\theta) + \int_0^t X(K(\theta + s\omega), \theta + s\omega) ds \tag{79}$$

For a whisker of the torus, $W(\eta, \theta)$, the invariance is given by

$$F_t(W(\eta, \theta), \theta) = W(\Lambda_t(\eta, \theta), \theta + t\omega) \ , \tag{80}$$

where Λ_t is a flow on the manifold, such that $\Lambda_t(0, \theta) = 0$. That is, there exists a vector field

$$\begin{aligned} \dot{\eta} &= A(\eta, \theta) \ , \\ \dot{\theta} &= \omega \ , \end{aligned}$$

such that

$$\begin{aligned} &\frac{d}{dt}\Lambda_t(\eta, \theta) = A(\Lambda_t(\eta, \theta), \theta + t\omega), \\ &\Lambda_0(\eta, \theta) = \eta \ . \end{aligned}$$

The infinitesimal version of (80) is the equation

$$X(W(\eta, \theta), \theta) = \mathrm{D}_\eta W(\eta, \theta) A(\eta, \theta) + \mathrm{D}_\theta W(\eta, \theta)\omega \ , \tag{81}$$

where W and A are unknown functions, and $A(0,\theta)=0$. The integral form is given by

(82)

$$\begin{aligned} W(\Lambda_t(\eta,\theta),\theta+t\omega) =& W(\eta,\theta)+\int_0^t X(W(\Lambda_s(\eta,\theta),\theta+s\omega),\theta+s\omega)ds\ , \\ \Lambda_t(\eta,\theta) =& \eta+\int_0^t A(\Lambda_s(\eta,\theta),\theta+s\omega)ds\ . \end{aligned}$$

5.3.1. *The one dimensional case.* For the sake of simplicity, we will analyze here the simplest case in which the whisker is one dimensional (in the vertical variable), and trivial as a bundle over the torus. Moreover, we will assume that the dynamics on the manifold can be reduced to constant coefficients $\dot\eta=\lambda\eta$. In this case, (81) reads

$$X(W(\eta,\theta),\theta)=\mathrm{D}_\eta W(\eta,\theta)\lambda\eta+\mathrm{D}_\theta W(\eta,\theta)\omega\ , \tag{83}$$

where the unknowns are W and λ.

Similarly as was done in Section 2.3, we write

$$W(\eta,\theta)=W^{\le}(\eta,\theta)+W^{>}(\eta,\theta)\ ,$$

where

$$W^{\le}(\eta,\theta)=\sum_{i=0}^{L}W^i(\theta)\eta^i$$

and the high order part of the function W satisfies

$$\frac{\partial^i W^{>}}{\partial\eta^i}(0,\theta)=0 \text{ for } i=0,\dots,L.$$

We seek the coefficients $W^0,\dots,W^L$ of $W^{\le}$ and the remainder $W^{>}$ from the equation (2), which leads to a hierarchy

$$\begin{aligned} & X(W^0(\theta),\theta)=\mathrm{D}_\theta W^0(\theta)\omega\ \text{, which gives } W^0(\theta)=K(\theta) \\ & \mathrm{D}_x X(K(\theta),\theta)W^1(\theta)=\lambda W^1(\theta)+\mathrm{D}_\theta W^1(\theta)\omega \\ & \mathrm{D}_x X(K(\theta),\theta)W^2(\theta)+P^2(\theta)=2\lambda W^2(\theta)+\mathrm{D}_\theta W^2(\theta)\omega \\ & \vdots \\ & \mathrm{D}_x X(K(\theta),\theta)W^L(\theta)+P^L(\theta)=L\lambda W^L(\theta)+\mathrm{D}_\theta W^L(\theta)\omega, \end{aligned} \tag{84}$$

where P^i stands for a polynomial expression in $W^1,\dots,W^{i-1}$ for $i=2,\dots L$ whose coefficients are derivatives of X of order up to i evaluated at $(W^0(\theta),\theta)$.

The high order part $W^{>}$ satisfies

$$\mathrm{D}_x X(K(\theta),\theta)W^{>}(\eta,\theta)+P^{>}(\eta,\theta)=\lambda\mathrm{D}_\eta W^{>}(\eta,\theta)+\mathrm{D}_\theta W^{>}(\eta,\theta)\omega\ , \tag{85}$$

where $P^{>}$ contains terms which vanish to order higher than L.

The hierarchy of equations (84) can be solved by recursion in the degree of the polynomials matched, provided that some non-resonance conditions are satisfied.

The analysis of these equations is harder than the corresponding analysis in the discrete case, due to the appearance of derivatives that "change" the space of the functions in the left and right hand side. In numerical applications, these equations can be solved by using Fourier expansions, up to arbitrarily high degree L.

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