KAM FOR THE NON-LINEAR SCHRÖDINGER EQUATION

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Abstract. We consider the d-dimensional nonlinear Schrödinger equation under periodic boundary conditions:

$$-i\dot{u} = \Delta u + V(x) * u + \varepsilon \frac{\partial F}{\partial \bar{u}}(x, u, \bar{u}), \quad u = u(t,x),\ x \in \mathbb{T}^d$$

where $V(x) = \sum \hat{V}(a) e^{i\langle a,x\rangle}$ is an analytic function with $\hat{V}$ real and F is a real analytic function in $\Re u$, $\Im u$ and x. (This equation is a popular model for the 'real' NLS equation, where instead of the convolution term $V * u$ we have the potential term Vu.) For $\varepsilon = 0$ the equation is linear and has time–quasi-periodic solutions u,

$$u(t,x) = \sum_{s \in \mathcal{A}} \hat{u}_0(a) e^{i(|a|^2 + \hat{V}(a))t} e^{i\langle a,x\rangle}, \quad 0 < |\hat{u}_0(a)| \le 1,$$

where $\mathcal{A}$ is any finite subset of $\mathbb{Z}^d$. We shall treat $\omega_a = |a|^2 + \hat{V}(a)$, $a \in \mathcal{A}$, as free parameters in some domain $U \subset \mathbb{R}^{\mathcal{A}}$.

This is a Hamiltonian system in infinite degrees of freedom, degenerate but with external parameters, and we shall describe a KAM-theory which, in particular, will have the following consequence:

If $|\varepsilon|$ is sufficiently small, then there is a large subset U' of U such that for all $\omega \in U'$ the solution u persists as a time–quasi-periodic solution which has all Lyapounov exponents equal to zero and whose linearized equation is reducible to constant coefficients.

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1. Introduction

We consider the d-dimensional nonlinear Schrödinger equation

$$-i\dot{u} = \Delta u + V(x) * u + \varepsilon \frac{\partial F}{\partial \bar{u}}(x, u, \bar{u}), \quad u = u(t, x) \quad (*)$$

under the periodic boundary condition $x \in \mathbb{T}^d$. The convolution potential $V : \mathbb{T}^d \to \mathbb{C}$ must have real Fourier coefficients $\hat{V}(a),\ a \in \mathbb{Z}^d$, and we shall suppose it is analytic. F is an analytic function in $\Re u$, $\Im u$ and x.

The non-linear Schrödinger as an ∞-dimensional Hamiltonian system. If we write

$$\begin{cases} u(x) = \sum_{a\in\mathbb{Z}^d} u_a e^{i<a,x>} \\ \overline{u(x)} = \sum_{a\in\mathbb{Z}^d} v_a e^{i<a,x>}, \end{cases}$$

and let

$$\zeta_a = \begin{pmatrix} \xi_a \\ \eta_a \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(u_a + v_a) \\ \frac{-i}{\sqrt{2}}(u_a - v_a) \end{pmatrix},$$

then, in the symplectic space

$$\{(\xi_a, \eta_a) : a \in \mathbb{Z}^d\} = \mathbb{C}^{\mathbb{Z}^d} \times \mathbb{C}^{\mathbb{Z}^d}, \quad \sum_{a\in\mathbb{Z}^d} d\xi_a \wedge d\eta_a,$$

the equation becomes a real Hamiltonian system with an integrable part

$$\frac{1}{2}\sum_{a\in\mathbb{Z}^d}(|a|^2 + \hat{V}(a))(\xi_a^2 + \eta_a^2)$$

plus a perturbation.

Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^d$ and fix

$$0 < p_a(0), \quad a \in \mathcal{A}$$

The $(\#\mathcal{A})$-dimensional torus

$$\begin{array}{ll} \frac{1}{2}(\xi_a^2 + \eta_a^2) = p_a(0) & a \in \mathcal{A} \\ \xi_a = \eta_a = 0 & a \in \mathcal{L} = \mathbb{Z}^d \setminus \mathcal{A}, \end{array}$$

is invariant for the Hamiltonian flow when $\varepsilon = 0$. In a neighborhood of this torus we introduce action-angle variables (φ_a, r_a)

$$\begin{aligned} \xi_a &= \sqrt{2(r_a(0) + r_a)}\cos(\varphi_a) \\ \eta_a &= \sqrt{2(r_a(0) + r_a)}\sin(\varphi_a). \end{aligned}$$

The integrable Hamiltonian now becomes

$$h = \sum_{a\in\mathcal{A}} \omega_a r_a + \frac{1}{2}\sum_{a\in\mathcal{L}} \Omega_a(\xi_a^2 + \eta_a^2),$$

where

$$\omega_a = |a|^2 + \hat{V}(a), \quad a \in \mathcal{A},$$

are the *basic frequencies*, and

$$\Omega_a = |a|^2 + \hat{V}(a), \quad a \in \mathcal{L},$$

are the *normal frequencies* (of the invariant torus). The perturbation $\varepsilon f(\varphi, r, \xi, \eta)$ will be a function of all variables.

This is a standard form for the perturbation theory of lower-dimensional (isotropic) tori with one exception: it is strongly degenerate. We therefore need external parameters to control the basic frequencies and the simplest choice is to let the basic frequencies (i.e. the potential itself) be our free parameters.

The parameters will belong to a set

$$U \subset \{\omega \in \mathbb{R}^{\mathcal{A}} : |\omega| \le C\}\,.$$

The normal frequencies will be assumed to verify

$$\begin{array}{ll} |\Omega_a| \ge C' & \forall\, a,b \in \mathcal{L}\,, \\ |\Omega_a + \Omega_b| \ge C' & \forall\, a,b \in \mathcal{L}\,, \\ |\Omega_a - \Omega_b| \ge C' & \forall\, a,b \in \mathcal{L}, |a| \ne |b|. \end{array}$$

We define the complex domain

$$\mathcal{O}^\gamma(\sigma,\rho,\mu) = \left\{ \begin{array}{l} \|\zeta\|_\gamma = \sqrt{\sum_{a\in\mathcal{L}}(|\xi_a|^2 + |\eta_a|^2)\langle a\rangle^{2m_*}e^{2\gamma|a|}} < \sigma \\ |\Im\varphi| < \rho \\ |r| < \mu, \end{array} \right.$$

$\langle a\rangle = \max(|a|,1)$ and $m_* > \frac{d}{2}$. In this space the Hamitonian equations have a well-defined local flow.

Theorem A. *Under the above assumptions, for ε sufficiently small there exist a subset $U' \subset U$, which is large in the sense that*

$$\mathrm{Leb}\,(U \setminus U') \le \mathrm{cte}.\varepsilon^{exp_1}\,,$$

and for each $\omega \in U'$, a real analytic symplectic diffeomorphism Φ

$$\mathcal{O}^0(\frac{\sigma}{2},\frac{\rho}{2},\frac{\mu}{2}) \to \mathcal{O}^0(\frac{\sigma}{2}+\varepsilon^{1/2},\frac{\rho}{2}+\varepsilon^{1/2},\frac{\mu}{2}+\varepsilon^{1/2})$$

and a vectro ω' such that $(h_{\omega'}+\varepsilon f)\circ\Phi$ equals

$$c + <\omega, r> + \frac{1}{2}<\xi, Q_1\xi> + <\xi, Q_2\eta> + \frac{1}{2}<\eta, Q_1\eta> + \varepsilon f'\,,$$

where

$$f' \in \mathcal{O}(|r|^2\,, |r|\,\|\zeta\|_0\,, \|\zeta\|_0^3)$$

and $Q = Q_1 + iQ_2$ is a Hermitian and block-diagonal matrix with finite-dimensional blocks.

The consequences of the theorem are well-known. The dynamics of the Hamiltonian vector field of $h_{\omega'} + \varepsilon f$ on $\Phi(\{0\}\times\mathbb{T}^d\times\{0\})$ is the same as that of

$$<\omega, r> + \frac{1}{2}<\xi, Q_1\xi> + <\xi, Q_2\eta> + \frac{1}{2}<\eta, Q_1\eta>\ .$$

The torus $\{\zeta = r = 0\}$ is invariant, since the Hamiltonian vector field on it is

$$\left\{ \begin{array}{l} \dot\zeta = 0 \\ \dot\varphi = \omega \\ \dot r = 0, \end{array} \right.$$

and the flow on the torus is linear

$$t \mapsto \varphi + t\omega.$$

Moreover, the linearized equation on this torus becomes

$$\begin{cases} \dot{\zeta} = \begin{pmatrix} {}^tQ_2(\omega) & Q_1(\omega) \\ -Q_1(\omega) & -Q_2(\omega) \end{pmatrix} \zeta + a(\varphi + t\omega, \omega) r \\ \dot{\varphi} = <a(\varphi + t\omega, \omega), \zeta> + c(\varphi + t\omega, \omega) r \\ \dot{r} = 0, \end{cases}$$

where $a = J\varepsilon\partial_r\partial_\zeta f'$ and $b = \varepsilon\partial_r^2 f'$. Since $Q_1 + iQ_2$ is Hermitian and block diagonal the eigenvalues of the ζ-linear part are purely imaginary

$$\pm i\Omega'_a, \quad a \in \mathcal{L}.$$

The linearized equation is reducible to constant coefficients if the eigenvalues Ω'_a are non-resonant with respect to ω, something which can be assumed of we restrict the set U' arbitrarily little. Then the ζ-component (and of course also the r-component) will have only quasi-periodic (in particular bounded) solutions. The φ-component may have a linear growth in t, the growth factor (the "twist") being linear in r.

Reducibility. Reducibility is not only an important outcome of KAM but also an essential ingredient in the proof. It simplifies the iteration since it makes possible to reduce all approximate linear equations to constant coefficients. But it does not come for free. It requires a lower bound on small divisors of the form

$$(**) \qquad |<k, \omega> + \Omega'_a - \Omega'_b|, \quad k \in \mathbb{Z}^{\mathcal{A}}, \ a, b \in \mathcal{L}.$$

The basic frequencies ω will be keept fixed during the iteration – that's what the parameters are there for – but the normal frequencies will vary. Indeed $\Omega'_a(\omega)$ and $\Omega'_b(\omega)$ are perturbations of Ω_a and Ω_b which are not known a priori but are determined by the approximation process. [1]

This is a lot of conditions for a few parameters ω. It is usually possible to make a (scale dependent) restriction of $(**)$ to

$$|k|, \ |a - b| \leq \Delta = \Delta_\varepsilon$$

which improves the situation a bit. Indeed, in one space-dimension ($d = 1$) it improves a lot, and $(**)$ reduces to only finitely many conditions. Not so however when $d \geq 2$, in which case the number of conditions in $(**)$ remains infinite.

[1] A lower bound on $(**)$, often known as the second Melnikov condition, is strictly speaking not necessary at all for reducibility. It is necessary, however, or reducibility with a reducing transformation close to the identity.

To cope with this problem we shall exploit the Töplitz-Lipschitz-property which allows for a sort of compactification of the dimensions and reduces the infinitely many conditions $(**)$ to finitely many. These can then be controlled by an appropriate choice of ω.

The Töplitz-Lipschitz property. The Töplitz-Lipschitz property is defined for infinite-dimensional matrices with exponential decay. We say that a matrix

$$A : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$$

is Töplitz at ∞ if, for all $a, b, c \in \mathbb{Z}^d$ the limit

$$\lim_{t\to\infty} A_{a+tc}^{b+tc} \ \exists \quad =: \ A_a^b(c).$$

The Töplitz-limit $A(c)$ is a new matrix which is c-invariant

$$A_{a+c}^{b+c}(c) = A_a^b(c).$$

So it is a simpler object because it is "more constant".

The approach to the Töplitz-limit in direction c is controlled by a Lipschitz-condition. This control does not take place everywhere,but on a certain subset

$$D_\Lambda(c) \in \mathcal{L} \times \mathcal{L}$$

– the Lipschitz domain. Λ is a parameter which, together with $|c|$, determines the size of the domain.

The Töplitz-Lipschitz property permits us to verify certain bounds of the matrix-coefficients or functions of these, like determinants of sub-matrices, in the Töplitz-limit and then recover these bounds for the matrix restricted to the Lipschitz domain.

The matrices we shall consider will not be scalar-valued but $gl(2, \mathbb{C})$-valued

$$A : \mathcal{L} \times \mathcal{L} \to gl(2, \mathbb{C})$$

and we shall define a Töplitz-Lipschitz property for such matrices also. These matrices constitute an algebra: one can multiply them and solve linear differential equations. A function f is said to the have the Töplitz-Lipschitz property if its Hessian (with respect to ζ) is Töplitz-Lipschitz. If this is the case, as it is for the perturbation f of the non-linear Schrödinger, then this is also true of the linear part of our KAM–transformations and for the transformed Hamiltonian. This will permit us to formulate an inductive statement which, as usual in KAM, gives Theorem A.

Some references. For finite dimensional Hamiltonian systems the first proof of persistence of stable (i.e. vanishing of all Lyapunov exponents) lower dimensional invariant tori was obtained in [Eli85, Eli88] and there are now many works on this subjects. There are also many works on reducibility (see for example [Kri99, Eli01]) and the situation in finite dimension is now pretty well understood. Not so, however, in infinite dimension.

If $d = 1$ and the space-variable x belongs to a finite segment supplemented by Dirichlet or Neumann boundary conditions, this result was obtained in [Kuk88] (also see [Kuk93, Pös96]). The case of periodic boundary conditions was treated in [Bou96], using another multi–scale scheme, suggested by Fröhlich–Spencer in their work on the Anderson localization [FS83]. This approach, often referred to as the Craig-Wayne scheme, is different from KAM. It avoids the, sometimes, cumbersome condition ($**$) but to a high cost: the approximate linear equations are not of constant coefficients. Moreover, it gives persistence of the invariant tori but no reducibility and no information on the linear stability. A KAM-theorem for periodic boundary conditions has recently been proved in [GY05] (with a perturbation F independent of x) and the perturbation theory for quasi-periodic solutions of one-dimensional Hamiltonian PDE is now sufficiently well developed (see for example [Kuk93, Cra00, Kuk00]).

The study of the corresponding problems for $d \geq 2$ is at its early stage. Developing further the scheme, suggested by Fröhlich–Spencer, Bourgain proved persistence for the case $d = 2$ [Bou98]. More recently, the new techniques developped by him and collaborators in their work on the linear problem has allowed him to prove persistence in any dimension d[Bou04]. (In this work he also treats the wave equation.)

Description of the paper. The paper is divided into three parts. The first part deals with linear algebra of Töplitz-Lipschitz matrices and the analysis of functions with the Töplitz-Lipschitz property. In Section 2 we introduce Töplitz-Lipschitz matrices and prove a product formula. This part is treated in greater generality in [EK05]. In Section 3 we analyze functions with the Töplitz-Lipschitz property.

The second part deals with the small divisor condition (**) which occurs in the solution of the homological equation. In Section 4 we analyze the block decomposition of the lattice $\mathbb{Z}^d$ and in Section 4 we study the small divisors. In Section 6 we solve the homological equations. This part is independent of the first part except for basic definitions and properties given in Sections 2.3 and 2.4.

The third part treats KAM-theory with Töplitz-Lipschitz property and contains a general KAM-theorem, Theorem 7.1. This theorem is applied to the non-linear Schrödinger to give Theorem 7.2 of which the theorem above is a variant.

Notations. $<,>$ is the standard scalar product in $\mathbb{R}^d$. $\| \ \|$ is an operator or l^2-norm. $| \ |$ will in general denote a supremum norm, with a notable exception: for a lattice vector $a \in \mathbb{Z}^d$ we use $|a|$ for the l^2-norm.

$\mathcal{L}$ and $\mathcal{A}$ are subsets of $\mathbb{Z}^d$, $\mathcal{A}$ being finite. A matrix on $\mathcal{L}$ is just a mapping $A : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ or $gl(2, \mathbb{C})$. Its components will be denoted A_a^b.

The dimension d will be fixed and m_* will be a fixed constant $> \frac{d}{2}$.

$\lesssim$ means $\leq$ modulo a multiplicative constant that only, unless otherwise specified, depends on d, m_* and $\#\mathcal{A}$.

The points in the lattice $\mathbb{Z}^d$ will be denoted $a, b, c, \ldots$. Also d will sometimes be used, without confusion we hope.

For a vector $c \in \mathbb{Z}^d$, $c^\perp$ will denote the $\perp$ complement of c in $\mathbb{Z}^d$ or in $\mathbb{R}^d$, depending on the context. If $c \neq 0$, for any $a \in \mathbb{Z}^d$ we let

$$\underline{a_c} \in (a + \mathbb{R}c) \cap \mathbb{Z}^d$$

be the lattice point b on the line $a + \mathbb{R}c$ with smallest norm, i.e. that minimizes

$$|<b, c>|$$

– if there are two such b's we choose the one with $<b, c> \geq 0$. It is the"$\perp$ projection of a to $c^\perp$".

Greek letter $\alpha, \beta, \ldots$ will mostly be used for bounds. Exceptions are φ which will denote an element in the torus – an angle – and ω, Ω.

For two subsets X and Y of a metric space,

$$\underline{\text{dist}}(X, Y) = \inf_{x \in X, y \in Y} d(x, y).$$

(This is not a metric.) X_ε is the ε-neighborhood of X, i.e.

$$\{y : \underline{\text{dist}}(y, X) < \varepsilon\}.$$

Let $B_\varepsilon(x)$ be the ball $\{y : d(x, y) < \varepsilon\}$. Then X_ε is the union, over $x \in X$, of all $B_\varepsilon(x)$.

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PART I. THE TÖPLITZ-LIPSCHITZ PROPERTY

In this part we consider

$$\mathcal{L} \subset \mathbb{Z}^d$$

and matrices $A : \mathcal{L} \times \mathcal{L} \to gl(2, \mathbb{C})$. We define: the sup-norms $|\cdot|_\gamma$; the notion of being Töplitz at ∞; the Lipschitz-domains $D_\Delta^\pm(c)$; the Lipschitz- norm $< \cdot >_{\Lambda,\gamma}$ and the notion of being Töplitz-Lipschitz. (For a more general exposition see [EK05].) We define the Töplitz-Lipschitz property for functions and the norms $[\cdot]_{\Lambda,\gamma,\sigma}$.

2. Töplitz-Lipschitz matrices

2.1. **Spaces and matrices.**

We denote by $l^2_\gamma(\mathcal{L}, \mathbb{C})$, $\gamma \geq 0$, the following weighted l_2-spaces:

$$l^2_\gamma(\mathcal{L}, \mathbb{C}) = \{\zeta = (\xi, \eta) \in \mathbb{C}^{\mathcal{L}} \times \mathbb{C}^{\mathcal{L}} : \|\zeta\|_\gamma < \infty\},$$

where

$$\|\zeta\|^2_\gamma = \sum_{a\in\mathcal{L}} (|\xi_a|^2 + |\eta_a|^2) e^{2\gamma|a|} \langle a \rangle^{2m_*}, \quad \langle a \rangle = \max(|a|, 1).$$

We provide $l^2_\gamma(\mathcal{L}, \mathbb{C})$ with the symplectic form

$$\sum_{a\in\mathcal{L}} d\xi_a \wedge d\eta_a.$$

Using the pairing

$$<\zeta, \zeta'> = \sum_{a\in\mathcal{L}} (\xi_a \xi'_a + \eta_a \eta'_a)$$

we can write the symplectic form as

$$<\cdot, J\cdot>$$

where $J : l^2_\gamma(\mathcal{L}, \mathbb{C})\gamma \to l^2_\gamma(\mathcal{L}, \mathbb{C})$ is the standard involution.

We consider the space $gl(2, \mathbb{C})$ of all complex 2×2-matrices provided with the scalar product

$$Tr({}^t\bar{A}B).$$

Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

and consider the orthogonal projection π of $gl(2, \mathbb{C})$ onto the subspace

$$M = \mathbb{C}I + \mathbb{C}J.$$

It is easy to verify that

$$\begin{cases} M \times M, M^\perp \times M^\perp \subset M \\ M \times M^\perp, M^\perp \times M \subset M^\perp \end{cases}$$

and

$$\begin{cases} \pi(AB) = \pi A \pi B + (I-\pi)A(I-\pi)B \\ (I-\pi)(AB) = (I-\pi)A\pi B + \pi A(I-\pi)B. \end{cases}$$

If $A = (A_i^j)_{i,j=1}^2$ $B = (B_i^j)_{i,j=1}^2$ we define

$$[A] = (|A_i^j|)_{i,j=1}^2,$$

and

$$A \le B \iff |A_i^j| \le B_i^j, \quad \forall i, j.$$

Since any euclidean space E is naturally isomorphic to its dual E^*, the canonical relations

$$E \otimes E \simeq E^* \otimes E^* \simeq Hom(E, E^*) \simeq Hom(E, E)$$

permits the identification of the tensor product $\zeta \otimes \zeta'$ with a 2×2-matrix

$$(\zeta \otimes \zeta')_i^j = \zeta_i \zeta'_j.$$

2.2. **Matrices with exponential decay.**
Consider now an infinite-dimensional $gl(2,\mathbb{C})$-valued matrix

$$A : \mathcal{L} \times \mathcal{L} \to gl(2,\mathbb{C}), \quad (a,b) \mapsto A_a^b.$$

We define matrix multiplication through

$$(AB)_a^b = \sum_d A_a^d B_d^b,$$

and, for any subset $\mathcal{D}$ of $\mathcal{L} \times \mathcal{L}$, the semi-norms

$$|A|_{\mathcal{D}} = \sup_{(a,b)\in\mathcal{D}} \left| A_a^b \right|.$$

We define πA through

$$(\pi A)_a^b = \pi A_a^b, \quad \forall a, b.$$

Clearly we have

$$\begin{aligned} &\pi(A+B) = \pi A + \pi B \\ &\pi(AB) = \pi A \pi B + (I-\pi)A(I-\pi)B \\ &(I-\pi)(AB) = (I-\pi)A\pi B + \pi A(I-\pi)B. \end{aligned} \tag{1}$$

We define

$$A \le B \iff A_a^b \le B_a^b, \quad \forall a, b,$$

and

$$(\mathcal{E}_\gamma^{\pm} A)_a^b = [A_a^b] e^{\gamma|a\mp b|}, \quad \forall a, b.$$

All operators $\mathcal{E}_\gamma^{\pm}$ commute and we have

$$\begin{cases} \mathcal{E}_\gamma^x(A+B) \le \mathcal{E}_\gamma^x A + \mathcal{E}_\gamma^x B, & x \in \{+,-\} \\ \mathcal{E}_\gamma^{xy}(AB) \le (\mathcal{E}_\gamma^x A)(\mathcal{E}_\gamma^y B), & x, y \in \{+,-\}. \end{cases}$$

2

We define the norm

$$|A|_\gamma = \max(|\mathcal{E}^+_\gamma \pi A^b_a|_{\mathcal{L}\times\mathcal{L}}, |\mathcal{E}^-_\gamma (1-\pi) A^b_a|_{\mathcal{L}\times\mathcal{L}}).$$

We have, by *Young's inequality* (see [Fol76]), that

$$\|A\zeta\|_{\gamma'} \lesssim (\frac{1}{\gamma-\gamma'})^{d+m_*} |A|_\gamma \|\zeta\|_{\gamma'}, \quad \forall \gamma' < \gamma. \tag{2}$$

It follows that if $|A|_\gamma < \infty$, then A defines a bounded operator on any $l^2_{\gamma'}(\mathcal{L}, \mathbb{C}),\ \gamma' < \gamma$.

Truncations. Let

$$(\mathcal{T}^\pm_\Delta) A^b_a = \begin{cases} A^b_a & \text{if } |a \mp b| \le \Delta \\ 0 & \text{if not,} \end{cases}$$

and

$$\mathcal{T}_\Delta A = \mathcal{T}^+_\Delta \pi A + \mathcal{T}^-_\Delta (I-\pi) A.$$

It is clear that

$$|\mathcal{T}_\Delta A|_\gamma \le |A|_\gamma \quad \text{and} \quad |A - \mathcal{T}_\Delta A|_{\gamma'} \le e^{-\Delta(\gamma-\gamma')} |A|_\gamma. \tag{3}$$

Tensor products. For any two elements $\zeta, \zeta' \in l^2_\gamma(\mathcal{L}, \mathbb{C})$, their tensor product $\zeta \otimes \zeta'$ is a matrix on $\mathcal{L} \times \mathcal{L}$, and it is easy to verify that

$$|\zeta \otimes \zeta'|_\gamma \lesssim \|\zeta\|_\gamma \|\zeta'\|_\gamma. \tag{4}$$

Multiplication. We have

$$|AB|_{\gamma'} + |BA|_{\gamma'} \lesssim (\frac{1}{\gamma-\gamma'})^d |A|_\gamma |B|_{\gamma'}, \quad \forall \gamma' < \gamma. \tag{5}$$

Linear differential equation. Consider the linear system

$$\begin{cases} X' = A(t) X \\ X(0) = I. \end{cases}$$

It follows from (5) that the series

$$I + \sum_{n=1}^{\infty} \int_0^{t_0} \int_0^{t_1} \dots \int_0^{t_{n-1}} A(t_1) A(t_2) \dots A(t_n) dt_n \dots dt_2 dt_1,$$

as well as its derivative with respect to t_0, converges to a solution which verifies, for $\gamma' < \gamma$,

$$|X(t) - I|_{\gamma'} \lesssim (\gamma-\gamma')^d (\exp(\text{cte.}(\frac{1}{\gamma-\gamma'})^d |t| \alpha(t)) - 1), \tag{6}$$

[2] We use the sign convention that $xy = +$ whenever x and y are equal and $xy = -$ whenever they are different.

where

$$\alpha(t) = \sup_{0 \le |s| \le |t|} |A(s)|_\gamma .$$

2.3. **Töplitz-Lipschitz matrices ($d = 2$).**
A matrix

$$A : \mathcal{L} \times \mathcal{L} \to gl(2, \mathbb{C})$$

is said to be *Töplitz at* ∞ if, for all a, b, c, the two limits

$$\lim_{t \to +\infty} A_{a+tc}^{b \pm tc} \ \exists \ = A_a^b(\pm, c).$$

It is easy to verify that if $|A|_\gamma < \infty$ and $|B|_\gamma < \infty$, then

$$(\pi A)(-, c) = (I - \pi) A(+, c) = 0$$

and

$$\begin{array}{ll} & \pi(AB)(+,c) = \\ (7) & \pi A(+,c)\pi B(+,c) + (I-\pi)A(-,c)(I-\pi)B(-,-c) \\ & (I-\pi)(AB)(-,c) = \\ & (I-\pi)A(-,c)\pi B(+,-c) + \pi A(+,c)(I-\pi)B(-,c). \end{array}$$

We define

$$(\mathcal{M}_c A)_a^b = (\max(\frac{|a|}{|c|}, \frac{|b|}{|c|}) + 1)[A_a^b], \quad \forall a, b.$$

The operators $\mathcal{M}_c$ and $\mathcal{E}_\gamma^\pm$ all commute and

$$\mathcal{M}_c(AB) \le (\mathcal{M}_c A)(\mathcal{M}_c B).$$

Lipschitz domains. For a non-negative constant Λ, let

$$D_\Lambda^+(c) \subset \mathcal{L} \times \mathcal{L}$$

be the set of all (a, b) such that there exist $a',\ b' \in \mathbb{Z}^d$ and $t \ge 0$ such that

$$\left\{ \begin{array}{lcl} |a = a' + tc| & \ge & \Lambda(|a'| + |c|)\,|c| \\ |b = b' + tc| & \ge & \Lambda(|b'| + |c|)\,|c| \end{array} \right.$$

and

$$\frac{|a|}{|c|}, \quad \frac{|b|}{|c|} \ \ge \ 2\Lambda^2.$$

Lemma 2.1. *Let* $t \ge 0$.

(i) *For* $\Lambda \ge 1$,

$$t \ge \Lambda\,|c| \ge \Lambda$$

if $|a = a' + tc| \ge \Lambda(|a'| + |c|)\,|c|$.

(ii) *For* $\Lambda > 1$,

$$\left\{ \begin{array}{ll} |a'| \le \frac{t}{\Lambda - 1} - |c| & \textit{if } |a = a' + tc| \ge \Lambda(|a'| + |c|)\,|c| \\ |a'| \ge \frac{t}{\Lambda + 1} - |c| & \textit{if not.} \end{array} \right.$$

(iii) *For* $\Lambda > 1$,

$$\left|\frac{|a|}{|c|} - t\right| \leq \frac{t}{\Lambda - 1} \ \ and \ \ \left|\frac{<a,c>}{|c|^2} - t\right| \leq \frac{t}{\Lambda - 1},$$

if $|a = a' + tc| \geq \Lambda(|a'| + |c|)\,|c|$.

(iv) *For* $\Omega \geq (\Lambda + 1)(|a - b| + 1)$,

$$|b| \geq \Lambda(|a' + b - a| + |c|)\,|c|$$

if $|a = a' + tc| \geq \Omega(|a'| + |c|)\,|c|$.

Proof. This is a direct computation. □

Corollary 2.2. *Let* $\Lambda \geq 3$.

(i)

$$(a,b) \in D_\Lambda^+(c) \Longrightarrow \frac{|a|}{|c|} \approx \frac{|b|}{|c|} \approx \frac{<a,c>}{|c|^2} \approx \frac{<b,c>}{|c|^2} \gtrsim \Lambda\,|c|\,.$$

(ii)

$$(a,b) \in D_\Lambda^+(c) \Longrightarrow (a + tc, b + tc) \in D_\Lambda^+(c) \quad \forall t \geq 0.$$

(iii)

$$(a,b) \in D_\Lambda^+(c) \ \Longrightarrow \ (\tilde a, \tilde b) \in D_\Omega^+(c),$$

where

$$\Omega = \Lambda - \max(|\tilde a - a|,\ |\tilde b - b|) - 2.$$

(iv)

$$(a,b) \in D_{\Lambda+3}^+(c),\ (a,d) \notin D_\Lambda^+(c) \Longrightarrow |a - d|\,,\ |b - d| \gtrsim \frac{1}{\Lambda^2}\frac{|a|}{|c|}.$$

Proof. (i) follows from Lemma 2.1 (i)+(iii) if we just observe that

$$t \approx t + \frac{t}{\Lambda - 1} \approx t - \frac{t}{\Lambda - 1}.$$

In order to see (ii) we write $a = a' + sc,\ s \geq 0$, with $|a| \geq \Lambda(|a'| + |c|)\,|c|$. Then

$$|a + tc|^2 = |a|^2 + t^2|c|^2 + 2t <a,c> = |a|^2 + t^2|c|^2 + 2ts|c|^2 + 2t <a',c> .$$

By Lemma 2.1(ii)

$$2ts|c|^2 + 2t <a',c> \geq 2ts(1 - \frac{1}{\Lambda - 1})|c|^2 \geq 0.$$

Hence

$$|a + tc|^2 = |a|^2 + t^2|c|^2 \geq |a|^2 \geq \Lambda(|a'| + |c|)\,|c|\,.$$

Moreover, for all $t \geq 0$

$$\frac{|a+tc|}{|c|} \geq \frac{|a|}{|c|} \leq 2\Lambda^2.$$

The same argument applies to b.

To see (iii), let $\Delta = \max(|\tilde{a}-a|,\ |\tilde{b}-b|)+2$ and write $a = a'+tc$ with $|a| \geq \Lambda(|a'|+|c|)|c|$. Then $\tilde{a} = a'+\tilde{a}-a+tc$, and if

$$|\tilde{a}| < \Omega(|a'+\tilde{a}-a|+|c|)|c|$$

then by Lemma 2.1(ii)

$$|\tilde{a}-a| \geq \frac{t\Delta}{(\Omega+1)(\Lambda-1)}.$$

This implies that $t \leq (\Omega+1)(\Lambda-1)$ and, hence,

$$\frac{|a|}{|c|} < 2\Lambda^2$$

which is impossible. Therefore

$$|\tilde{a}| \geq \Omega(|a'+\tilde{a}-a|+|c|)|c|.$$

Moreover

$$\frac{|\tilde{a}|}{|c|} \geq \frac{|a|}{|c|} - \frac{\Delta}{|c|} \geq 2\Lambda^2 - \Delta \geq 2\Omega^2.$$

The same argument applies to b.

To see (iv), assume that $\frac{|d|}{|c|} < 2\Lambda^2$. As $\frac{|b|}{|c|} \geq 2(\Lambda+3)^2$ it follows that

$$\frac{|b-d|}{|c|} \geq 12\Lambda$$

and we are done unless

$$\frac{|b|}{|c|} \approx \frac{|a|}{|c|} \gtrsim \Lambda^3|c|.$$

In this case we must have

$$\frac{|d|}{|c|} \lesssim \frac{1}{\Lambda|c|}\frac{|b|}{|c|}$$

which implies that

$$\frac{|b-d|}{|c|} \geq (1-\text{cte.}\frac{1}{\Lambda|c|})\frac{|b|}{|c|} \geq \frac{1}{\Lambda^2|c|^2}\frac{|a|}{|c|}$$

and we are done again.

Therefore we can assume that $\frac{|d|}{|c|} \geq 2\Lambda^2$. Now the assumption that $(a,d) \notin D^+_\Lambda(c)$ leads to the conclusion by Lemma 2.1 (i)+(ii). $\square$

Lipschitz constants and norms. Define the Lipschitz-constants

$$\mathrm{Lip}^x_{\Lambda,\gamma} A = \sup_c |\mathcal{E}^x_\gamma \mathcal{M}_c(A - A(x,c))|_{D^x_\Lambda(c)}, \quad x \in \{+,-\},$$

(see the notations of Section 2.2) and the Lipschitz-norm

$$< A >_{\Lambda,\gamma} = \max(\mathrm{Lip}^+_{\Lambda,\gamma} \pi A, \mathrm{Lip}^-_{\Lambda,\gamma}(1-\pi)A) + |A|_\gamma \,.$$

Here we have used

$$(a,b) \in D^-_\Lambda(c) \iff (a,-b) \in D^+_\Lambda(c).$$

The matrix A is *Töplitz-Lipschitz* if it is Töplitz at ∞ and $< A >_{\Lambda,\gamma} < \infty$ for some Λ, γ.

Truncations. It is easy to see that

$$\begin{array}{lcl} < \mathcal{T}_\Delta A >_{\Lambda,\gamma} & \le & < A >_{\Lambda,\gamma} \\ < A - \mathcal{T}_\Delta A >_{\Lambda,\gamma'} & \le & e^{-\Delta(\gamma-\gamma')} < A >_{\Lambda,\gamma} \,. \end{array} \tag{8}$$

Tensor products. It is easy to verify that

$$< \zeta \otimes \zeta' >_{\Lambda,\gamma} \lesssim \|\zeta\|_\gamma \|\zeta'\|_\gamma \,. \tag{9}$$

Multiplications and differential equations are more delicate and we shall need the following proposition.

Proposition 2.3. *For all $x, y \in \{+,-\}$ and all $\gamma' < \gamma$*

(i)

$$\begin{array}{rl} \left|\mathcal{E}^{xy}_{\gamma'} \mathcal{M}_c(AB)\right|_{D^{xy}_{\Lambda+3}} \lesssim & (\frac{1}{\gamma-\gamma'})^d \left|\mathcal{E}^x_{\gamma_1} \mathcal{M}_c(A)\right|_{D^x_\Lambda} \left|\mathcal{E}^y_{\gamma_2} B\right|_{\mathcal{L}\times\mathcal{L}} + \\ & \Lambda^2 (\frac{1}{\gamma-\gamma'})^{d+1} \left|\mathcal{E}^x_{\gamma_1} A\right|_{\mathcal{L}\times\mathcal{L}} \left|\mathcal{E}^y_{\gamma_2} B\right|_{\mathcal{L}\times\mathcal{L}} , \end{array}$$

where one of γ_1, γ_2 is $= \gamma$ and the other one is $= \gamma'$. The same bound holds for BA.

(ii)

$$\begin{array}{rl} \left|\mathcal{E}^{xyz}_{\gamma'} \mathcal{M}_c(ABC)\right|_{D^{xyz}_{\Lambda+6}} \lesssim & (\frac{1}{\gamma-\gamma'})^{2d} \left|\mathcal{E}^x_{\gamma_1} A\right|_{\mathcal{L}\times\mathcal{L}} \left|\mathcal{E}^y_{\gamma_2} \mathcal{M}_c(B)\right|_{D^y_\Lambda} \left|\mathcal{E}^z_{\gamma_3} C\right|_{\mathcal{L}\times\mathcal{L}} + \\ & \Lambda^2 (\frac{1}{\gamma-\gamma'})^{2d+1} \left|\mathcal{E}^x_{\gamma_1} A\right|_{\mathcal{L}\times\mathcal{L}} \left|\mathcal{E}^y_{\gamma_2} B\right|_{\mathcal{L}\times\mathcal{L}} \left|\mathcal{E}^z_{\gamma_3} C\right|_{\mathcal{L}\times\mathcal{L}} , \end{array}$$

where two of $\gamma_1, \gamma_2, \gamma_3$ are $= \gamma$ and the third one is $= \gamma'$.

Proof. To prove (i), let first $x = y = +$. We shall only prove the estimate for AB – the estimate for BA being the same. Notice that for $(a,b) \in D^+_{\Lambda+3}(c)$ we have, by Corollary 2.2(i), that

$$M_c(a,b) = \max(\frac{|a|}{|c|}, \frac{|b|}{|c|}) + 1 \approx \frac{|a|}{|c|} + 1.$$

Now

$$\begin{array}{l} (\mathcal{E}^+_{\gamma'} \mathcal{M}_c(AB))^b_a \le \sum_d M_c(a,b)[A^d_a][B^b_d] e^{\gamma'|a-b|} = \\ \sum_{(a,d)\in D^+_\Lambda(c)} \cdots + \sum_{(a,d)\notin D^+_\Lambda(c)} \cdots = (I) + (II). \end{array}$$

In the domain of (I) we have, by Corollary 2.2(i), that

$$M_c(a,b) \approx \frac{|a|}{|c|} + 1 \approx M_c(d,b),$$

so

$$(I) \lesssim \left|\mathcal{E}^+_{\gamma_1}\mathcal{M}_c A\right|_{D^+_\Lambda(c)} \left|\mathcal{E}^+_{\gamma_2} B\right|_{\mathcal{L}\times\mathcal{L}} \sum_d e^{-(\gamma_1-\gamma')|a-d|-(\gamma_2-\gamma')|d-b|}.$$

Since one of $\gamma_1-\gamma'$ and $\gamma_2-\gamma'$ is $\gamma-\gamma'$ the sum is

$$\lesssim (\frac{1}{\gamma-\gamma'})^d.$$

In the domain of (II) we have, by Corollary 2.2(iv), that

$$|a-d|,\ |b-d| \gtrsim \frac{1}{\Lambda^2}\frac{|a|}{|c|},$$

so (II) is

$$\lesssim \left|\mathcal{E}^+_{\gamma_1} A\right|_{\mathcal{L}\times\mathcal{L}} \left|\mathcal{E}^+_{\gamma_2} B\right|_{\mathcal{L}\times\mathcal{L}} \times$$
$$\textstyle\sum_{|a-d|,|d-b|\gtrsim \frac{1}{\Lambda^2}\frac{|a|}{|c|}} (\frac{|a|}{|c|}+1) e^{-(\gamma_1-\gamma')|a-d|-(\gamma_2-\gamma')|d-b|}.$$

Since one of $\gamma_1-\gamma'$ and $\gamma_2-\gamma'$ is $\gamma-\gamma'$ the sum is

$$\lesssim \Lambda^2(\frac{1}{\gamma-\gamma'})^{d+1}.$$

The three other cases of (i) are treated in the same way.

To prove (ii), let first $x=y=z=+$. Notice that for $(a,b)\in D^+_{\Lambda+6}(c)$ we have, by Corollary 2.2(i), that

$$M_c(a,b) = \max(\frac{|a|}{|c|},\frac{|b|}{|c|}) + 1 \approx \frac{|a|}{|c|} + 1.$$

Now

$$(\mathcal{E}^+_{\gamma'}\mathcal{M}_c(ABC))_a^b \le \textstyle\sum_{d,e} M_c(a,b)[A_a^d][B_d^e][C_e^b]e^{\gamma'|a-b|} \le$$
$$\textstyle\sum_{|d|\ge|e|}\dots + \sum_{|e|\ge|d|}\dots.$$

We shall only consider the first of these sums – the second one being analogous. We decompose this sum as

$$\sum_{\substack{(a,d)\in D^+_{\Lambda+3}(c)\\ (d,e)\in D^+_\Lambda(c)}} \dots + \sum_{\substack{(a,d)\in D^+_{\Lambda+3}(c)\\ (d,e)\notin D^+_\Lambda(c)}} \dots + \sum_{(a,d)\notin D^+_{\Lambda+3}(c)} \dots = (I)+(II)+(III).$$

In the domain of (I) we have, by Corollary 2.2(i), that

$$M_c(d,e) \approx M_c(a,b),$$

so (I) is

$$\lesssim \left|\mathcal{E}^+_{\gamma_1}A\right|_{\mathcal{L}\times\mathcal{L}} \left|\mathcal{E}^+_{\gamma_2}\mathcal{M}_cB\right|_{D^+_\Lambda(c)} \left|\mathcal{E}^+_{\gamma_3}C\right|_{\mathcal{L}\times\mathcal{L}} \times \sum_d e^{-(\gamma_1-\gamma')|a-d|-(\gamma_2-\gamma')|d-e|-(\gamma_3-\gamma')|e-b|}.$$

Since two of $\gamma_1-\gamma'$, $\gamma_2-\gamma'$ and $\gamma_3-\gamma'$ are $\gamma-\gamma'$ the sum is

$$\lesssim (\frac{1}{\gamma-\gamma'})^{2d}.$$

By Corollary 2.2(iv) we have, in the domain of (II),

$$|a-d|,\ |d-e| \gtrsim \frac{1}{\Lambda^2}\frac{|a|}{|c|}.$$

and, in the domain of (III),

$$|a-d|,\ |d-b| \gtrsim \frac{1}{\Lambda^2}\frac{|a|}{|c|}.$$

Hence in both these domains we have

$$s(d,e) = \max(|a-d|,|d-e|,|e-b|) \gtrsim \frac{1}{\Lambda^2}\frac{|a|}{|c|},$$

so $(II)+(III)$ is

$$\lesssim \left|\mathcal{E}^+_{\gamma_1}A\right|_{\mathcal{L}\times\mathcal{L}} \left|\mathcal{E}^+_{\gamma_2}B\right|_{\mathcal{L}\times\mathcal{L}} \left|\mathcal{E}^+_{\gamma_3}C\right|_{\mathcal{L}\times\mathcal{L}} \times \sum_{s(d,e)\gtrsim\frac{1}{\Lambda^2}\frac{|a|}{|c|}} (\frac{|a|}{|c|}+1)e^{-(\gamma_1-\gamma')|a-d|-(\gamma_2-\gamma')|d-e|-(\gamma_3-\gamma')|e-b|}.$$

Since two of $\gamma_1-\gamma'$, $\gamma_2-\gamma'$ and $\gamma_3-\gamma'$ are $\gamma-\gamma'$ the sum is

$$\lesssim \Lambda^2(\frac{1}{\gamma-\gamma'})^{2d+1}.$$

The seven other cases of (ii) are treated in the same way. $\square$

We give a more compact and slightly weaker formulation of this result.

Corollary 2.4. *For all $x,y\in\{+,-\}$ and all $\gamma'<\gamma$*

(i)

$$\begin{aligned}\left|\mathcal{E}^{xy}_{\gamma'}\mathcal{M}_c(AB)\right|_{D^{xy}_{\Lambda+3}} \lesssim\ & \Lambda^2(\tfrac{1}{\gamma-\gamma'})^{d+1}[\left|\mathcal{E}^x_{\gamma_1}A\right|_{\mathcal{L}\times\mathcal{L}}+\\ & \left|\mathcal{E}^x_{\gamma_1}\mathcal{M}_c(A)\right|_{D^x_\Lambda}]\left|\mathcal{E}^y_{\gamma_2}B\right|_{\mathcal{L}\times\mathcal{L}},\end{aligned}$$

where one of γ_1,γ_2 is $=\gamma$ and the other one is $=\gamma'$. The same bound holds for BA.

(ii)

$$\left|\mathcal{E}^{xyz}_{\gamma'}\mathcal{M}_c(ABC)\right|_{D^{xyz}_{\Lambda+6}} \lesssim \Lambda^2\left(\tfrac{1}{\gamma-\gamma'}\right)^{2d+1}\left|\mathcal{E}^{x}_{\gamma_1}A\right|_{\mathcal{L}\times\mathcal{L}}\left[\left|\mathcal{E}^{y}_{\gamma_2}\mathcal{M}_c(B)\right|_{D^y_\Lambda} + \left|\mathcal{E}^{y}_{\gamma_2}B\right|_{\mathcal{L}\times\mathcal{L}}\right]\left|\mathcal{E}^{z}_{\gamma_3}C\right|_{\mathcal{L}\times\mathcal{L}},$$

where two of $\gamma_1, \gamma_2, \gamma_3$ *are* $= \gamma$ *and the third one is* $= \gamma'$.

Multiplication. Using the relation (1) and (7) we obtain immediately from Proposition 2.3(i) that a product of two Töplitz-Lipschitz matrices is again Töplitz-Lipschitz and for all $\gamma' < \gamma$

$$(10)\qquad \begin{array}{c} <AB>_{\Lambda+3,\gamma'}\lesssim \\ \Lambda^2\left(\frac{1}{\gamma-\gamma'}\right)^{d+1}\left[<A>_{\Lambda,\gamma_1}\left|B\right|_{\gamma_2} + \left|A\right|_{\gamma_1}<B>_{\Lambda,\gamma_2}\right], \end{array}$$

where one of γ_1, γ_2 is $= \gamma$ and the other one is $= \gamma'$.

This formula cannot be iterated without consecutive loss of the Lipschitz domain. However Proposition 2.3(ii) together with (5) gives for all $\gamma' < \gamma$

$$(11)\qquad \begin{array}{c} <A_1\cdots A_n>_{\Lambda+6,\gamma'}\le \\ (\text{cte.})^n\Lambda^2\left(\frac{1}{\gamma-\gamma'}\right)^{(n-1)d+1}\left[\sum_{1\le k\le n}\prod_{\substack{1\le j\le n\\ j\neq k}}\left|A_j\right|_{\gamma_j}<A_k>_{\Lambda,\gamma_k}\right], \end{array}$$

where all $\gamma_1, \ldots, \gamma_n$ are $= \gamma$ except one which is $= \gamma'$.

Linear differential equation. Consider the linear system

$$\begin{cases} \frac{d}{dt}X = A(t)X \\ X(0) = I. \end{cases}$$

where $A(t)$ is Töplitz-Lipschitz with exponential decay. The solution verifies

$$X(t_0) = I + \sum_{n=1}^{\infty}\int_0^{t_0}\int_0^{t_1}\cdots\int_0^{t_{n-1}} A(t_1)A(t_2)\ldots A(t_n)dt_n\ldots dt_2dt_1.$$

Using (11) we get for $\gamma' < \gamma$

$$(12)\qquad \begin{array}{c} <X(t)-I>_{\Lambda+6,\gamma'}\lesssim \\ \Lambda^2\left(\frac{1}{\gamma-\gamma'}\right)|t|\exp\left(\text{cte.}\left(\frac{1}{\gamma-\gamma'}\right)^d|t|\alpha(t)\right)\sup_{|s|\le|t|}<A(s)>_{\Lambda,\gamma}, \end{array}$$

where

$$\alpha(t) = \sup_{0\le|s|\le|t|}\left|A(s)\right|_{\gamma}.$$

2.4. **Töplitz-Lipschitz matrices ($d \geq 2$).**

Let

$$A : \mathcal{L}\times\mathcal{L} \to gl(2,\mathbb{C})$$

be a matrix. We say that A is 1-Töplitz if all Töplitz-limits $A(\pm, c)$ exist, and we define, inductively, that A is n-Töplitz if all Töplitz-limits $A(\pm, c)$ are (n-1)-Töplitz. We say that A is *Töplitz* if it is (d-1)-Töplitz.

In Section 2.3 we have defined $<A>_{\Lambda,\gamma}$ which we shall now denote by

$$^{1}\!<A>_{\Lambda,\gamma}\ .$$

We define, inductively,

$$^{n}\!<A>_{\Lambda,\gamma}=\sup_{c}(^{n-1}\!<A(+,c)>_{\Lambda,\gamma},{}^{n-1}\!<A(-,c)>_{\Lambda,\gamma})$$

and we denote

$$<\!A\!>_{\Lambda,\gamma}=\ ^{d-1}\!<A>_{\Lambda,\gamma}\ .$$

The matrix A is *Töplitz-Lipschitz* if it is Töplitz at ∞ and $<A>_{\Lambda,\gamma}<\infty$ for some Λ,γ.

Proposition 2.3, Corollary 2.4 and (9-12) remain valid with this norm in any dimension d.

3. Functions with Töplitz-Lipschitz property

3.1. **Töplitz-Lipschitz property.**

Let $\mathcal{O}^{\gamma}(\sigma)$ be the set of vectors in the complex space $l^2_{\gamma}(\mathcal{L},\mathbb{C})$ of norm less than σ, i.e.

$$\mathcal{O}^{\gamma}(\sigma)=\{\zeta\in\mathbb{C}^{\mathcal{L}}\times\mathbb{C}^{\mathcal{L}}:\|\zeta\|_{\gamma}<\sigma\}.$$

Our functions $f:\mathcal{O}^0(\sigma)\to\mathbb{C}$ will be defined and real analytic on the domain $\mathcal{O}^0(\sigma)$. [3]

Its first differential

$$l^2_0(\mathcal{L},\mathbb{C})\ni\hat\zeta\mapsto<\hat\zeta,\partial_\zeta f(\zeta)>$$

defines a unique vector $\partial_\zeta f(\zeta)$ in $l^2_0(\mathcal{L},\mathbb{C})$, and its second differential

$$l^2_0(\mathcal{L},\mathbb{C})\ni\hat\zeta\mapsto<\hat\zeta,\partial^2_\zeta f(\zeta)\hat\zeta>$$

defines a unique symmetric matrix $\partial^2_\zeta f(\zeta)$ $\mathcal{L}\times\mathcal{L}\to gl(2,\mathbb{C})$. A matrix $A:\mathcal{L}\times\mathcal{L}\to gl(2,\mathbb{C})$ is symmetric if

$$^{t}\!A_a^b=A_b^a.$$

We say that f is *Töplitz at* ∞ if the matrix $\partial^2_\zeta f(\zeta)$ is Töplitz at ∞ for all $\zeta\in\mathcal{O}^0(\sigma)$. We define the norm

$$[f]_{\Lambda,\gamma,\sigma}$$

[3]The space $l^2_{\gamma}(\mathcal{L},\mathbb{C})$ is the complexification of the space $l^2_{\gamma}(\mathcal{L},\mathbb{R})$ of real sequences. "real analytic" means that it is a holomorphic function which is real on $\mathcal{O}^0(\sigma)\cap l^2_{\gamma}(\mathcal{L},\mathbb{R})$.

to be the smallest C such that

$$\begin{cases} |f(\zeta)| \le C & \forall \zeta \in \mathcal{O}^0(\sigma) \\ \|\partial_\zeta f(\zeta)\|_{\gamma'} \le \frac{1}{\sigma} C & \forall \zeta \in \mathcal{O}^{\gamma'}(\sigma),\ \forall \gamma' \le \gamma, \\ <\partial_\zeta^2 f(\zeta)>_{\Lambda,\gamma'} \le \frac{1}{\sigma^2} C & \forall \zeta \in \mathcal{O}^{\gamma'}(\sigma),\ \forall \gamma' \le \gamma. \end{cases}$$

Proposition 3.1. (i)

$$[fg]_{\Lambda,\gamma,\sigma} \lesssim [f]_{\Lambda,\gamma,\sigma} [g]_{\Lambda,\gamma,\sigma}.$$

(ii) *If* $g(\zeta) = <c, \partial_\zeta f(\zeta)>$, *then*

$$[g]_{\Lambda,\gamma,\sigma'} \lesssim \frac{1}{\sigma - \sigma'} \|c\|_\gamma [f]_{\Lambda,\gamma,\sigma}$$

for $\sigma' < \sigma$.

(iii) *If* $g(\zeta) = <C\zeta, \partial_\zeta f(\zeta)>$, *then*

$$\begin{aligned}[g]_{\Lambda+3,\gamma',\sigma'} \lesssim\ & ((1 + \tfrac{\sigma'}{\sigma-\sigma'})(\tfrac{1}{\gamma-\gamma'})^{d+m_*} |C|_\gamma \\ & + \Lambda^2 (\tfrac{1}{\gamma-\gamma'})^{d+1} < C >_{\Lambda,\gamma}) [f]_{\Lambda,\gamma,\sigma}\end{aligned}$$

for $\sigma' < \sigma$ *and* $\gamma' < \gamma$.

Proof. We have

$$\begin{aligned} & fg(\zeta) = f(\zeta) g(\zeta) \\ & \partial_\zeta fg(\zeta) = f(\zeta)\partial_\zeta g(\zeta) + \partial_\zeta f(\zeta) g(\zeta) \\ & \partial_\zeta^2 fg(\zeta) = f(\zeta)\partial_\zeta^2 g(\zeta) + \partial_\zeta^2 f(\zeta) g(\zeta) + 2(\partial_\zeta f(\zeta) \otimes \partial_\zeta g(\zeta)). \end{aligned}$$

(i) now follows from (9).

For $\zeta \in \mathcal{O}^0(\sigma')$ we have

$$|g(\zeta)| \le \|c\|_0 \|\partial_\zeta f(\zeta)\|_0 \le \|c\|_0 \frac{1}{\sigma}\alpha,$$

where $\alpha = [f]_{\Lambda,\gamma,\sigma}$.

Let $\zeta \in \mathcal{O}^{\gamma'}(\sigma')$ and $h(z) = \partial_\zeta f(\zeta + zc)$. h is a holomorphic function (with values in the Hilbert-space $l^2_{\gamma'}(\mathcal{L}, \mathbb{C})$) in the disk $|z| < \frac{\sigma-\sigma'}{\|c\|_{\gamma'}}$ and

$$\|h(z)\|_{\gamma'} \le \frac{1}{\sigma}\alpha.$$

Since $\partial_\zeta g(\zeta) = \partial_z h(0)$, we get by a Cauchy estimate that

$$\|\partial_\zeta g(\zeta)\|_{\gamma'} \le \frac{1}{\sigma'}(\frac{\sigma'}{\sigma}\frac{1}{\sigma-\sigma'} \|c\|_{\gamma'} \alpha).$$

Let $\zeta \in \mathcal{O}^{\gamma'}(\sigma')$ and $k(z) = \partial_\zeta^2 f(\zeta + zc)$. k is a holomorphic function (with values in the Banach-space of matrices with the norm $<\cdot>_{\gamma',\Lambda}$) in the disk $|z| < \frac{\sigma-\sigma'}{\|c\|_{\gamma'}}$ and

$$<k(z)>_{\Lambda,\gamma'} \le \frac{1}{\sigma^2}\alpha.$$

Since $\partial_\zeta^2 g(\zeta) = \partial_\zeta k(0)$, we get by a Cauchy estimate that

$$<\partial_\zeta g(\zeta)>_{\Lambda,\gamma'} \leq (\frac{1}{\sigma'})^2((\frac{\sigma'}{\sigma})^2 \frac{1}{\sigma-\sigma'} \left\|c\right\|_{\gamma'} \alpha).$$

This proves (ii).

To see (iii) we replace c by $C\zeta$ and notice that

$$\partial_\zeta g(\zeta) = \partial_z h(0) + {}^tC\partial_\zeta f(\zeta)$$

and

$$\partial_\zeta^2 g(\zeta) = \partial_z k(0) + {}^tC\partial_\zeta^2 f(\zeta) + {}^t\partial_\zeta^2 f(\zeta)C.$$

$\partial_z h(0)$ and $\partial_z k(0)$ are estimated as above and $\left\|C\zeta\right\|_{\gamma'}$ with Young's inequality (2). The matrix products are estimated by (10). □

3.2. **Truncations.**

Let Tf be the Taylor polynomial of order 2 of f at $\zeta = 0$.

Proposition 3.2. (i)

$$[Tf]_{\Lambda,\gamma,\sigma} \lesssim [f]_{\Lambda,\gamma,\sigma}.$$

(ii)

$$[f - Tf]_{\Lambda,\gamma,\sigma'} \lesssim (\frac{\sigma'}{\sigma})^3 \frac{\sigma}{\sigma-\sigma'} [f]_{\Lambda,\gamma,\sigma}.$$

Proof. Let $\zeta \in \mathcal{O}^0(\sigma')$ and let $g(z) = f(z\zeta)$. Then g is a real holomorphic function in the disk of radius $\frac{\sigma}{\sigma'}$ and bounded by $\alpha = [f]_{\Lambda,\gamma,\sigma}$. Since $Tf(z\zeta) = g(0) + g'(0)z + \frac{1}{2}g''(0)z^2$ we get by a Cauchy estimate that

$$|(f - Tf)'\zeta)| \leq (\frac{\sigma'}{\sigma})^3 \frac{\sigma}{\sigma-\sigma'}\alpha.$$

Let $\zeta \in \mathcal{O}^{\gamma'}(\sigma')$ and let $h(z) = \partial_\zeta f(z\zeta)$. Then h is a holomorphic function in the disk of radius $\frac{\sigma}{\sigma'}$ and bounded by $\frac{\alpha}{\sigma}$. Since $\partial_\zeta Tf(\zeta) = h(0) + h'(0)z$ we get by a Cauchy estimate that

$$\left\|\partial_\zeta(f - Tf)(\zeta)\right\|_{\gamma'} \leq (\frac{\sigma'}{\sigma})^2 \frac{\sigma}{\sigma-\sigma'}\frac{\alpha}{\sigma}.$$

Let $\zeta \in \mathcal{O}^{\gamma'}(\sigma')$ and let $k(z) = \partial_\zeta^2 f(z\zeta)$. Then k is a holomorphic function in the disk of radius $\frac{\sigma}{\sigma'}$ and bounded by $\frac{\alpha}{\sigma^2}$. Since $\partial_\zeta^2 Tf(\zeta) = k(0)$ we get by a Cauchy estimate that

$$<\partial_\zeta^2(f - Tf)(\zeta)>_{\Lambda,\gamma'} \leq (\frac{\sigma'}{\sigma}) \frac{\sigma}{\sigma-\sigma'}\frac{\alpha}{\sigma^2}.$$

This gives (ii).

The first statement is obtained by taking $\sigma' = \frac{1}{2}\sigma$. Since f is a quadratic polynomial it satisfies the same (modulo a constant) estimate on σ as on $\frac{1}{2}\sigma$. □

3.3. **Poisson brackets.**

The Poisson bracket of two functions f and g is defined by

$$\{f,g\}(\zeta) = <\partial_\zeta f(\zeta), J\partial_\zeta g(\zeta)> .$$

Proposition 3.3. (i) *If g is a quadratic polynomial, then*

$$[\{f,g\}]_{\Lambda+3,\gamma',\sigma'} \lesssim [\frac{1}{\sigma_1\sigma_2} + \Lambda^2(\frac{1}{\gamma-\gamma'})^{d+1}(\frac{\sigma'}{\sigma_1\sigma_2})^2][f]_{\Lambda,\gamma,\sigma_1}[g]_{\Lambda,\gamma,\sigma_2},$$

for $0<\sigma_1-\sigma\approx\sigma_1,\ 0<\sigma_2-\sigma\approx\sigma_2$ *and* $\gamma'<\gamma$.

(ii) *If g is a quadratic polynomial and $f(\zeta) = <\zeta, A\zeta>$, then*

$$[\{f,g\}]_{\Lambda+3,\gamma',\sigma'} \lesssim [(\frac{1}{\gamma-\gamma'})^{d+m_*}\frac{1}{\sigma_1^2} + \Lambda^2(\frac{1}{\gamma-\gamma'})^{d+1}\frac{1}{\sigma_1^2})][f]_{\Lambda,\gamma,\sigma_1}[g]_{\Lambda,\gamma,\sigma_2},$$

for $0<\sigma_1-\sigma\approx\sigma_1,\ 0<\sigma_2-\sigma\approx\sigma_2$ *and* $\gamma'<\gamma$.

Proof. We have

$$\partial_\zeta\{f,g\}(\zeta) = \partial_\zeta^2 f(\zeta)J\partial_\zeta g(\zeta) + \partial_\zeta^2 g(\zeta)J\partial_\zeta f(\zeta)$$

and

$$\begin{aligned}\partial_\zeta^2\{f,g\}(\zeta) = \partial_\zeta^3 f(\zeta)J\partial_\zeta g(\zeta) + \partial_\zeta^3 g(\zeta)J\partial_\zeta f(\zeta) +\\ \partial_\zeta^2 f(\zeta)J\partial_\zeta^2 g(\zeta) + {}^t\partial_\zeta^2 f(\zeta)J\partial_\zeta^2 g(\zeta).\end{aligned}$$

For $\zeta\in\mathcal{O}^0(\sigma')$ we get, by Cauchy-Schwartz, that

$$|\{f,g\}(\zeta)| \le \|\partial_\zeta f(\zeta)\|_0\,\|\partial_\zeta g(\zeta)\|_0 \le (\frac{\alpha\beta}{\sigma_1\sigma_2}),$$

where $\alpha=[f]_{\Lambda,\gamma,\sigma_1}$ and $\beta=[g]_{\Lambda,\gamma,\sigma_2}$.

For $\zeta\in\mathcal{O}^{\gamma'}(\sigma')$, let $h(z)=\partial_\zeta f(\zeta+zJ\partial_\zeta g(\zeta))$. For $|z|<\frac{\sigma_1-\sigma'}{\|\partial_\zeta g(\zeta)\|_{\gamma'}}$ we have

$$\|h(z)\|_{\gamma'} \le \frac{\alpha}{\sigma_1}.$$

Since $\partial_z h(0) = \partial_\zeta^2 f(\zeta)J\partial_\zeta g(\zeta)$, we get by a Cauchy estimate that

$$\left\|\partial_\zeta^2 f(\zeta)J\partial_\zeta g(\zeta)\right\|_{\gamma'} \le \frac{1}{\sigma_1^2\sigma_2}\alpha\beta.$$

The same estimate holds with f and g interchanged.

For $\zeta\in\mathcal{O}^{\gamma'}(\sigma')$, let $k(z)=\partial_\zeta^2 f(\zeta+zJ\partial_\zeta g(\zeta))$. By a Cauchy-estimate we get as above that

$$<\partial_\zeta^3 f(\zeta)J\partial_\zeta g(\zeta)>_{\Lambda,\gamma'} \le \frac{1}{\sigma_1^3\sigma_2}\alpha\beta.$$

The same estimate holds with f and g interchanged.

Finally, for $\zeta\in\mathcal{O}^{\gamma'}(\sigma')$ we get by (10) that

$$<\partial_\zeta^2 f(\zeta)J\partial_\zeta^2 g(\zeta)>_{\Lambda+3,\gamma'} \lesssim \Lambda^2(\gamma-\gamma')^{-d-1} <\partial_\zeta^2 f(\zeta)>_{\Lambda,\gamma'}<\partial_\zeta^2 g(\zeta)>_{\Lambda,\gamma} .$$

By hypothesis we have

$$<\partial_\zeta^2 g(\zeta)>_{\Lambda,\gamma}\leq \frac{\beta}{\sigma_2^2}$$

for ζ only in $\mathcal{O}^\gamma(\sigma')$. But since g is quadratic, $\partial_\zeta^2 g(\zeta)$ is independent of ζ and, hence, this also holds in the larger domain $\zeta\in\mathcal{O}^{\gamma'}(\sigma')$.

The second part follows directly from Proposition 3.1(iii). □

3.4. **The flow map.**

Consider the linear system

$$\dot\zeta = J\partial_\zeta f_t(\zeta)$$

where $f_t(\zeta) = <\zeta, a_t> + <\zeta, A_t\zeta>$, and let

$$\alpha(t)=\sup_{|s|\leq|t|}|A_s|_\gamma \quad \text{and} \quad \beta(t)=\sup_{|s|\leq|t|}\|a_s\|_{\gamma'}\,.$$

Consider the non-linear system

$$\dot z = g(\zeta, z)$$

where $g(\zeta,z)$ is real analytic in $\mathcal{O}^0(\sigma)\times\mathbb{D}(\mu)$. $\mathbb{D}(\mu)$ is the disk of radius μ in $\mathbb{C}$.

Proposition 3.4. (i) *The flow map of the linear system has the form*

$$\zeta_t:\zeta\mapsto\zeta+b_t+B_t\zeta,$$

and for $\gamma'<\gamma$

$$\|\zeta_t(\zeta)-\zeta\|_{\gamma'}\lesssim$$
$$(\tfrac{1}{\gamma-\gamma'})^{m_*}[e^{\mathrm{cte.}(\frac{1}{\gamma-\gamma'})^d|t|\alpha(t)}|t|\beta(t)+[e^{\mathrm{cte.}(\frac{1}{\gamma-\gamma'})^d|t|\alpha(t)}-1]\,\|\zeta\|_{\gamma'}]$$

and

$$<B_t>_{\Lambda+6,\gamma'}\lesssim$$
$$\Lambda^2(\tfrac{1}{\gamma-\gamma'})|t|e^{\mathrm{cte.}(\frac{1}{\gamma-\gamma'})^d|t|\alpha(t)}\sup\nolimits_{|s|\leq|t|}<A_s>_{\Lambda,\gamma}\,.$$

(ii) *For* $|z|<\mu'$, *the flow of the non-linear system is defined for* $|t|\leq\frac{\mu-\mu'}{2\varepsilon}$ *and*

$$[z_t(\cdot,z)-z]_{\Lambda,\gamma,\sigma}\lesssim$$
$$(1+\tfrac{\mu-\mu'}{\varepsilon}(e^{\mathrm{cte.}|t|\frac{1}{\mu-\mu'}\varepsilon}-1))^2\varepsilon,$$

where

$$\varepsilon=\sup_{z\in\mathbb{D}(\mu)}[g(\cdot,z)]_{\Lambda,\gamma,\sigma}.$$

Proof. (i) We have

$$b_t = \sum_{n=1}^{\infty} \int_0^t \cdots \int_0^{t_{n-1}} JA_{t_1} \dots JA_{t_{n-1}} Ja_{t_n} dt_n dt_{n-1} \dots dt_1$$

and

$$B_t = \sum_{n=1}^{\infty} \int_0^t \cdots \int_0^{t_{n-1}} JA_{t_1} \dots JA_{t_n} dt_n \dots dt_1.$$

By (5) we have

$$|B_t|_{\gamma'} \lesssim (\gamma-\gamma')^d(\delta(t)-1), \quad \delta(t) = \exp(\mathrm{cte.}(\gamma-\gamma')^{-d}|t|\alpha(t))$$

and by (2) we have

$$\|B_t\zeta\|_{\gamma'} \lesssim (\frac{1}{\gamma-\gamma'})^{m_*}(\delta(t)-1)\,\|\zeta\|_{\gamma'}\,.$$

By (2+5) we have

$$\|b_t\|_{\gamma'} \lesssim (\frac{1}{\gamma-\gamma'})^{m_*}\delta(t)|t|\beta(t).$$

By (12) we have

$$< B_t >_{\Lambda+6,\gamma'} \lesssim \Lambda^2(\gamma-\gamma')^{-1}\delta(t) \sup_{|s|\le|t|} < A_s >_{\Lambda,\gamma}\,.$$

The proof of (ii) easier. We have

$$\partial_\zeta \dot z_t = \partial_\zeta g(\dots) + \partial_z g(\dots)\partial_\zeta z_t$$

which implies that

$$\partial_\zeta z_t = \int_0^t e^{\int_s^t \partial_z g(\zeta,z_r)dr}\partial_\zeta g(\zeta,z_s)ds$$

This is easy to estimate.

We also have

$$\partial_\zeta^2 \dot z_t = \partial_\zeta^2 g(\dots) + \partial_z\partial_\zeta g(\dots)\otimes\partial_\zeta z_t + \partial_z g(\dots)\partial_\zeta^2 z_t$$

which is treated in the same way. □

Remark. The same result holds for $z = (z_1,\dots,z_n) \in \mathbb{D}(\mu)^n$ and $g = (g_1,\dots,g_n)$.

Remark. If $|t| \le 1$ and

$$\sup_{|s|\le|t|} |A_s|_\gamma \lesssim (\gamma-\gamma')^d,$$

then

$$\|\zeta_t(\zeta)-\zeta\|_{\gamma'} \lesssim (\frac{1}{\gamma-\gamma'})^{m_*}\sup_{|s|\le|t|}\|a_s\|_{\gamma'} + (\frac{1}{\gamma-\gamma'})^{m_*+d}\sup_{|s|\le|t|}|A_s|_\gamma\,\|\zeta\|_{\gamma'}$$

and

$$<B_t>_{\Lambda+6,\gamma'} \lesssim \Lambda^2(\frac{1}{\gamma-\gamma'}) \sup_{|s|\leq|t|} <A_s>_{\Lambda,\gamma} .$$

If $|t| \leq 1$ and

$$\varepsilon = \sup_{z\in\mathbb{D}(\mu)} [g(\cdot,z)]_{\Lambda,\gamma,\sigma} \lesssim \mu-\mu',$$

then

$$[z_t(\cdot,z)-z]_{\Lambda,\gamma,\sigma} \lesssim \varepsilon.$$

3.5. **Compositions.**
Let $f(\zeta,z)$ be a real analytic function on $\mathcal{O}^0(\sigma)\times\mathbb{D}(\mu)$ and

$$\sup_{z\in\mathbb{D}(\mu)} [f(\cdot,z)]_{\Lambda,\gamma,\sigma} < \infty.$$

Let

$$\Phi(\zeta,z) = b(z)+\zeta+B(z)\zeta$$

with

$$\|b(z)+B(z)\zeta\|_{\gamma'} < \sigma-\sigma', \quad \forall(\zeta,z)\in\mathcal{O}^{\gamma'}(\sigma')\times\mathbb{D}(\mu')$$

for all $\gamma'\leq\gamma$. This implies that

$$\Phi : \mathcal{O}^{\gamma'}(\sigma') \to \mathcal{O}^{\gamma'}(\sigma), \quad \forall\ \gamma'\leq\gamma, \quad \forall z\in\mathbb{D}(\mu').$$

Let $g(\zeta,z)$ be a real holomorphic function on $\mathcal{O}^0(\sigma')\times\mathbb{D}(\mu')$ such that

$$|g| \leq \frac{1}{2}(\mu-\mu').$$

Proposition 3.5. *For all $z\in\mathbb{D}(\mu')$ and $\gamma'<\gamma$*

$$\begin{array}{c}[f(\Phi(\cdot),z+g(\cdot))]_{\Lambda+6,\gamma',\sigma'} \lesssim \\ \max(1,\alpha,\Lambda^2(\frac{1}{\gamma-\gamma'})\alpha^2)\sup_{z\in\mathbb{D}(\mu)}[f(\cdot,z)]_{\Lambda,\gamma,\sigma}\end{array}$$

where

$$\alpha = \frac{1}{\mu-\mu'}[g]_{\Lambda,\gamma,\sigma'} + (\frac{1}{\gamma-\gamma'})^{d+m_*} <B>_{\Lambda,\gamma} .$$

Proof. Let $\varepsilon = \sup_{z\in\mathbb{D}(\mu)}[f(\cdot,z)]_{\Lambda,\gamma,\sigma}$ and $\beta = \sup_{z\in\mathbb{D}(\mu')}[g(\cdot,z)]_{\Lambda,\gamma,\sigma'}$.

Let $h(\zeta,z) = f(\Phi(\zeta,z),z+g(\zeta,z))$. Then

$$\partial_\zeta h = \partial_z f(\ldots)\partial_\zeta g + {}^tB\partial_\zeta f(\ldots)$$

and

$$\begin{aligned}\partial_\zeta^2 h = \ & \partial_z^2 f(\ldots)(\partial_\zeta g\otimes\partial_\zeta g) + \partial_z f(\ldots)\partial_\zeta^2 g + \\ & {}^tB(\partial_\zeta\partial_z f(\ldots)\otimes\partial_\zeta g) + {}^tB\partial_\zeta^2 f(\ldots)B.\end{aligned}$$

For $(\zeta,z)\in\mathcal{O}^0(\sigma')\times\mathbb{D}(\mu')$ we get: $|h(\zeta)|\leq\varepsilon$.

For $(\zeta,z)\in\mathcal{O}^{\gamma'}(\sigma')\times\mathbb{D}(\mu')$ we get:

$$\|\partial_z f(\ldots)\partial_\zeta g\|_{\gamma'}=|\partial_z f(\ldots)|\,\|\partial_\zeta g\|_{\gamma'}\lesssim(\frac{1}{\mu-\mu'})\varepsilon\frac{\beta}{\sigma'};$$

$$\left\|{}^tB\partial_\zeta f(\ldots)\right\|_{\gamma'}\lesssim(\frac{1}{\gamma-\gamma'})^{d+m_*}|B|_\gamma\,\frac{\varepsilon}{\sigma}$$

by Young's inequality (2).

For $(\zeta,z)\in\mathcal{O}^{\gamma'}(\sigma')\times\mathbb{D}(\mu')$ we get:

$$<\partial_z^2 f(\ldots)\partial_\zeta g\otimes\partial_\zeta g>_{\Lambda,\gamma'}\lesssim(\frac{1}{\mu-\mu'})^2\varepsilon(\frac{\beta}{\sigma'})^2$$

by (9);

$$<\partial_z f(\ldots)\partial_\zeta^2 g>_{\Lambda,\gamma'}\lesssim(\frac{1}{\mu-\mu'})\varepsilon(\frac{\beta}{(\sigma')^2});$$

$$<{}^tB(\partial_\zeta\partial_z f(\ldots)\otimes\partial_\zeta g)>_{\Lambda+3,\gamma'}\lesssim\Lambda^2(\frac{1}{\gamma-\gamma'})^{d+1}<B>_{\Lambda,\gamma}(\frac{1}{\mu-\mu'})\varepsilon\frac{\beta}{\sigma\sigma'}$$

by (9-10);

$$<{}^tB\partial_\zeta^2 f(\ldots)B>_{\Lambda+6,\gamma'}\lesssim\Lambda^2(\frac{1}{\gamma-\gamma'})^{2d+1}<B>_{\Lambda,\gamma}^2\,\frac{\varepsilon}{\sigma^2}$$

by (11). □

Remark. The same result holds for $z=(z_1,\ldots,z_n)\in\mathbb{D}(\mu)^n$ and $g=(g_1,\ldots,g_n)$.

Remark. If, for $z\in\mathbb{D}(\mu')$,

$$[g(\cdot),z]_{\Lambda,\gamma,\sigma'}\lesssim(\mu-\mu')\min(1,\sqrt{\gamma-\gamma'})\Lambda^2$$

and

$$<B(z)>_{\Lambda,\gamma}\lesssim(\gamma-\gamma')^{d+m_*}\Lambda^2,$$

then

$$[f(\Phi(\cdot,z),z+g(\cdot,z)]_{\Lambda+6,\gamma',\sigma'}\lesssim\Lambda^6\varepsilon.$$

PART II. THE HOMOLOGICAL EQUATIONS

In this part we consider scalar-valued matrices $Q:\mathcal{L}\times\mathcal{L}\to\mathbb{C}$ which we consider as a sub-class of $gl(2,\mathbb{C})$-valued matrices through the identification

$$Q_a^b=Q_a^bI.$$

We will only consider the Lipschitz domains $D_\Lambda^+(c)$ which we denote by $D_\Lambda(c)$.

We define the block decomposition $\mathcal{E}_\Delta$ together with the blocks $[\,\cdot\,]_\Delta$ and the bound d_Δ of the block diameter.

We consider parameters $U \subset \mathbb{R}^{\mathcal{A}}$, $\mathcal{A} = \mathbb{Z}^d \setminus \mathcal{L}$, and define the norms $|\cdot|_{\{\gamma \atop U\}}$ and $<\ \cdot\ >_{\{\Lambda,\gamma \atop U\}}$.

4. Decomposition of $\mathcal{L}$

In this section $d \geq 2$. For a non-negative integer Δ we define an *equivalence relation* on $\mathcal{L}$ generated by the pre-equivalence relation

$$a \sim b \iff \begin{cases} |a|^2 = |b|^2 \\ |a-b| \leq \Delta. \end{cases}$$

Let $[a]_\Delta$ denote the equivalence class (*block*) of a, and let $\mathcal{E}_\Delta$ be the set of equivalence classes. It is trivial that each block $[a]$ is finite with cardinality

$$\lesssim |a|^{d-1}$$

that depends on a. But there is also a uniform Δ-dependent bound. Indeed, let d_Δ be the supremum of all block diameters. We will see (Proposition 4.1)

$$d_\Delta \lesssim \Delta^{\frac{(d+1)!}{2}}.$$

Δ will be fixed in this section and we will write $[\ \cdot\ ]$ for $[\ \cdot\ ]_\Delta$.

4.1. **Blocks.**

For any $X \subset \mathbb{Z}^d$ we define its *rank* to be the dimension of the smallest affine subspace in $\mathbb{R}^d$ containing X.

Proposition 4.1. *Let $c \in \mathbb{Z}^d$ and rank$[c] = k$, $k = 1, \ldots, d$. Then the diameter of $[c]$ is*

$$\lesssim\ \Delta^{\frac{(k+1)!}{2}}.$$

Proof. Let $\Delta_j,\ j \geq 1$ be an increasing sequence of numbers.

Assume that for any $1 \leq l \leq k$

$$(*)_l \qquad \mathrm{rank}(B_{\Delta_l}(c) \cap [c]) \geq l \quad \forall c \in [c],$$

where $B_r(c)$ is the ball of radius r centered at c. This means that for any $c \in [c]$, there exist linearly independent vectors $a_1, \ldots, a_l$ in $\mathbb{Z}^d$ such that

$$c + a_j \in [c] \text{ and } |a_j| \leq \Delta_l, \quad 1 \leq j \leq l.$$

$(*)_l$ implies that the $\perp$ projection $\tilde{c}$ of c onto $\sum \mathbb{R} a_j$ verifies

$$(**) \qquad |\tilde{c}| \lesssim \begin{cases} \Delta_l & l = 1 \\ \Delta_l^{l+1} & l \geq 2. \end{cases}$$

Proof. In order to see this we observe that, since $|c+a_j|^2 = |c|^2$ for each j, the (row) vector c verifies

$$cM = -\frac{1}{2}(|a_1|^2 \dots |a_l|^2),$$

where M is the $d \times l$-matrix whose columns are ${}^t a_1, \dots {}^t a_l$. Now there exists an orthogonal matrix Q such that

$$QM = \begin{pmatrix} B \\ 0 \end{pmatrix},$$

where B is an invertible $l \times l$-matrix. We have

$$(\det B)^2 = \det({}^t BB) = \det({}^t MM) \geq 1,$$

and (the absolute values of) the entires of B are bounded by $\lesssim \Delta_l$.

Define now x by

$$\begin{cases} (x_1 \dots x_l) = \frac{1}{2}(|a_1|^2 \dots |a_l|^2)B^{-1} \\ x_{l+1} = \dots = x_d = 0, \end{cases}$$

and $y = xQ$. Then $c - y \perp \sum \mathbb{R}a_j$, so $|\tilde{c}| \leq |y|$. An easy computation gives

$$|y| = |x| \lesssim \Delta_l^{l+1} \quad \text{and} \quad \lesssim \Delta_1 \text{ (if } l = 1).$$

This is $\leq \Delta_l$. □

We shall now determine Δ_l so that $(*)_l$ holds. This will be done by induction on l. For $l = 1$ $\Delta_1 = \Delta$ works, so let us assume that $(*)_l$ holds for some $1 \leq l < k$. If $(*)_{l+1}$ does not hold, it is violated for some c. Let us fix this $c \in [c]$, and let X be the real subspace generated by $(B_{\Delta_{l+1}}(c)) \cap [c]) - c$. X has rank $= l$.

For any $b \in [c]$ with $|b - c| \leq \Delta_{l+1} - \Delta_l$ we have

$$B_{\Delta_l}(b) \cap [c] \subset B_{\Delta_{l+1}}(c) \cap [c].$$

By the induction assumption the $\perp$ projection $\tilde{b}$ of b onto X verifies $(**)$.

Take now $b \in [c]$ such that $\Delta_{l+1} - \Delta_l - \Delta \leq |b - c| \leq (\Delta_{l+1} - \Delta_l)$ — such a b exists since rank of $[c]$ is $\geq l+1$. Since $b - c$ is parallel to X we have

$$\Delta_{l+1} - \Delta_l - \Delta \leq |b - c| = |\tilde{b} - \tilde{c}| \lesssim \begin{cases} \Delta_l & l = 1 \\ \Delta_l^{l+1} & l \geq 2. \end{cases}$$

So if we take $\Delta_{l+1} \approx$ the RHS, then the assumption that $(*)_{l+1}$ does not hold leads to a contradiction. Hence with this choice $(*)_l$ hold for all $l \leq k$.

To conclude we observe now that $[c] \subset c + X$ where X is a subspace of dimension k. Clearly the diameter of $[c]$ is the same as the diameter of its $\perp$ projection onto X, and, by $(**)$, the diameter of the projection is $\leq \Delta_k$. $\square$

We say that $[a]$ and $[b]$ have the same *block-type* if there are $a' \in [a]$ and $b' \in [b]$ such that

$$[a] - a' = [b] - b'.$$

It follows from the proposition that there are only finitely many block-types. We say that the block-type of $[a]$ is *orthogonal to c* if

$$[a] - a \perp c.$$

Description of blocks when $d = 2, 3$. For $d = 2$, we have outside $\{|a| :\leq d_\Delta \approx \Delta^3\}$

- $\star$ rank[a]=1 if, and only if, $a \in \frac{b}{2} + b^\perp$ for some $0 < |b| \leq \Delta$ – then $[a] = \{a, a-b\}$;
- $\star$ rank[a]=0 – then $[a] = \{a\}$.

For $d = 3$, we have outside $\{|a| :\leq d_\Delta \approx \Delta^{12}\}$

- $\star$ rank[a]=2 if, and only if, $a \in \frac{b}{2} + b^\perp \cap \frac{c}{2} + c^\perp$ for some $0 < |b|, |c| \leq \Delta_2$ linearly independent – then $[a] \supset \{a, a-b, a-c\}$;
- $\star$ rank[a]=1 if, and only if, $a \in \frac{b}{2} + b^\perp$ for a unique(!) $0 < |b|, \leq \Delta$ – then $[a] = \{a, a-b\}$;
- $\star$ rank[a]=0 – then $[a] = \{a\}$.

4.2. **Neighborhood at ∞.**

Proposition 4.2. *For any $|a| \gtrsim \Lambda^{2d-1}$, there exist $c \in \mathbb{Z}^d$,*

$$|c| \lesssim \Lambda^{d-1},$$

such that

$$|a| \geq \Lambda(|a_c| + |c|)\,|c|\,, \ \ <a, c> \geq 0.$$

(a_c is the lattice element on $a + \mathbb{R}c$ closest to the origin.)

Proof. For all $K > 0$ there is a $c \in \mathbb{Z}^d \cap \{|x| \leq K\}$ such that

$$\delta = dist(c, \mathbb{R}a) \leq C_1(\frac{1}{K})^{\frac{1}{d-1}}$$

where C_1 only depends on d.

To see this we consider the segment $\Gamma = [0, \frac{K}{|a|}a]$ in $\mathbb{R}^d$ and a tubular neighborhood Γ_ε of radius ε:

$$\mathrm{vol}(\Gamma_\varepsilon) \approx K\varepsilon^{d-1}.$$

The projection of $\mathbb{R}^d$ onto $\mathbb{T}^d$ is locally injective and locally volume-preserving. If $\varepsilon \gtrsim (\frac{1}{K})^{\frac{1}{d-1}}$, then the projection of Γ_ε cannot be injective (for volume reasons), so there are two different points $x, x' \in \Gamma_\varepsilon$ such that

$$x - x' = c \in Z^d \setminus 0.$$

Then

$$|a_c| \leq \frac{|a|}{|c|}\delta.$$

Now

$$\Lambda(|a_c| + |c|)\,|c| \leq 2\Lambda K^2 + C_1 \frac{\Lambda}{K^{\frac{1}{d-1}}}\,|a|\,.$$

If we choose $K = (2C_1\Lambda)^{d-1}$, then this is $\leq |a|$. □

Corollary 4.3. *For any N, the subset*

$$\{|a| + |b| \gtrsim \Lambda^{2d-1}\} \cap \{|a - b| \leq N\} \subset \mathbb{Z}^d \times \mathbb{Z}^d$$

is contained in

$$\bigcup_{|c| \lesssim \Lambda^{d-1}} D_\Omega(c)$$

for any

$$\Omega \leq \frac{\Lambda}{N+1} - 1.$$

Proof. Let $|a| \gtrsim \Lambda^{2d-1}$. Then there exists $|c| \lesssim \Lambda^{d-1}$ such that $|a| \geq \Lambda(|a_c| + |c|)\,|c|$. Clearly (because $d \geq 2$)

$$\frac{|a|}{|c|} \geq 2\Lambda^2 \geq 2\Omega^2.$$

If we write $a = a_c + tc$ then $b = a_c + b - a + tc$. According to Lemma 2.1(iv)

$$|b| \geq \Omega(|a_c + b - a| + |c|)|c|,$$

and moreover

$$\frac{|b|}{|c|} \geq \frac{|a|}{|c|} - N \geq \Lambda^2 - N \geq 2\Omega^2.$$

□

Remark. This corollary is essential. It says that any neighborhood

$$\{(a, b) : |a - b| \leq N\} \subset \mathbb{Z}^d \times \mathbb{Z}^d$$

of the diagonal, outside some *finite* set, is covered by *finitely* many Lipschitz domains.

4.3. **Lines** $(a+\mathbb{R}c)\cap\mathbb{Z}^d$.

Proposition 4.4. (i) *If* $[a+tc]=[b+tc]$ *for all* $t\gg 1$*, then* $[a+tc]=[b+tc]$ *for all* t*.*
(ii) $[a+tc]-(a+tc)$ *is constant and* $\perp$ *to* c *for all* t *such that*

$$|a+tc|\geq d_\Delta^2(|a_c|+|c|)\,|c|\,.$$

Proof. To prove (i) we observe that

$$|a+tc|=|b+tc|\quad \forall t\gg 1,$$

which clearly implies that

$$|a+tc|=|b+tc|\quad \forall t.$$

If $|a-b|\leq\Delta$ then this implies that $[a+tc]=[b+tc]$ for all t. Otherwise, for all $t\gg 1$ there is a $d_t\notin\{a,b\}$ such that

$$[d_t+tc]=[a+tc].$$

Since the diameter of each block is $\leq d_\Delta$, it follows that $|d_t-a|\leq d_\Delta$. Since there are infinitely many t:s and only finitely many d_t:s, there is some d such that $d=d_t$ for at least three different t:s. Then

$$|d+tc|=|a+tc|\quad\forall t.$$

If now $|a-d|\leq\Delta$ and $|d-b|\leq\Delta$, then $[a+tc]=[b+tc]$ for all t. Otherwise, for all $t\gg 1$ there is a $e_t\notin\{a,b,d\}$ such that

$$[e_t+tc]=[a+tc],$$

and the statement follows by a finite induction.

To prove (ii) it is enough to consider $a=a_c$. Let $b\in[a+tc]-(a+tc)$ for some $t=t_0$, such that $|a+tc|\geq d_\Delta^2(|a_c|+|c|)\,|c|$. Then $|a+tc+b|^2=|a+tc|^2$, i.e.

$$2t<b,c>+2<b,a>+|b|^2=0.$$

If $<b,c>\neq 0$, then

$$|a+tc|\leq|a|+|t<b,c>|\,|c|\leq|a|+(|<b,a>|+\frac{1}{2}\,|b|^2)\,|c|$$

which, in view of Proposition 4.1 is less than

$$((d_\Delta+1)\,|a|+\frac{1}{2}d_\Delta^2)\,|c|\,.$$

But this is impossible under the assumption on $a+tc$. Therefore $<b,c>=0$, i.e. $[a+tc]-(a+tc)\perp$ to c.

Moreover it follows that $|a+tc+b|=|a+tc|$ for all t. If $|b|\leq\Delta$ it follows that $[a+b+tc]=[a+tc]$ for all t. If not, there is a sequence of points $0=b_1,b_2,\ldots,b_k=b$ in $[a+tc]-(a+tc)$ such that $|b_{j+1}-b_j|\leq\Delta$

for all j. By a finite induction it follows that $[a+b+tc]=[a+tc]$ for all t. Hence

$$[a+tc]=(t-t_0)c+[a+t_0c]$$

for all $t\geq t_0$. $\square$

More on Töplitz-Lipschitz matrices. For a matrix $Q:\mathcal{L}\times\mathcal{L}\to\mathbb{C}$ we denote by $Q(tc)$ the matrix whose elements are

$$Q_a^b(tc)=:Q(tc)_a^b=Q_{a+tc}^{b+tc}.$$

[4] Clearly for any subset I,J of $\mathcal{L}$

$$Q_I^J(tc)=:Q(tc)_I^J=Q_{I+tc}^{J+tc}$$

in an obvious sense.

Corollary 4.5. *Let $\Lambda\geq d_\Delta^2$. If $(a,b)\in D_\Lambda(c)$, then*

$$Q_{[a]_\Delta}^{[b]_\Delta}(tc)=Q_{[a+tc]_\Delta}^{[b+tc]_\Delta}$$

for all $t\geq 0$. In particular, if Q is Töplitz at ∞, then

$$\lim_{t\to\infty}\left\|Q_{[a]_\Delta}^{[b]_\Delta}(tc)-Q_{[a]_\Delta}^{[b]_\Delta}(\infty c)\right\|=0.$$

Proof. This follows immediately from Proposition 4.4(ii). $\square$

5. Small Divisor Estimates

Let $\omega\in U\subset\mathbb{R}^{\mathcal{A}}$ be a set in

$$\{|\omega|\leq C_1\}. \tag{13}$$

If $A:\mathcal{L}\times\mathcal{L}\to gl(2,\mathbb{C})$ depends on the parameters $\omega\in U$ we define

$$|A|_{\left\{\substack{\gamma\\ U}\right\}}=\sup_{\omega\in U}(|A(\omega)|_\gamma\,,|\partial_\omega A(\omega)|_\gamma),$$

where the derivative should be understood in the sense of Whitney. [5] If the matrices $A(\omega)$ and $\partial_\omega A(\omega)$ are Töplitz at ∞ for all $\omega\in U$, then we can define

$$<A>_{\left\{\substack{\Lambda,\gamma\\ U}\right\}}=\sup_{\omega\in U}(<A(\omega)>_{\Lambda,\gamma},<\partial_\omega A(\omega)>_{\Lambda,\gamma}).$$

It is clear that if $<A>_{\left\{\substack{\Lambda,\gamma\\ U}\right\}}$ is finite, then the convergence to the Töplitz-limits takes place in the $\mathcal{C}^1$-norm in ω.

[4] Notice the abuse of notation. In order to avoid confusion we shall in this section denote the Töplitz-limit in the direction c by $Q(\infty c)$.

[5] This implies that $<A>_{\left\{\substack{\Lambda,\gamma\\ U}\right\}}$ bounds a $\mathcal{C}^1$-extension of $A(\omega)$ to the full ball $\{|\omega|\leq C_1\}$.

5.1. **Normal form matrices.**
A matrix $A:\mathcal{L}\times\mathcal{L}\to gl(2,\mathbb{C})$ is on *normal form* – denoted $\mathcal{NF}_\Delta$ – if

(i) A is real valued;
(ii) A is symmetric, i.e. $A_b^a={}^t(A_a^b)$;
(iii) $\pi A=A$;
(iv) A is block-diagonal over $\mathcal{E}_\Delta$, i.e. $A_a^b=0$ for all $[a]_\Delta\neq[b]_\Delta$.

For a normal form matrix A the quadratic form $\frac{1}{2}<\zeta,A\zeta>$ takes the form

$$\frac{1}{2}<\xi,A_1\xi>+<\xi,A_2\eta>+\frac{1}{2}<\eta,A_1\eta>$$

where A_1+iA_2 is a Hermitian (scalar-valued) matrix.

Let

$$w=\begin{pmatrix}u_a\\ v_a\end{pmatrix}=C^{-1}\begin{pmatrix}\xi_a\\ \eta_a\end{pmatrix}\qquad C=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}}&\frac{i}{\sqrt{2}}\end{pmatrix}$$

and define ${}^tCAC:\mathcal{L}\times\mathcal{L}\to gl(2,\mathbb{C})$ through

$$({}^tCAC)_a^b={}^tCA_a^bC.$$

Then A is on normal form if, and only if,

$$\frac{1}{2}<w,{}^tCACw>=\frac{1}{2}<u,Qv>,$$

where

(i) Q is Hermitian;
(ii) Q is *block-diagonal* over over $\mathcal{E}_\Delta$.

We denote for any subset I of $\mathcal{L}$

$$Q_I=Q_I^I=Q|_{I\times I}.$$

We say that a scalar-valued matrix with this propetries Q is on *normal form*, denoted $\mathcal{NF}_\Delta$. Notice that scalar valued normal form matrix Q will in general not become a $gl(2,\mathbb{R})$-valued normal form matrix through the identification $Q_a^b=Q_a^bI$.

5.2. **Small divisor estimates.** Let $\Omega=\Omega(\omega):\mathcal{L}\times\mathcal{L}\to\mathbb{R}$ be a real scalar valued diagonal matrix with diagonal elements

$$\Omega_a(\omega).$$

Assume that for all $a\in\mathcal{L}$ and all $\omega\in U$

$$\left|\partial_\omega^\nu(\Omega_a(\omega)-|a|^2)\right|\leq C_2e^{-C_3|a|},\quad C_3>0,\quad \nu=0,1,\tag{14}$$

and for all $k \in \mathbb{Z}^n \setminus 0$

$$\left\{ \begin{array}{l} |\partial_\omega(<k,\omega> + \Omega_a(\omega))| \geq C_4 \\ |\partial_\omega(<k,\omega> + \Omega_a(\omega) + \Omega_b(\omega))| \geq C_4 \\ |\partial_\omega(<k,\omega> + \Omega_a(\omega) - \Omega_b(\omega))| \geq C_4 \quad |a| \neq |b| \, . \end{array} \right. \tag{15}$$

Let $H = H(\omega) : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ be $\mathcal{NF}_\Delta$ for $\omega \in U$ and assume that

$$\|\partial_\omega H(\omega)\| \leq \frac{C_4}{4}, \quad \omega \in U. \tag{16}$$

Here $\| \cdot \|$ is the operator norm.

Let us first formulate and prove the easy case.

Proposition 5.1. *Let $\Delta' > 0$ and $\kappa > 0$. Assume that U verifies (13), that $\Omega(\omega)$ is real diagonal and verifies (14)+(15) and that $H(\omega)$ is $\mathcal{NF}_\Delta$ and verify (16) for all $\omega \in U$.*

Then there exists a closed set $U' \subset U$,

$$Leb(U \setminus U') \leq \text{cte.}(\max(\Delta', d_\Delta^2))^{d+\#\mathcal{A}-1} \kappa C_1^{2d-1}$$

such that for all $\omega \in U'$, all $0 < |k| \leq \Delta'$ and for all

$$[a]_\Delta, [b]_\Delta \tag{17}$$

we have

$$|<k,\omega>| \geq \kappa, \tag{18}$$

$$|<k,\omega> + \alpha(\omega)| \geq \kappa, \quad \forall \ \alpha(\omega) \in \sigma((\Omega + H(\omega))_{[a]_\Delta}) \tag{19}$$

and

$$|<k,\omega> + \alpha(\omega) + \beta(\omega)| \geq \kappa \quad \forall \left\{ \begin{array}{l} \alpha(\omega) \in \sigma((\Omega + H)(\omega)_{[a]_\Delta}) \\ \beta(\omega) \in \sigma((\Omega + H)(\omega)_{[b]_\Delta}). \end{array} \right. \tag{20}$$

Moreover the κ-neighborhood of $U' \subset U$ satisfies the same estimate.

The constant cte. *depends on the dimensions d and $\#\mathcal{A}$ and on C_4.*

Proof. It is enough to prove the statement for $\Delta' \geq d_\Delta^2$. Let us prove the estimate (20), the other two being the same, but easier.

Since $|k| \leq \Delta'$, $|<k,\omega>| \lesssim C_1\Delta'$. [6] If the block I intersects $\{|c| \gtrsim \sqrt{C_1\Delta'}\}$, then any eigenvalue α of $(\Omega + H)_I(\omega)$ verifies

$$|\alpha| \gtrsim C_1\Delta'.$$

Hence

$$|< k,\omega > + \alpha + \beta| \gtrsim 1.$$

[6] In this proof $\lesssim$ depends on $d, \#\mathcal{A}$ and on C_4.

So it suffices to consider pair of eigenvalues $\alpha \in \sigma((\Omega + H)_I(\omega))$ and $\beta \in \sigma((\Omega + H)_J(\omega))$ with blocks

$$I, J \subset \{|c| \lesssim \sqrt{C_1 \Delta'}\}.$$

(Here we used that $\Delta' \geq d_\Delta^2$.) These are at most

$$\lesssim (C_1 \Delta')^d$$

many possibilities.

Now, $(<k, \omega> + \alpha + \beta)$ is an eigenvalue of the Hermitian operator $<k, \omega> + \mathcal{H}(\omega)$,

$$\mathcal{H}(\omega) : X \mapsto (\Omega + H)_I(\omega) X + (\Omega + H)_J(\omega) X$$

which extends $\mathcal{C}^1$ to $\{|\omega| < C_1\}$. Assumption (16) shows that

$$\|\partial_\omega H(\omega)\| \leq \frac{C_4}{4}$$

and (15), via Proposition 9.3 (Appendix), now implies that the inverse of $\mathcal{H}(\omega)$ is bounded from above by $\frac{1}{\kappa}$ – this gives a lower bound for its eigenvalues – outside a set of Lebesgue measure

$$\lesssim \frac{\kappa}{|k|} C_1^{d-1}.$$

Summing now over all these blocks I, J and all $|k| \leq \Delta'$ gives the result. □

We now turn to the main problem.

Proposition 5.2. *Let $\Delta' > 0$ and $\kappa > 0$. Assume that U verifies (13), that $\Omega(\omega)$ is real diagonal and verifies (14)+(15) and that $H(\omega)$ and $\partial H(\omega)$ are Töplitz at ∞ and $\mathcal{NF}_\Delta$ and verify (16) for all $\omega \in U$.*

Then there exists a subset $U' \subset U$,

$$Leb(U \setminus U') \leq \text{cte.} \max(\Delta', d_\Delta^2, \Lambda)^{exp + \#\mathcal{A} - 1} (1 + <H>_{\left\{\substack{\Lambda \\ U}\right\}})^d \kappa^{\frac{1}{d}} C_1^{d-1},$$

such that, for all $\omega \in U'$, $0 < |k| \leq \Delta'$ and all

$$\text{dist}([\mathrm{a}]_\Delta, [\mathrm{b}]_\Delta) \leq \Delta' \tag{21}$$

we have

$$|<k, \omega> + \alpha(\omega) - \beta(\omega)| \geq \kappa \quad \forall \begin{cases} \alpha(\omega) \in \sigma((\Omega + H)(\omega)_{[a]_\Delta}) \\ \beta(\omega) \in \sigma((\Omega + H)(\omega)_{[b]_\Delta}). \end{cases} \tag{22}$$

Moreover the κ-neighborhood of $U \setminus U'$ satisfies the same estimate.

The exponent exp depends only on d. The constant cte. *depends on the dimensions d and $\#\mathcal{A}$ and on C_2, C_3, C_4.*

Proof. The proof goes in the following way: first we prove an estimate in a large finite part of $\mathcal{L}$ (this requires parameter restriction); then we assume an estimate "at ∞" of $\mathcal{L}$ and we prove, using the Lipschitz-property, that this estimate propagate from "∞" down to the finite part (this requires no parameter restriction); in a third step we have to prove the assumption at ∞. This will be done by reducing it to an estimate in dimension $d-1$. In order to carry out the induction we must carry out the argument for a full rank lattice in $\mathbb{Z}^d$ and not only for $\mathbb{Z}^d$ itself.

So $\mathcal{L}$ is a lattice in $\mathbb{Z}^d$, minus a finite part.

Let us notice that it is enough to prove the statement for $\Delta' \geq \max(\Lambda, d_\Delta^2)$. We let $[\ ]$ denote $[\ ]_\Delta$. Let $\Omega \approx (\Delta')^3$.

1. Finite part. For the finite part, let us suppose a belongs to

$$\{a \in \mathcal{L} : |a| \lesssim (1 + \frac{1}{\kappa'} d_\Delta^d < H >_{\{\substack{\Lambda \\ U}\}})\Omega^{2d-1}\}, \tag{23}$$

[7] where $\kappa' = \kappa^{\frac{d-1}{d}}$. These are finitely many possibilities and $(22)_\kappa$ is fulfilled, for all $[a]$ satisfying (23), all $[b]$ with $|a-b| \lesssim \Delta'$ and all $|k| \leq \Delta'$, outside a set of Lebesgue measure

$$\lesssim\ (1 + d_\Delta^d < H >_{\{\substack{\Lambda \\ U}\}})^d \Omega^{2(2d-1)} (\Delta')^{d+n-1} \frac{\kappa}{\kappa'} C_1^{d-1}. \tag{24}$$

(This is the same argument as in Proposition 5.1.)

Let us now get rid of the diagonal terms $\hat{V}(a,\omega) = \Omega_a(\omega) - |a|^2$ which, by (14), are

$$\lesssim\ C_2 e^{-|a| C_3}.$$

We include them into H. Since they are diagonal, H will remain on normal form. Due to the exponential decay, H and $\partial_\omega H$ will remain Töplitz at ∞. The estimate of the Lipschitz norm gets worse but this is innocent in view of the estimates. Also the estimate of the Lipschitz norm gets worse, but if a is outside (23) then condition (16) remains true with a slightly worse bound, say

$$\|\partial_\omega H(\omega)\| \leq \frac{3C_4}{8}, \quad \omega \in U.$$

So from now on, a is outside (23) and

$$\Omega_a = |a|^2.$$

2. Condition at ∞. For each vector $c \in \mathbb{Z}^d$ such that $|c| \leq \Omega^{d-1}$, we suppose that the Töplitz limit $H(c,\omega)$ verifies $(22)_{\kappa'}$ for (21) and for

$$([a]_{[\Delta]} - [b]_{[\Delta]}) \perp c. \tag{25}$$

[7] In this proof $\lesssim$ depends on $d, \#\mathcal{A}$ and on C_2, C_3, C_4.

It will become clear in the next part why we only need $(22)_{\kappa'}$ + (21) under the supplementary restriction (25).

3. Propagation of the condition at ∞. We must now prove that for $|b-a| \lesssim \Delta'$ and an $a \in \mathcal{L}$ outside (23), $(22)_\kappa$ is always fulfilled.

By the Corollary 4.3 we get

$$[a]\times[a] \text{ and } [b]\times[b] \subset \bigcup_{|c|\lesssim \Omega^{d-1}} D_{\Omega'}(c), \quad \Omega' \approx \frac{\Omega}{\Delta'},$$

and by Proposition 4.4 – notice that $\Omega' \geq d_\Delta^2$ –

$$[a+tc] = [a]+tc \quad \text{and} \quad [b+tc] = [b]+tc$$

for $t \geq 0$ and

$$[a]-a,\ [b]-b \ \perp\ c.$$

It follows that

$$\lim_{t\to\infty} H_{[a+tc]}(\omega) = H_{[a]}(c,\omega) \quad \text{and} \quad \lim_{t\to\infty} H_{[b+tc]}(\omega) = H_{[b]}(c,\omega).$$

The matrices $\Omega_{[a+tc]}$ and $\Omega_{[b+tc]}$ do not have limits as $t\to\infty$. However, for any $(\#[a]\times\#[b])$-matrix X,

$$\Omega_{[a+tc]}X - X\Omega_{[b+tc]} \ = \ \Omega_{[a]}X - X\Omega_{[b]} + 2t <a-b,c> X$$

for $t \geq 0$, and we must discuss two different cases according to if $< c, b-a >= 0$ or not.

Consider for $t \geq 0$ a pair of eigenvalues

$$\begin{cases} \alpha_t \in \sigma((\Omega + H(\omega))_{[a+tc]}) \\ \beta_t \in \sigma((\Omega + H(\omega))_{[b+tc]}) \end{cases}$$

Case I: $<c, b-a>= 0$. Here

$$(\Omega + H(\omega))_{[a+tc]}X - X(\Omega + H(\omega))_{[b+tc]}$$

equals

$$(|a|^2 + H(\omega))_{[a+tc]}X - X(|b|^2 + H(\omega))_{[b+tc]}$$

– the linear and quadratic terms in t cancel!

By continuity of eigenvalues,

$$\lim_{t\to\infty}(\alpha_t - \beta_t) = (\alpha_\infty - \beta_\infty),$$

where

$$\begin{cases} \alpha_\infty \in \sigma((|a|^2 + H(c,\omega))_{[a]}) \\ \beta_\infty \in \sigma((|b|^2 + H(c,\omega))_{[b]}) \end{cases}$$

Since $[a]$ and $[b]$ verify (25), our assumption on $H(c,\omega)$ implies that $(\alpha_\infty - \beta_\infty)$ verifies $(22)_{\kappa'}$.

For any two $a, a' \in [a]$ we have, since a violates (23) and $|a-a'| \leq d_\Delta$,

$$\frac{|a'|}{|c|} \approx \frac{|a|}{|c|}.$$

Hence

$$\left\| H_{[a]}(\omega) - H_{[a]}(c,\omega) \right\| \frac{|a|}{|c|} \lesssim d_\Delta^d < H >_{\{ \substack{\Lambda \\ U} \}},$$

and the same for $[b]$. Recalling that a and, hence, b violates (23) this implies

$$\left\| H_{[d]}(\omega) - H_{[d]}(c,\omega) \right\| \leq \frac{\kappa'}{4}, \quad d = a, b.$$

By Lipschitz-dependence of eigenvalues (of Hermitian operators) on parameters, this implies that

$$|(\alpha_0 - \beta_0) - (\alpha_\infty - \beta_\infty)| \leq \frac{\kappa'}{2}$$

and we are done.

Case II: $<c, b-a> \neq 0$. We write $a = a_c + \tau c$. Since

$$|a| \geq \Omega'(|a_c| + |c|)\, |c|\,,$$

it follows that

$$|a_c| \lesssim \frac{1}{\Omega'} \frac{|a|}{|c|}.$$

Now, $\alpha_0 - \beta_0$ is close to $|a|^2 - |b|^2$ and

$$|a|^2 - |b|^2 = -2\tau <c, b-a> -2 <a_c, b-a> - |b-a|^2\,.$$

Since $| <c, b-a> | \geq 1$ it follows that

$$\tau \lesssim |\alpha_0 - \beta_0| + |a_c|\Delta' + (\Delta')^2.$$

If now $|\alpha_0 - \beta_0| \lesssim \Delta'$ then

$$|a| \leq |a_c| + |\tau\, c| \lesssim |a_c|\, \Delta'\, |c| + (\Delta')^2\, |c| \lesssim (\Delta')^2 |c|.$$

Since $\Omega' \gtrsim (\Delta')^2$ this is impossible.

Therefore $|\alpha_0 - \beta_0| \gtrsim \Delta'$ and $(22)_\kappa$ holds.

4. Condition at ∞ – Reduction of dimension. We consider the Töplitz limit $H(c)$. This matrix is, by the assumption on H, Töplitz at ∞ and, by construction, Töplitz in the direction c, i.e.

$$H_{a+tc}^{b+tc}(c) = H_a^b(c), \quad \forall a, b, a+tc, b+tc \in \mathcal{L}.$$

Moreover, it is Hermitian and, by Proposition 4.4(ii), block diagonal over $\mathcal{E}_\Delta$. The same is true for its first order ω-derivatives.

We want to prove that for each $c \in \mathcal{L}$, $|c| \leq \Omega^{d-1}$, $H(c,\omega)$ verifies $(22)_{\kappa'}$ for (21) and (25) with $\omega \in U \setminus U'$. This last condition implies that both $[a]-a$ and $[b]-b$ are $\perp c$ and therefore $[a]$ and $[b]$ belongs to one and the same affine sub-lattice

$$f+\mathcal{L}^c, \quad \mathcal{L}^c \perp c \text{ and } f \in \mathcal{L}.$$

It is clear that it suffices to consider only primitive vectors c.

4.1 Special case. Let us first consider the case when c is one of the standard basis vectors, for example

$$c = e^d =: (0, \dots, 0, 1).$$

In this case $\mathcal{L}^c \subset \mathbb{Z}^{d-1}$ and we can chose f to be parallell to c, i.e. orthogonal to $\mathcal{L}^c$.

For a matrix Q on $\mathcal{L} \times \mathcal{L}$ we define a matrix Q^f on $\mathcal{L}^c \times \mathcal{L}^c$ by

$$(Q^f)_{a'}^{b'} = Q_{(a',0)+f}^{(b',0)+f}.$$

If $\Omega^f + H^f(c)$ verifies $(21)+(22)_{\kappa'}$ for all f, then $\Omega + H(c)$ verifies $(21)+(22)_{\kappa'}+(25)$.

Ω^f is diagonal with diagonal elements

$$\Omega_{a'}^f = |a'|^2 + |f|^2,$$

and $H^f(c)$ is independent of f since $H(c)$ is Töplitz in the direction c. Therefore $(21)+(22)_{\kappa'}$ holds for all f if and only if it holds for one f. Let us take $f=0$.

The diagonal elements of $\Omega^f(\omega)$ verify (14)+(15) for all $\omega \in U$. The matrices $H^f(c,\omega)$ and $\partial_\omega H^f(c,\omega)$ are $\mathcal{NF}_\Delta$ for all $\omega \in U$ because

$$[a']_\Delta = [b']_\Delta \Longrightarrow [(a',0)+f]_\Delta = [(b',0)+f]_\Delta.$$

They are Töplitz at ∞, (16) remains true and

$$< H^f(c) >_{\left\{\substack{\Lambda \\ U}\right\}} \le < H(c) >_{\left\{\substack{\Lambda \\ U}\right\}} .$$

So we can, by induction, apply the Proposition to $\Omega^f + H^f(c)$ which implies that, outside a set of Lebesgue measure

$$\lesssim (\Delta')^{\exp + \#\mathcal{A} - 1} (1 + < H(c) >_{\left\{\substack{\Lambda \\ U}\right\}})^{d-1} (\kappa')^{\frac{1}{d-1}} C_1^{d-1} \tag{26}$$

$(21)+(22)_{\kappa'}+(25)$ holds.

4.2 General Case. Let $c^1, c^2, \dots, c^d$. be an orthogonal set of vectors in $\mathbb{Z}^d$ such that c^d is parallel to c and all c_j have the same length

$$\Gamma \lesssim |c|^{\exp_d} \lesssim (\Delta')^{\exp'_d}.$$

Let

$$C = \left(\begin{array}{c} c^1 \dots c^d \end{array} \right),$$

i.e. $C \in GL(\mathbb{Z}^d)$ is an orthogonal matrix multiplied by the positive integer Γ.

Now we make the linear transformation

$$\tilde{x} = {}^tCx \quad \Longleftrightarrow \quad x = \frac{1}{\Gamma^2}C\tilde{x}.$$

It takes the lattice $\mathcal{L} \subset \mathbb{Z}^d$ to a lattice $\tilde{\mathcal{L}} \subset \mathbb{Z}^d$. (Notice that $\tilde{\mathcal{L}}$ is only a lattice in $\mathbb{Z}^d$ even if $\mathcal{L} = \mathbb{Z}^d$.) The transformation takes c to $\tilde{c} = |c|\Gamma e^d$ and, hence, $\mathcal{L}^c$ to $\tilde{\mathcal{L}}^{\tilde{c}} \subset \mathbb{Z}^{d-1} \times \{0\}$. The quadratic form $|\cdot|_d^2$ on $\mathbb{Z}^d$ will verify

$$|a|_d^2 = \frac{1}{\Gamma^2}\,|\tilde{a}|_d^2\,,$$

and when restricted to the affine sub-lattice $\tilde{f} + \tilde{\mathcal{L}}^{\tilde{c}}$ it becomes

$$\frac{1}{\Gamma^2}(|\tilde{a}'|_{d-1}^2 + |\tilde{f}_d|^2), \quad \tilde{a} = (\tilde{a}', \tilde{f}_d).$$

The equivalence classes $\mathcal{E}_\Delta \cap \mathcal{L}$ are mapped to the equivalence classes $\mathcal{E}_{\tilde{\Delta}} \cap \tilde{\mathcal{L}}$, $\tilde{\Delta} = \Gamma\Delta$, which have diameter $\leq \tilde{d}_\Delta = \Gamma d_\Delta$. The Lipschitz domain $D_\Lambda(c)$ is mapped to the Lipschitz domain $D_{\tilde{\Lambda}}(\tilde{c})$, $\tilde{\Lambda} = \Gamma\Lambda$.

For a matrix $Q(\omega)$, $\omega \in U$, on $\mathcal{L} \times \mathcal{L}$ we define

$$\tilde{Q}_{\tilde{a}}^{\tilde{b}}(\tilde{\omega}) = \frac{1}{\Gamma^2}Q_a^b(\omega)$$

where

$$\tilde{\omega} = \Gamma^2\omega \subset \tilde{U} \subset \{|\tilde{\omega}| \leq \tilde{C}_1 = \Gamma^2 C_1\}.$$

Since

$$\tilde{\Omega}_{\tilde{a}} = \Omega_{\tilde{a}}$$

it follows that

$$\sigma((\Omega + \tilde{H}(c,\tilde{\omega}))_{[\tilde{a}]_{\tilde{\Delta}}} = \Gamma^2\sigma(\Omega + H(c,\omega))_{[a]_\Delta}.$$

Therefore $H(c,\omega)$ verifies

$$(21)_{\Delta,\Delta'} + (22)_{\Delta,\kappa'} + (25)_{\Delta,c}$$

if $\tilde{H}(\tilde{c},\tilde{\omega})$ verifies

$$(21)_{\tilde{\Delta},\tilde{\Delta}'} + (22)_{\tilde{\Delta},\tilde{\kappa}'} + (25)_{\tilde{\Delta},\tilde{c}},$$

where $\tilde{\Delta}' = \Gamma\Delta'$, $\tilde{\kappa}' = \Gamma^2\kappa'$.

$\tilde{H}(c,\tilde{\omega})$ and $\partial_{\tilde{\omega}}\tilde{H}(c,\tilde{\omega})$ are Töplitz at ∞ and $\mathcal{NF}_\Delta$ and verifies

$$\left\|\partial_{\tilde{\omega}}\tilde{H}(c,\tilde{\omega})\right\| = |\partial_\omega H(c,\omega)|$$

for all $\tilde{\omega} \in \tilde{U}$. Moreover $\tilde{H}(c)$ is Töplitz in direction $\tilde{c}$. We are therefore back in in previous case. Hence, outside a set of $\tilde{\omega}$ of Lebesgue measure

$$\lesssim (\tilde{\Delta}')^{\exp + \#\mathcal{A} - 1}(1 + <\tilde{H}(c)>_{\left\{\substack{\tilde{\Lambda}\\ \tilde{U}}\right\}}^{\tilde{\Delta}})^{d-1}(\tilde{\kappa'})^{\frac{2}{d-1}}\tilde{C}_1^{d-1},$$

the matrix $\tilde{H}(\tilde{c},\tilde{\omega})$ verifies $(21)_{\tilde{\Delta},\tilde{\Delta}'} + (22)_{\tilde{\Delta},\tilde{\kappa}'} + (25)_{\tilde{\Delta},\tilde{c}}$. To conclude we just observe that

$$<\tilde{H}(c)>_{\left\{\substack{\tilde{\Lambda}\\ \tilde{U}}\right\}} \leq \; \Gamma^2 <H(c)>_{\left\{\substack{\Lambda\\ U}\right\}} .$$

This finishes the induction step, and to conclude we must only verify the first step $d = 1$.

5. *Case* $d = 1$. Consider

$$\alpha \in \sigma((\Omega + H)_{[a]}) \text{ and } \beta \in \sigma((\Omega + H)_{[b]})$$

with $|a - b| \lesssim \Delta'$. $\alpha - \beta$ differs from

$$|a|^2 - |b|^2 = (a - b)(a + b)$$

by at most $2|H|$. If $\left||a|^2 - |b|^2\right|$ is $\gtrsim \Delta'$ we are done (here we use that $|<k,\omega>| \leq C_1\Delta'$), so therefore we only need to consider

$$|(a - b)(a + b)| \lesssim \Delta'.$$

If $[b] = [a] = \{a\}$, then $\alpha = \beta$ and condition (22) reduces to

$$|<k,\omega>| > \kappa,$$

i.e. only one possibility (for each k). If not, there will always be a pair $a \in [a]$ and $b \in [b]$ such that $a - b \neq 0$ and ($|a - b| \lesssim \Delta'$) hence

$$|a| \lesssim \Delta'.$$

This gives finitely many possibilities so (22) will hold outside at set of measure

$$\lesssim \; (\Delta')^d(\Delta')^{d+\#\mathcal{A}-1}\kappa.$$

□

6. The homological equations

6.1. **A first equation.**

For $k \in \mathbb{Z}^n$ consider the equation

$$i <k,\omega> S + i(\Omega(\omega) + H(\omega))S = F(\omega), \tag{27}$$

where $F(\omega) : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ and $\partial_\omega F(\omega)$ are elements in $l^2_\gamma(\mathcal{L},\mathbb{C}) = \{\xi = (\xi_a)_{a\in\mathcal{L}} : \|\xi\|_\gamma : \|\xi\|_\gamma < \infty\}$, where

$$\|\xi\|_\gamma = \sqrt{\sum_{a\in\mathcal{L}} |\xi_a|^2 e^{2\gamma|a|}\langle a\rangle^{2m_*}}$$

($\langle a\rangle = \max(1,|a|)$). Denote

$$\|F\|_{\left\{\substack{\gamma\\ U}\right\}} = \sup_{\omega\in U}(\|F(\omega)\|_\gamma\,, \|\partial_\omega F(\omega)\|_\gamma).$$

Let $U' \subset U$ be a set such that for all $\omega \in U$ the small divisor condition (19) holds, i.e. for all a

$$|\langle k,\omega\rangle + \alpha(\omega)| \geq \kappa, \quad \forall\ \alpha(\omega) \in \sigma((\Omega + H)(\omega)_{[a]_\Delta}).$$

Proposition 6.1. *Assume that $\Omega(\omega)$ is real diagonal and verifies (14) and that $H(\omega)$ and $\partial_\omega H(\omega)$ are $\mathcal{NF}_\Delta$ for all $\omega \in U$. Then the equation (27) has for all $\omega \in U'$ a unique solution $S(\omega)$ such that*

$$\|S\|_{\left\{\substack{\gamma \\ U'}\right\}} \leq \text{cte}.d_\Delta^{2d}(1 + |k| + |H|_U)\frac{1}{\kappa^2}\|F\|_{\left\{\substack{\gamma \\ U'}\right\}}.$$

The constant cte. *only depends on $d, \#\mathcal{A}$ and C_2, C_3.*

Proof. This is a standard results. There exists a $\kappa' \approx \kappa$ [8] such that $(27)_{\frac{\kappa}{2}}$ holds on $U'_{\kappa'}$. Let χ be a cut off function which is 1 on U' and 0 outside $(U')_{\kappa'}$. We now first solve the equation on $(U')_{\kappa'}$ to get a solution $\tilde{S}$ and then we define $S = \chi\tilde{S}$. □

6.2. **Truncations.**
For a matrix $Q : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ consider three truncations (defined in terms of the block-decomposition $\mathcal{E}_\Delta$)

$\mathcal{T}_{\Delta'}^{\Delta} Q = Q$ restricted to $\{(a,b) : \text{dist}([a]_\Delta, [b]_\Delta) \leq \Delta'\}$
$\mathcal{P}_c^{\Delta} Q = Q$ restricted to $\{(a,b) : ([a]_\Delta - [b]_\Delta) \perp c\}$
$\mathcal{D}_{\Delta'}^{\Delta} Q = Q$ restricted to $\{(a,b) : \text{dist}([a]_\Delta, [b]_\Delta) \leq \Delta'$ and $|a| = |b|\}$.

These truncations all commute. Moreover,

Lemma 6.2. (i)

$$\begin{cases} \left|\mathcal{T}_{\Delta'}^{\Delta} Q\right|_{\left\{\substack{\gamma \\ U}\right\}} & \leq & |Q|_{\left\{\substack{\gamma \\ U}\right\}} \\ \langle\mathcal{T}_{\Delta'}^{\Delta} Q\rangle_{\left\{\substack{\Lambda,\gamma \\ U}\right\}} & \leq & \langle Q\rangle_{\left\{\substack{\Lambda,\gamma \\ U}\right\}}, \end{cases}$$

for any $\Lambda \geq d_\Delta^2$. Moreover

$$(\mathcal{T}_{\Delta'}^{\Delta} Q)(c) = \mathcal{T}_{\Delta'}^{\Delta}(Q(c))$$

for all c.

(ii) *The same result holds for $\mathcal{P}_c^{\Delta}$.*

(iii)

$$\begin{cases} \left|\mathcal{D}_{\Delta'}^{\Delta} Q\right|_{\left\{\substack{\gamma \\ U}\right\}} & \leq & |Q|_{\left\{\substack{\gamma \\ U}\right\}} \\ \langle\mathcal{D}_{\Delta'} Q\rangle^{\Delta}_{\left\{\substack{\Lambda,\gamma \\ U}\right\}} & \leq & \langle Q\rangle_{\left\{\substack{\Lambda,\gamma \\ U}\right\}}, \end{cases}$$

for any $\Lambda \geq (d_{\Delta'})^2$. Moreover

$$(\mathcal{D}_{\Delta'}^{\Delta} Q)(c) = \mathcal{D}_{\Delta'}^{\Delta}(Q(c))$$

for all c.

---

[8] $\lesssim$ depends in this proof on $d, \#\mathcal{A}$ and C_2, C_3.

Proof. By Proposition 4.4(ii)

$$[a+tc]_\Delta = [a]_\Delta + tc \text{ and } [b+tc]_\Delta = [b]_\Delta + tc$$

for all $t \geq 0$ if $(a,b) \in D_\Lambda(c) \subset D_{d_\Delta^2}(c)$. Therefore either

$$(\mathcal{T}_{\Delta'}^{\Delta} Q)_{[a+tc]_\Delta}^{[b+tc]_\Delta} = Q_{[a+tc]_\Delta}^{[b+tc]_\Delta} \quad \forall t \geq 0,$$

or

$$(\mathcal{T}_{\Delta'}^{\Delta} Q)_{[a+tc]_\Delta}^{[b+tc]_\Delta} = 0 \quad \forall t \geq 0.$$

(ii) follows by exactly the same argument and (iii) is similar. Indeed, for (a,b) such that $\mathrm{dist}([a]_\Delta,[b]_\Delta) \leq \Delta'$ we have

$$|a| = |b| \iff [a]_{\Delta'} = [b]_{\Delta'}.$$

□

6.3. **A second equation,** $k \neq 0$.

For $k \in \mathbb{Z}^n \setminus \{0\}$ consider the equation

$$i <k,\omega> S + i[(\Omega(\omega) + H(\omega)), S] = \mathcal{T}_{\Delta'}^{\Delta} F(\omega) \tag{28}$$

where $F(\omega) : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ and $\partial_\omega F(\omega)$ are Töplitz at ∞.

Let $U' \subset U$ be a set such that for all $\omega \in U$ the small divisor condition (21) + (22) holds, i.e.

$$|<k,\omega> + \alpha(\omega) - \beta(\omega)| \geq \kappa \quad \forall \left\{ \begin{array}{l} \alpha(\omega) \in \sigma((\Omega + H)(\omega)_{[a]_\Delta}) \\ \beta(\omega) \in \sigma((\Omega + H)(\omega)_{[b]_\Delta}) \end{array} \right.$$

for

$$\mathrm{dist}([\mathrm{a}]_\Delta, [\mathrm{b}]_\Delta) \leq \Delta'.$$

These conditions depend on κ and Δ', and in order to simplify the estimate a little we shall assume that $\kappa < 1$.

Proposition 6.3. *Assume that U verifies (13), that $\Omega(\omega)$ is real diagonal and verifies (14) and that $H(\omega)$ and $\partial H(\omega)$ are Töplitz at ∞ and $\mathcal{NF}_\Delta$ for all $\omega \in U$. Then the equation*

$$(28) \quad \textit{and} \quad S = \mathcal{T}_{\Delta'}^{\Delta} S$$

has for all $\omega \in U'$ a unique solution $S(\omega)$ verifying

(i)

$$|S|_{\{\substack{\gamma \\ U'}\}} \leq \text{cte.}\ \frac{1}{\kappa^2} d_\Delta^{2d} e^{2\gamma d_\Delta} (1 + |k| + |H|_U) \, |F|_{\{\substack{\gamma \\ U'}\}}\,;$$

(ii) *$S(\omega)$ and $\partial_\omega S(\omega)$ are Töplitz at ∞ and the Töplitz-limits verify*

$$\left\{ \begin{array}{l} i <k,\omega> S + i[(\Omega(\omega) + H(c,\omega)), S] = \mathcal{T}_{\Delta'}^{\Delta} \mathcal{P}_c^{\Delta} F(c,\omega) \\ S = \mathcal{T}_{\Delta'}^{\Delta} S; \end{array} \right.$$

(iii)

$$< S >^{\Delta}_{\left\{\substack{\Lambda'+d_\Delta+2,\gamma \\ U'}\right\}} \le \text{cte.}$$

$$\frac{1}{\kappa^3} d_\Delta^{2d} e^{2\gamma d_\Delta}(1+|k|+< H >_{\left\{\substack{\Lambda \\ U'}\right\}}) < F >_{\left\{\substack{\Lambda',\gamma \\ U'}\right\}}$$

for any

$$\Lambda' \gtrsim \max(\Lambda, d_\Delta^2, \Delta', \sup_U \|H(\omega)\|).$$

The constant cte. *only depends on the dimensions* d *and* $\#\mathcal{A}$ *and on* C_1, C_2, C_3, C_4.

Proof. Let us first get rid of the diagonal terms $\hat{V}(a,\omega) = \Omega_a(\omega) - |a|^2$ which by (14) are

$$\lesssim C_2 e^{-|a|C_3}.$$

[9] We include them into H – in view of the estimates of the proposition this is innocent. Let us also notice that it is enough to prove the statement for $\Lambda \ge d_\Delta^2$. We first assume that $F = \mathcal{T}_{\Delta'}^{\Delta} F$.

So from now on we assume $\Omega_a = |a|^2$ and $\Lambda \ge d_\Delta^2$. We shall denote the blocks $[\ ]_\Delta$ by $[\ ]$.

Block decompose the operator $\mathcal{H}(\omega) : S \mapsto [(\Omega(\omega) + H(\omega)), S]$ over $\mathcal{E}_\Delta$:

$$(\mathcal{H}(\omega)S)_{[a]}^{[b]} = (\Omega + H)(\omega)_{[a]} S_{[a]}^{[b]} - S_{[a]}^{[b]}(\Omega + H)(\omega)_{[b]}.$$

Then the equation becomes

$$\left\{ \begin{array}{ll} i < k,\omega > S_{[a]}^{[b]} + i\mathcal{H}(\omega) S_{[a]}^{[b]} = F_{[a]}^{[b]} & \text{if } \operatorname{dist}([a],[b]) \le \Delta' \\ S_{[a]}^{[b]} = 0 & \text{if not.} \end{array} \right. \tag{29}$$

Since $\Omega + H$ is Hermitian, under condition (21) + (22) equation (29) has a unique solution which is $\mathcal{C}^1$ in ω and verifies

$$\left\| S_{[a]}^{[b]} \right\| \le \frac{1}{\kappa} \left\| F_{[a]}^{[b]} \right\|,$$

hence

$$|S|_\gamma \le \frac{1}{\kappa} d_\Delta^d e^{2\gamma d_\Delta} |F|_\gamma . \tag{30}$$

1. Töplitz at ∞. Let Q be a matrix on $\mathcal{L}$ and denote by $Q(tc)$ the matrix whose elements are

$$Q_a^b(tc) = Q_{a+tc}^{b+tc}.$$

[9] In this proof $\lesssim$ depends on $d, \#\mathcal{A}$ and on C_2, C_3, C_4.

[10]

By Proposition 4.4, for $(a,b) \in D_{\Lambda'}(c)$ – notice that $\Lambda' \geq d_\Delta^2$ –

$$[a+tc] = [a] + tc \quad \text{and} \quad [b+tc] = [b] + tc$$

for $t \geq 0$ and

$$[a] - a,\ [b] - b \ \perp\ c.$$

It follows that

$$\begin{aligned} i <k,\omega> S_{[a]}^{[b]}(tc) + i(\Omega + H)_{[a]}(tc) S_{[a]}^{[b]}(tc) - \\ S_{[a]}^{[b]}(tc)(\Omega + H)_{[b]}(tc)) = F_{[a]}^{[b]}(tc) \end{aligned} \tag{31}$$

for all $t \geq 0$.

Moreover $H_{[a]}(tc),\ H_{[b]}(tc)$ and $F_{[a]}^{[b]}(tc)$ have limits as $t \to \infty$. $\Omega_{[a]}(tc)$ and $\Omega_{[b]}(tc)$ do not have limits, and we must analyze two different cases according to if $<c, a-b>= 0$ or not.

Case I: $<c, a-b>= 0$. We have that $\Omega_{[a]}(tc)X - X\Omega_{[b]}(tc)$ (for any $(\#[a] \times \#[b])$-matrix X) equals

$$|a|^2 X - X |b|^2$$

– the linear and quadratic terms in t cancel! Therefore equation (31) has a limit as $t \to \infty$:

$$i <k,\omega> X + i(\Omega_a + H_{[a]}(\infty c))X - X(\Omega_b + H_{[b]}(\infty c)) = F_{[a]}^{[b]}(\infty c).$$

Since eigenvalues are continuous in parameters we have

$$|<k,\omega> + \alpha - \beta| \geq \kappa \quad \forall \begin{cases} \alpha \in \sigma(|a|^2 + H_{[a]}(\infty c)) \\ \beta \in \sigma(|b|^2 + H_{[b]}(\infty c)). \end{cases}$$

Therefore the limit equation has a unique solution X which is $\mathcal{C}^1$ in ω and verifies

$$\|X\| \leq \frac{1}{\kappa} \left\| F_{[a]}^{[b]}(\infty c) \right\|.$$

Since $S_{[a]}^{[b]}(tc)$ is bounded, it follows from uniqueness that

$$S_{[a]}^{[b]}(tc) \to S_{[a]}^{[b]}(\infty c) = X$$

as $t \to \infty$.

Case II: $<c, a-b> \neq 0$. We have that $\Omega_{[a]}(tc)X - X\Omega_{[b]}(tc)$ equals

$$(2t <a,c> + |a|^2)X - X(2t <b,c> |b|^2)$$

– only the quadratic terms in t cancel! Dividing (31) by t and letting $t \to \infty$, the limit equation becomes

$$2 <c, a-b> X = 0.$$

[10]In order to avoid confusion we shall denote the Töplitz-limit in the direction c by $Q(\infty c)$.

It has the unique solution $X = 0$. For the same reason as in the previous case we have that

$$S_{[a]}^{[b]}(tc) \to S_{[a]}^{[b]}(\infty c) = 0$$

as $t \to \infty$.

We have thus shown that the solution S is Töplitz at ∞, and that its Töplitz limits $S(\infty c)$ verify

$$\begin{cases} (i <k,\omega> S(\infty c) + i\mathcal{H}(c,\omega)S(\infty c))_{[a]}^{[b]} & \\ = F_{[a]}^{[b]}(\infty c) & \text{if } \operatorname{dist}([a],[b]) \le \Delta', \text{ and} \\ & [a]-[b] \perp c \\ S(\infty c)_{[a]}^{[b]} = 0 & \text{if not} \end{cases}$$

and

$$|S(c)|_{\gamma} \le \frac{1}{\kappa} d_{\Delta}^{d} e^{2\gamma d_{\Delta}} |F(c)|_{\gamma}.$$

2. Estimate of Lipschitz norm. Consider the "derivative" ∂_c:

$$\partial_c Q_{[a]}^{[b]}(tc) = (Q_{[a]}^{[b]}(tc) - Q_{[a]}^{[b]}(\infty c)) \max_{d=a,b} \frac{|d|}{|c|}.$$

(Notice that the definition does not depend on the choice of representatives a and b in $[a]$ and $[b]$ respectively.) We shall "differentiate" equation (31) and estimate the solution of the "differentiated" equation over $[a] \times [b] \subset D_{\Lambda'}(c) \subset D_{\Lambda}(c)$. By Corollary 2.2(iii) this will provide us with an estimate of the Lipschitz constant $\operatorname{Lip}_{\Lambda'+d_\Delta+2,\gamma}$.

Since S is 0 at distances $\gtrsim \Delta' + d_\Delta$ from the diagonal we only need to treat $|a-b| \lesssim \Delta' + d_\Delta$. Again we must consider two cases.

Case I: $<c, a-b>= 0$. Subtracting the equation for $S_{[a]}^{[b]}(\infty c)$ from the equation for $S_{[a]}^{[b]}$ and multiplying by $\max(\frac{|a|}{|c|}, \frac{|b|}{|c|})$ gives

$$\begin{aligned} & i <k,\omega> \partial_c S_{[a]}^{[b]} + i(\Omega+H)_{[a]}\partial_c S_{[a]}^{[b]} - \partial_c S_{[a]}^{[b]}(\Omega+H)_{[b]} \\ & = \partial_c F_{[a]}^{[b]} - \partial_c H_{[a]} S_{[a]}^{[b]} + S_{[a]}^{[b]} \partial_c H_{[b]}, \end{aligned}$$

where the argument in all matrices is tc with $t = 0$. Now we get as for equation (31) that

$$\left\|\partial_c S_{[a]}^{[b]}\right\| \lesssim \frac{1}{\kappa}(\left\|\partial_c F_{[a]}^{[b]}\right\| + (\left\|\partial_c H_{[a]}\right\| + \left\|\partial_c H_{[b]}\right\|) \left\|S_{[a]}^{[b]}\right\|.$$

Since

$$[a] \times [b] \subset D_{\Lambda'}(c)$$

and $\Lambda' \gtrsim \Lambda + d_\Delta$ it follows by Corollary 2.2 (iii) that

$$[a] \times [a],\ [b] \times [b] \subset D_{\Lambda}(c)$$

and therefore

$$\left\|\partial_c H_{[a]}\right\| + \left\|\partial_c H_{[b]}\right\| \le d_\Delta^d < H >_\Lambda$$

and

$$\left\|\partial_c S_{[a]}^{[b]}\right\| e^{\gamma \mathrm{dist}([a],[b])} \lesssim \frac{1}{\kappa} < F >_{\Lambda',\gamma} + d_\Delta^d < H >_\Lambda |S|_\gamma .$$

Case II: $<c, a-b> \neq 0$. Then

$$|\,|a|^2 - |b|^2\,| \approx \frac{|a|}{|c|} |<c, a-b>| \approx \frac{|b|}{|c|} |<c, a-b>| \gtrsim \Lambda'.$$

Indeed $|a|^2 - |b|^2\,|$ can be written

$$|a' + \tau c|^2 - |b' + \tau c|^2 = |a'|^2 - |b'|^2 + 2\tau <c, a-b>,$$

and (recalling Lemma 2.1(ii))

$$\left| |a'|^2 - |b'|^2 \right| \le |a-b|\,(|a'| + |b'|) \le \mathrm{cte.}(\Delta' + d_\Delta)\frac{\tau}{\Lambda'}$$

and this is $\le \frac{1}{2}\tau$, since $\Lambda' \ge 2\mathrm{cte.}(\Delta' + d_\Delta)$. Moreover

$$\frac{|a|}{|c|} \approx \frac{|b|}{|c|} \approx \tau \ge \Lambda'.$$

Since $\Lambda' \gtrsim \|H\|$, assuring that $\|H\|$ is small compared with $|a|^2 - |b|^2\,|$, we have

$$|\alpha - \beta| \approx 2\,|<a-b, c>| \ge 2 \quad \forall \begin{cases} \alpha \in \sigma(\frac{1}{\tau}(\Omega + H)_{[a]}) \\ \beta \in \sigma(\frac{1}{\tau}(\Omega + H)_{[b]}). \end{cases}$$

Since $S_{[a]}^{[b]}(\infty c) = 0$, equation (31) can be written

$$\begin{aligned} &\tfrac{i}{\tau} <k,\omega> \partial_c S_{[a]}^{[b]} + \tfrac{i}{\tau}(\Omega + H)_{[a]} \partial_c S_{[a]}^{[b]} - \partial_c S_{[a]}^{[b]} \tfrac{1}{t}(\Omega + H)_{[b]} \\ &= F_{[a]}^{[b]} \tfrac{1}{\tau} \max_{d \in [a]\cup[b]} \tfrac{|d|}{|c|} \approx F_{[a]}^{[b]}, \end{aligned}$$

where the argument in all matrices is tc, $t = 0$ Since $\Lambda' \gtrsim \Delta'$, then the absolute value of the eigenvalues of the LHS-operator is $\gtrsim 2$ and it follows as in (31) that

$$\left\|\partial_c S_{[a]}^{[b]}\right\| \lesssim \frac{1}{\kappa} \left\|F_{[a]}^{[b]}\right\|.$$

Adding the estimate (30) and taking the supremum over all c, we get in both cases

$$< S >_{\Lambda' + d_\Delta + 2,\gamma} \lesssim d_\Delta^{2d} e^{2d_\Delta} \left(\frac{1}{\kappa} < F >_{\Lambda',\gamma} + \frac{1}{\kappa^2} < H >_\Lambda |F|_\gamma\right). \tag{32}$$

3. Estimate of ω-derivatives. In order to estimate the derivatives in ω we just differentiate (31) with respect to ω:

$$(i <k,\omega> +i(\Omega+H(\omega))_{[a]})\partial_\omega S_{[a]}^{[b]} - i\partial_\omega S_{[a]}^{[b]}(\Omega+H(\omega))_{[b]} =$$
$$= \partial_\omega F_{[a]}^{[b]} - i(k+\partial_\omega H(\omega)_{[a]}S_{[a]}^{[b]} - S_{[a]}^{[b]}\partial_\omega H(\omega)_{[b]})$$

where the argument in all matrices is tc with $t=0$.

Let $G_{[a]}^{[b]}$ be the matrix on RHS. Then

$$\left\|G_{[a]}^{[b]}\right\| \le \left\|\partial_\omega F_{[a]}^{[b]}\right\| + (|k| + \left\|\partial_\omega H_{[a]}\right\| + \left\|\partial_\omega H_{[b]}\right\|)\left\|\partial_\omega S_{[a]}^{[b]}\right\| \tag{33}$$

and

$$\left\|\partial_c G_{[a]}^{[b]}\right\| \le \left\|\partial_c\partial_\omega F_{[a]}^{[b]}\right\| + (|k| + \left\|\partial_\omega H_{[a]}\right\| + \left\|\partial_\omega H_{[b]}\right\|)\left\|\partial_c\partial_\omega S_{[a]}^{[b]}\right\| + (\left\|\partial_c\partial_\omega H_{[a]}\right\| + \left\|\partial_c\partial_\omega H_{[b]}\right\|)\left\|\partial_\omega S_{[a]}^{[b]}\right\|. \tag{34}$$

$\partial_\omega S_{[a]}^{[b]}$ is now estimated like $S_{[a]}^{[b]}$ and $\partial_c\partial_\omega S_{[a]}^{[b]}$ is now estimated like $\partial_c S_{[a]}^{[b]}$ but with G instead of F. Combining these estimates now gives the result in $d=2$ when $F=\mathcal{T}_{\Delta'}^{\Delta}F$. By Lemma 6.2(i) we get the result for $d=2$.

By (ii) the Töplitz limits of S verifies the same equation as S, with F replaced by $\mathcal{P}_c^{\Delta}F$, and we get the result for $d\ge 3$ by Lemma 6.2(i)+(ii). □

6.4. **A second equation, $k=0$.**

Consider the equation

$$i[(\Omega(\omega)+H(\omega)),S] = (\mathcal{T}_{\Delta'}^{\Delta} - \mathcal{D}_{\Delta'}^{\Delta})F(\omega) \tag{35}$$

where $F(\omega):\mathcal{L}\times\mathcal{L}\to\mathbb{C}$ and $\partial_\omega F(\omega)$ are Töplitz at ∞.

Let $U'\subset U$ be a set such that for all $\omega\in U$ the small divisor condition

$$\begin{cases} |\alpha(\omega)-\beta(\omega)|\ge\kappa \quad \forall \begin{cases}\alpha(\omega)\in\sigma((\Omega+H)(\omega)_{[a]_\Delta})\\ \beta(\omega)\in\sigma((\Omega+H)(\omega)_{[b]_\Delta})\end{cases} \\ \mathrm{dist}([\mathrm{a}]_\Delta,[\mathrm{b}]_\Delta)\le\Delta' \quad \text{and} \quad |\mathrm{a}|\ne|\mathrm{b}|. \end{cases} \tag{36}$$

holds. As before we shall assume that $\kappa<1$.

Proposition 6.4. *Assume that U verifies (13), that $\Omega(\omega)$ is real diagonal and verifies (14) and that $H(\omega)$ and $\partial H(\omega)$ are Töplitz at ∞ and $\mathcal{NF}_\Delta$ for all $\omega\in U$. Then the equation*

$$(35) \quad \textit{and} \quad S-\mathcal{T}_{\Delta'}^{\Delta}S = \mathcal{D}_{\Delta'}^{\Delta}S = 0$$

has for all $\omega \in U'$ a unique solution $S(\omega)$ verifying

(i)

$$|S|_{\left\{ \substack{\gamma \\ U'} \right\}} \le \text{cte. } \frac{1}{\kappa^2} d_\Delta^{2d} e^{2\gamma\Delta}(1+|H|_U)\,|F|_{\left\{ \substack{\gamma \\ U'} \right\}};$$

(ii) *$S(\omega)$ and $\partial_\omega S(\omega)$ are Töplitz at ∞ and the Töplitz-limits verify*

$$\begin{cases} i <k,\omega> S + i[(\Omega(\omega)+H(c,\omega)), S] = (\mathcal{T}_{\Delta'}^{\Delta} - \mathcal{D}_{\Delta'}^{\Delta})\mathcal{P}_c^{\Delta} F(c,\omega) \\ S - \mathcal{T}_{\Delta'}^{\Delta} S = \mathcal{D}_{\Delta'}^{\Delta} S = 0; \end{cases}$$

(iii)

$$<S>_{\left\{ \substack{\Lambda'+d_\Delta+2,\gamma \\ U'} \right\}} \le \text{cte. } \frac{1}{\kappa^3} d_\Delta^{2d} e^{2\gamma\Delta}(1+ <H>_{\left\{ \substack{\Lambda \\ U'} \right\}}) <F>_{\left\{ \substack{\Lambda',\gamma \\ U'} \right\}}$$

for any

$$\Lambda' \gtrsim \max(\Lambda, d_\Delta^2, (d_{\Delta'})^2, \sup_U \|H(\omega)\|).$$

The constant cte. *only depends on the dimensions d and $\#\mathcal{A}$ and on C_1, C_2, C_3, C_4.*

Proof. We first assume that $F = (\mathcal{T}_{\Delta'}^{\Delta} - \mathcal{D}_{\Delta'}^{\Delta})F(\omega)$. The proof is the same as in Proposition 6.3, with $k=0$, and gives a

$$\Lambda' \gtrsim \max(\Lambda, d_\Delta^2, \Delta', \|H\|).$$

In order to get the result we need to estimate $(\mathcal{T}_{\Delta'}^{\Delta} - \mathcal{D}_{\Delta'}^{\Delta})F(\omega)$ in terms of F. This is done by Lemma 6.2(iii) and requires a larger Λ'. □

6.5. **A third equation.**

Consider the equation

$$(37) \qquad i <k,\omega> S + i(\Omega(\omega)+H(\omega))S + iS\mathcal{I}(\Omega(\omega) + {}^tH(\omega)) = F(\omega)$$

where $F(\omega) : \mathcal{L}\times\mathcal{L} \to \mathbb{C}$ and $\partial_\omega F(\omega)$ are Töplitz at ∞ and $\mathcal{I}Q$ is defined by

$$(\mathcal{I}Q)_a^b = Q_{-a}^{-b}.$$

(This equation will be motivated in the proof of Proposition 6.7.)

Let $U' \subset U$ be a set such that for all $\omega \in U$ the small divisor condition (20) holds, i.e. for all a, b

$$|<k,\omega> + \alpha(\omega) + \beta(\omega)| \ge \kappa \quad \forall \begin{cases} \alpha(\omega) \in \sigma((\Omega+H)(\omega)_{[a]_\Delta}) \\ \beta(\omega) \in \sigma((\Omega+H)(\omega)_{[b]_\Delta}). \end{cases}$$

As before we shall assume that $\kappa < 1$.

Proposition 6.5. *Assume that U verifies (13), that $\Omega(\omega)$ is real diagonal and verifies (14) and that $H(\omega)$ and $\partial H(\omega)$ are Töplitz at ∞ and $\mathcal{NF}_\Delta$ for all $\omega \in U$. Then the equation* (37) *has for all $\omega \in U'$ a unique solution $S(\omega)$ verifying*

(i)

$$|S|_{\left\{ \substack{\gamma \\ U'} \right\}} \le \text{cte. } \frac{1}{\kappa^2} d_\Delta^{2d} e^{2\gamma\Delta} (1 + |k| + |H|_U) \, |F|_{\left\{ \substack{\gamma \\ U'} \right\}} ;$$

(ii) *$S(\omega)$ and $\partial_\omega S(\omega)$ are Töplitz at ∞ and all Töplitz-limits $S(c, \omega)$, $c \neq 0$, are $= 0$;*

(iii)

$$< S >_{\left\{ \substack{\Lambda' + d_\Delta + 2, \gamma \\ U'} \right\}} \le \text{cte.} \frac{1}{\kappa^3} d_\Delta^{2d} e^{2\gamma\Delta} (1 + |k| + < H >_{\left\{ \substack{\Lambda \\ U'} \right\}}) < F >_{\left\{ \substack{\Lambda', \gamma \\ U'} \right\}}$$

for any

$$\Lambda' \gtrsim \max(\Lambda, d_\Delta^2, \Delta', \sup_U \|H(\omega)\|).$$

The constant cte. *only depends on the dimensions d and $\#\mathcal{A}$ and on C_1, C_2, C_3, C_4.*

Proof. As before we reduce to $\Omega_a = |a|^2$ and we block decompose the equation over $\mathcal{E}_\Delta$:

$$i <k, \omega> S_{[a]}^{[b]} + i(\Omega + H)_{[a]} S_{[a]}^{[b]} + i S_{[a]}^{[b]} (\Omega + {}^t H)_{-[b]} = F_{[a]}^{[b]}.$$

We then repeat the proof as for Proposition 6.3. There is a difference in the computation of the Töplitz limits. The equation (31) becomes

$$i <k, \omega> + i((\Omega + H)_{[a]}(tc) S_{[a]}^{[b]}(tc) + S_{[a]}^{[b]}(tc)(\Omega + {}^t H)_{[-b]}(-tc)) = F_{[a]}^{[b]}(tc)$$

and now

$$\Omega_{[a]}(tc) X + X \Omega_{[-b]}(-tc)$$

equals

$$(t^2 |c|^2 + 2t <a, c> + |a|^2) X + X (t^2 |c|^2 - 2t <b, c> + |b|^2)$$

– the quadratic terms in t do not cancel! Dividing the equation by t^2 and letting $t \to \infty$, the limit equation becomes

$$2 |c|^2 X = 0,$$

which has the unique solution $X = 0$. Therefore

$$S_{[a]}^{[b]}(tc) \to S_{[a]}^{[b]}(\infty c) = 0$$

as $t \to \infty$, i.e. the Töplitz limits are always 0.

In order to estimate the Lipschitz-norm we only need to consider the analogue of Case II (even when $< c, a - b >= 0$). We have for $(a, b) \in D_{\Lambda'}(c)$

$$|a|^2 + |b|^2 \gtrsim (\frac{|a|}{|c|})^2 \approx (\frac{|b|}{|c|})^2 \gtrsim (\Lambda')^2.$$

To avoid any problems with $<k,\omega>$ and H it is sufficient that $(\Lambda')^2$ is $\gtrsim \Delta'$ and $\gtrsim \|H\|$. $\square$

6.6. **The homological equations.**

Let $\Omega(\omega) : \mathcal{L}\times\mathcal{L} \to gl(2,\mathbb{C})$ be a real diagonal matrix, i.e.

$$\Omega_a^b(\omega) = \begin{cases} \Omega_a(\omega) I & a=b \\ 0 & a\neq b \end{cases}$$

Assume that for all $a\in\mathcal{L}$ and all $\omega\in U$ (14)+(15) holds and

$$\begin{cases} |\Omega_a(\omega)| \geq C_4 \\ |\Omega_a(\omega)+\Omega_b(\omega)| \geq C_4 \\ |\Omega_a(\omega)-\Omega_b(\omega)| \geq C_4 \quad |a|\neq|b| \end{cases} \tag{38}$$

Assume $H(\omega)$ and $\partial_\omega H(\omega)$ are Töplitz at ∞ and $\mathcal{NF}_\Delta$ for all $\omega\in U$ and

$$\begin{cases} \|H(\omega)\| \leq \frac{C_4}{4} \\ \|\partial_\omega H(\omega)\| \leq \frac{C_5}{4} \\ < H >_{\left\{ {\Lambda \atop U} \right\}} \lesssim 1 \end{cases} \tag{39}$$

(Here $\|\ \|$ is the operator norm.)

Proposition 6.6. *Let $\Delta'>0$ and $\kappa>0$. Assume that U verifies (13), that $\Omega(\omega)$ is real diagonal and verifies* $(14)+(15)+(38)$ *and $H(\omega)$ and $\partial_\omega H(\omega)$ are $\mathcal{NF}_\Delta$ and verify (39) for all $\omega\in U$.*

Then there is a subset $U'\subset U$,

$$\mathrm{Leb}(U\setminus U') \ \leq\ \mathrm{cte.}(\Delta')^{\frac{d}{2}+\#\mathcal{A}-1}\kappa,$$

such that for all $\omega\in U'$ the following hold:

(i) *for any $|k|\leq\Delta'$*

$$|<k,\omega>| \geq \kappa.$$

(ii) *for any $|k|\leq\Delta'$ and for any vector $F(\omega)\in l^2_\gamma(\mathcal{L},\mathbb{C}^2)$ there exists a unique vector $S(\omega)\in l^2_\gamma(\mathcal{L},\mathbb{C}^2)$ such that*

$$i <k,\omega> S + J(\Omega+H)S = F$$

and satisfying

$$\|S\|_{\left\{ {\gamma \atop U'} \right\}} \leq \mathrm{cte.} d_\Delta^{2d} e^{2\gamma d_\Delta} \frac{1}{\kappa^2}\Delta' \|F\|_{\left\{ {\gamma \atop U'} \right\}}.$$

The constants cte. *only depend on $d,\#\mathcal{A},m_*$ and on $C_1,\dots,C_5$.*

Proof. Let

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

and define ${}^t\!CAC : \mathcal{L} \times \mathcal{L} \to gl(2, \mathbb{C})$ through

$$({}^t\!CAC)_a^b = {}^t\!CA_a^bC.$$

To see (ii) we change to complex coordinates $\tilde{S} = C^{-1}S$ and $\tilde{F} = C^{-1}F$. Then the equation becomes

$$i <k, \omega> \tilde{S} + iJ(\tilde{\Omega} + \tilde{H})\tilde{S} = \tilde{F}$$

with

$$(\tilde{\Omega} + \tilde{H}) = \begin{pmatrix} 0 & \Omega + Q \\ \Omega + {}^t\!Q & 0 \end{pmatrix},$$

where Q and ${}^t\!Q$ are (scalar-valued) normal form matrices.

This equation decouples into two equations for (scalar-valued) matrices of the type

$$i <k, \omega> R \pm i(\Omega + Q)R = G.$$

We get the results from Proposition 5.1, the first part of condition (38) and Proposition (6.1). □

Proposition 6.7. *Let $\Delta' > 0$ and $\kappa > 0$. Assume that U verifies (13), that $\Omega(\omega)$ is real diagonal and verifies* (14) + (15) + (27) *and $H(\omega)$ and $\partial_\omega H(\omega)$ are Töplitz at ∞ and $\mathcal{NF}_\Delta$ and verify (28) for all $\omega \in U$.*

Then there is a subset $U' \subset U$,

$$\mathrm{Leb}(U - U') \ \leq \ \mathrm{cte.} \max(\Lambda, \Delta, \Delta')^{\exp}(1 + <H>_{\left\{\substack{\Lambda \\ U'}\right\}})\kappa^{\frac{1}{d}},$$

such that for all $\omega \in U'$ the following hold:

for any $|k| \leq \Delta'$ and for any symmetric $gl(2, \mathbb{C})$-matrix $F(\omega)$,

$$(\pi F)_{[a]_\Delta}^{[b]_\Delta} = 0 \quad \textit{when} \ \ \underline{\mathrm{dist}}([a]_\Delta, [b]_\Delta) > \Delta',$$

there exist symmetric matrices $S(\omega)$ and $H'(\omega)$ such that

$$i <k, \omega> S + (\Omega + H)JS - SJ(\Omega + H) = F - H'$$

and satisfying – for any

$$\Lambda' \geq \mathrm{cte.} \max(\Lambda, d_\Delta^2, (d_{\Delta'})^2) \quad -$$

(i)

$$<S>^{\Delta}_{\left\{\substack{\Lambda' + d_\Delta + 2, \gamma \\ U'}\right\}} \ \leq \ \mathrm{cte.} \frac{1}{\kappa^3} \Delta' d_\Delta^{2d} e^{2\gamma d_\Delta} <F>_{\left\{\substack{\Lambda', \gamma \\ U'}\right\}},$$

(ii) *for $k \neq 0$ $H' = 0$ and for $k = 0$ H' and $\partial_\omega H'$ are $\mathcal{NF}_{\Delta'}$ and*

$$<H'>^{\Delta}_{\left\{\substack{\Lambda' + d_\Delta + 2 \\ U'}\right\}} \ \leq \ \mathrm{cte.} d_\Delta^{2d} <F>_{\left\{\substack{\Lambda' \\ U'}\right\}}.$$

The exponent exp *only depends on $d, \#\mathcal{A}$ and the constants* cte. *also depend on $C_1, \ldots, C_5$.*

Proof. Consider first $k \neq 0$. We change to complex coordinates $\tilde{S} = C^{-1}SC$ and $\tilde{F} = C^{-1}FC$. Then the equation becomes

$$i <k,\omega> \tilde{S} + i(\tilde{\Omega} + \tilde{H})J\tilde{S} - i\tilde{S}J(\tilde{\Omega} + \tilde{H}) = \tilde{F}$$

with

$$(\tilde{\Omega} + \tilde{H}) = \begin{pmatrix} 0 & \Omega + Q \\ \Omega + {}^tQ & 0 \end{pmatrix},$$

where Q and ${}^tQ$ are (scalar-valued) normal form matrices.

This equation decouples into four (scalar-valued) matrices of the types

$$i <k,\omega> R \pm i((\Omega + Q)R - R(\Omega + Q)) = G$$

and

$$i <k,\omega> R \pm i((\Omega + Q)R + R(\Omega + {}^tQ)) = G.$$

By the assumption on F we have $\mathcal{T}_{\Delta'}^{\Delta} G = G$ and the first type follows from Propositions 5.2 and 6.3.

To treat the second type let us consider the operators

$$(\mathcal{R}Q)_a^b = Q_a^{-b} \text{ and } (\mathcal{I}Q)_a^b = Q_{-a}^{-b}.$$

With $T = \mathcal{R}R$ the equation takes the form

$$i <k,\omega> T \pm i((\Omega + Q)T + T\mathcal{I}(\Omega + {}^tQ)) = \mathcal{R}G.$$

Then the result follows from Propositions 5.1 and 6.5.

It is clear that the solution S is unique if we impose that

$$(\pi S)_{[a]_\Delta}^{[b]_\Delta} = 0 \quad \text{when } \underline{\text{dist}}([a]_\Delta, [b]_\Delta) > \Delta'.$$

The symmetry follows from this.

For $k = 0$ the argument is similar, using the second and third part of condition (27) and Proposition 6.4. $\square$

PART III. KAM

7. A KAM THEOREM

7.1. Statement of the theorem.

Let

$$\mathcal{O}^\gamma(\sigma, \rho, \mu) = \mathcal{O}^\gamma(\sigma) \times \mathbb{T}_\rho^{\mathcal{A}} \times \mathbb{D}(\mu)^{\mathcal{A}}$$

be the set of all ζ, φ, r such that

$$\zeta = (\xi, \eta) \in \mathcal{O}^\gamma(\sigma), \ |\Im\varphi_a| < \rho, \ |r_a| < \mu \quad \forall a \in \mathcal{A}.$$

Let

$$h_\omega(z, r) = h(\zeta, r, \omega) = <\omega, r> + \frac{1}{2} <\zeta, (\Omega(\omega) + H(\omega))\zeta>

where $\Omega(\omega)$ is a real diagonal matrix with diagonal elements $\Omega_a(\omega)I$ and $H(\omega)$ and $\partial_\omega H(\omega)$ are Töplitz at ∞ and $\mathcal{NF}_\Delta$ for all $\omega \in U$. We recall (section 5.1) that a matrix $H : \mathcal{L}_t imes\mathcal{L} \to gl(2,\mathbb{C})$ is $\mathcal{NF}_\Delta$ if it is real, symmetric and can be written

$$H = \begin{pmatrix} Q_1 & Q_2 \\ {}^tQ_2 & Q_1 \end{pmatrix}$$

with $Q = Q_1 + iQ_2$ Hermitian and block-diagonal over the decomposition $\mathcal{E}_\Delta$ of $\mathcal{L}$.

We assume (13-15)+(38), i.e.

$$U \text{ is an open subset of } \{|\omega| < C_1\} \subset \mathbb{R}^{\#\mathcal{A}},$$

$$\left|\partial_\omega^\nu(\Omega_a(\omega) - |a|^2)\right| \le C_2 e^{-C_3|a|}, \quad C_3 > 0, \quad \nu = 0,1,$$

$$\begin{cases} |\Omega_a(\omega)| \ge C_4 \\ |\Omega_a(\omega) + \Omega_b(\omega)| \ge C_4 \\ |\Omega_a(\omega) - \Omega_b(\omega)| \ge C_4 \quad |a| \ne |b|\,, \end{cases}$$

and, for all $k \in \mathbb{Z}^n \setminus 0$,

$$\begin{cases} |\partial_\omega(<k,\omega> + \Omega_a(\omega))| \ge C_5 \\ |\partial_\omega(<k,\omega> + \Omega_a(\omega) + \Omega_b(\omega))| \ge C_5 \\ |\partial_\omega(<k,\omega> + \Omega_a(\omega) - \Omega_b(\omega))| \ge C_5 \quad |a| \ne |b|\,. \end{cases}$$

Remark. These conditions are quite generic. Indeed for any family $\{\Omega_a : a \in \mathcal{L}\}$ of analytic functions on U verifying (14) the following hold. For any $\delta > 0$ there is a set $U_\delta \subset U$, $\mathrm{Leb}(U \setminus U_\delta) < \delta$, and constants $C_3 = C_3(\delta), C_4 = C_4(\delta)$, such that (15)+(38) hold for each a unless $\Omega_a(\omega) \pm \Omega_b(\omega) - <k,\omega>$ is a constant function (on some connected component of U) for some b and k.

We also assume (39), i.e.

$$\begin{cases} \|H(\omega)\| \le \frac{C_4}{4} \\ \|\partial_\omega H(\omega)\| \le \frac{C_5}{4} \\ <H>_{\{\Lambda \atop U\}} \lesssim 1 \end{cases}$$

for some Λ. Here $\|\cdot\|$ is the operator norm.

Remark. For simplicity we shall assume that $\gamma, \sigma, \rho, \sigma$ are < 1 and that Δ, Λ are ≥ 3.

Let

$$f : \mathcal{O}^\gamma(\sigma,\rho,\mu) \times U \to \mathbb{C}$$

be real analytic in ζ, φ, r and $\mathcal{C}^1$ in $\omega \in U$ and let

$$[f]_{\left\{\substack{\Lambda,\gamma,\sigma \\ U,\rho,\mu}\right\}} = \sup_{\substack{\varphi \in \mathbb{T}^{\mathcal{A}}_{\rho} \\ r \in \mathbb{D}(\mu)^{\mathcal{A}}}} [f(\cdot, \varphi, r, \cdot)]_{\left\{\substack{\Lambda,\gamma,\sigma \\ U}\right\}}.$$

Theorem 7.1. *Assume that* U *verifies (13), that* $\Omega(\omega)$ *is real diagonal and verifies* $(14) + (15) + (38)$ *and* $H(\omega)$ *and* $\partial_\omega H(\omega)$ *are Töplitz at* ∞ *and* $\mathcal{NF}_\Delta$ *and verify (39) for all* $\omega \in U$.

Then there is a constant $\mathcal{C}$ *(only depending on* $d, \#\mathcal{A}, m_*, C_1, \ldots, C_5$*) and an exponent* exp *(only depending on* $d, \#\mathcal{A}, m_*$*) such that, if*

$$[f]_{\left\{\substack{\Lambda,\gamma,\sigma \\ U,\rho,\mu}\right\}} = \varepsilon \le \mathcal{C} \min(\gamma, \rho, \frac{1}{\Lambda}, \frac{1}{\Delta})^{\exp} \min(\sigma^2, \mu)^2$$

then there is a $U' \subset U$ *with*

$$\mathrm{Leb}(U \setminus U') \le \mathrm{cte.}\varepsilon^{\exp'}$$

such that for all $\omega \in U'$ *the following hold: there is an analytic symplectic diffeomorphism*

$$\Phi : \mathcal{O}^0(\frac{\sigma}{2}, \frac{\rho}{2}, \frac{\mu}{2}) \to \mathcal{O}^0(\sigma, \rho, \mu)$$

and a vector ω' *such that* $(h_{\omega'} + f) \circ \Phi$ *equals*

$$c + <\omega', r> + \frac{1}{2} <\zeta, (\Omega + H')(\omega)\zeta> + f'(\zeta, \varphi, r, \omega)$$

where

$$\partial_\zeta f' = \partial_r f' = \partial_\zeta^2 f' = 0 \quad \textit{for} \quad \zeta = r = 0$$

and

$$H' = \begin{pmatrix} Q_1' & Q_2' \\ {}^t Q_2' & Q_1' \end{pmatrix}$$

with $Q' = Q_1' + iQ_2'$ *Hermitian and block diagonal matrix (with finite-dimensional blocks).*

Moreover

$$\Phi : \mathcal{O}^0(\frac{\sigma}{2}, \frac{\rho}{2}, \frac{\mu}{2}) \to \mathcal{O}^0(\frac{\sigma}{2} + \varepsilon^{\frac{1}{2}}, \frac{\rho}{2} + \varepsilon^{\frac{1}{2}}, \frac{\mu}{2} + \varepsilon^{\frac{1}{2}}).$$

The exponent $\exp'$ *only depend on* $d, \#\mathcal{A}, m_*$ *while the constant* cte. *also depends on all* $C_1, \ldots, C_5$.

Remark. If $(\Omega')_a^b \neq 0$ then $|a| = |b|$. This gives a trivial bound on the block-dimension which is not uniform – they may be of arbitrarily large dimension. Due to this lack of uniformity we loose, in our estimates, all exponential decay in the space modes. However, if there were a uniform bound – as happens in some cases [GY06] – we would retain some exponential decay.

The consequences of the theorem are well-known and discussed in the introduction.

7.2. **Application to the Schrödinger equation.**
Consider a non-linear Schrödinger equation

$$-i\dot{u} = \Delta u + V(x) * u + \varepsilon \frac{\partial F}{\partial \bar{u}}(x, u, \bar{u}), \quad u = u(t,x),\ x \in \mathbb{T}^d, \qquad (*)$$

where $V(x) = \sum \hat{V}(a) e^{i\langle a,x\rangle}$ is an analytic function with $\hat{V}$ real and where F is real analytic in $\Re u, \Im u$ and in $x \in \mathbb{T}^d$.

Consider a function

$$\tilde{u}(\varphi, x) = \sum_{a \in \mathcal{A}} p_a e^{i\varphi_a} e^{i\langle a,x\rangle}, \quad p_a > 0,$$

where $\mathcal{A} \in \mathbb{Z}^d$ is a finite set. Let $\mathcal{L}$ be the complement of $\mathcal{A}$ and let

$$\begin{aligned} \omega &= \{\omega_a = |a|^2 + \hat{V}(a) : a \in \mathcal{A}\} \\ \Omega &= \{\omega_a = |a|^2 + \hat{V}(a) : a \in \mathcal{L}\} \end{aligned}$$

Then

$$u(t,x) = \tilde{u}(\varphi + t\omega, x)$$

is a solution of $(*)$ for $\varepsilon = 0$.

Let V depend $\mathcal{C}^1$ on a parameter $w \in W \subset \mathbb{R}^{\#\mathcal{A}}$ and assume that it satisfies conditions analogous to (13- 15)+(38), i.e.

$$W \text{ is an open subset of } \{|w| < C_1\} \subset \mathbb{R}^{\#\mathcal{A}},$$

$$\left|\partial_w^\nu (\Omega_a(w) - |a|^2)\right| \le C_2 e^{-C_3|a|}, \quad C_3 > 0, \quad \nu = 0, 1,$$

$$\begin{cases} |\Omega_a(w)| \ge C_4 \\ |\Omega_a(w) + \Omega_b(w)| \ge C_4 \\ |\Omega_a(w) - \Omega_b(w)| \ge C_4 \quad |a| \ne |b|, \end{cases}$$

and, for all $k \in \mathbb{Z}^n \setminus 0$,

$$\begin{cases} |\partial_w(\langle k, \omega(w)\rangle + \Omega_a(w))| \ge C_5 \\ |\partial_w(\langle k, \omega(w)\rangle + \Omega_a(w) + \Omega_b(w))| \ge C_5 \\ |\partial_w(\langle k, \omega(w)\rangle + \Omega_a(w) - \Omega_b(w))| \ge C_5 \quad |a| \ne |b|. \end{cases}$$

We also assume that the mapping

$$W \ni w \mapsto \omega(w) = \{\omega_a = |a|^2 + \hat{V}(a, w); a \in \mathcal{A}\} \in U$$

is a diffeomorphism whose inverse is bounded in the $\mathcal{C}^1$-norm, i.e.

$$\left|\omega^{-1}\right|_{\mathcal{C}^1} \le C_6. \tag{40}$$

Let $f(u,\bar u)=\int_{\mathbb{T}^d}F(x,u(x)\bar u(x))dx$. Then one verifies easily that there exists γ,σ,ρ,μ such that f is real analytic on $\mathcal{O}^\gamma(\sigma,\rho,\mu)$ and that f has the Töplitz-Lipschitz-property:

$$[f]_{\left\{\substack{\Lambda,\gamma,\sigma\\ U,\rho,\mu}\right\}}\le C_7 \tag{41}$$

for some constant C_7.

Theorem 7.2. *For ε sufficiently small, there is a subset $W'\subset W$,*

$$\mathrm{Leb}(W\setminus W')\le \mathrm{cte}.\varepsilon^{\exp},$$

such that on W' there is an $\tilde u'(\varphi,x)$, analytic in $\varphi\in\mathbb{T}^d_{\frac{\rho}{2}}$ and $\mathcal{C}^{m_*-d}$ in $x\in\mathbb{T}^d$, with

$$\sup_{|\Im\varphi|<\frac{\rho}{2}}\|\tilde u'(\varphi,\cdot)-\tilde u(\varphi,\cdot)\|_0\le\varepsilon^{\frac12},$$

and there is a $\omega':W'\to U$,

$$|\omega'-\omega|\le\varepsilon^{\frac12},$$

such that

$$u'(t,x)=\tilde u'(\varphi+t\omega'(w),x)$$

is a solution of $(*)$ for any $w\in W'$.

Moreover, the linearized equation

$$-i\dot v=\Delta v+V(x)*v+\varepsilon\frac{\partial^2F}{\partial\bar u^2}(x,u'(t,x),\bar{u'}(t,x))\bar v+ \\ \varepsilon\frac{\partial^2F}{\partial u\partial\bar u}(x,u'(t,x),\bar{u'}(t,x))v$$

is reducible to constant coefficients and has only time-quasi-periodic solutions – except for a $(\#\mathcal{A})$-dimensional subspace where solutions may increase at most linearly in t.

Proof. We write

$$\begin{cases} u(x)=\sum_{a\in\mathbb{Z}^d}u_ae^{i<a,x>}\\ \overline{u(x)}=\sum_{a\in\mathbb{Z}^d}v_ae^{i<a,x>},\end{cases}$$

and let

$$\zeta_a=\begin{pmatrix}\xi_a\\ \eta_a\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt2}(u_a+v_a)\\ \frac{-i}{\sqrt2}(u_a-v_a)\end{pmatrix}.$$

In the symplectic space

$$\{(\xi_a,\eta_a):a\in\mathbb{Z}^d\}=\mathbb{R}^{\mathbb{Z}^d}\times\mathbb{R}^{\mathbb{Z}^d},\quad \sum_{a\in\mathbb{Z}^d}d\xi_a\wedge d\eta_a,$$

the equation becomes a Hamiltonian equation in infinite degrees of freedom. The Hamiltonian function has an integrable part

$$\frac12\sum_{a\in\mathbb{Z}^d}(|a|^2+\hat V(a))(\xi_a^2+\eta_a^2)$$

plus a perturbation.

In a neighborhood of the unperturbed solution

$$\frac{1}{2}(\xi_a^2+\eta_a^2)=p_a, \quad a\in\mathcal{A},$$

we introduce the action angle variables (φ_a, r_a) (notice that each $p_a>0$ by assumption), defined through the relations

$$\begin{aligned}\xi_a&=\sqrt{2(r_a-p_a)}\cos(\varphi_a)\\ \eta_a&=\sqrt{2(r_a-p_a)}\sin(\varphi_a).\end{aligned}$$

The integrable part of the Hamiltonian becomes

$$h(\zeta,r,\omega)=<\omega,r>+\frac{1}{2}\sum_{a\in\mathcal{L}}\Omega_a(\omega)(\xi_a^2+\eta_a^2),$$

while the perturbation εf will be a function of ζ,φ,r.

The assumptions of Theorem 7.1 are now fulfilled and gives the result. □

8. Proof of theorem

8.1. **Preliminaries.**

Let

$$f:\mathcal{O}^\gamma(\sigma,\rho,\mu)\times U\to\mathbb{C}$$

[11] be real analytic in ζ,φ,r and $\mathcal{C}^1$ in $\omega\in U$ and consider

$$[f]_{\left\{\substack{\Lambda,\gamma,\sigma\\ U,\rho,\mu}\right\}}.$$

Notation. We let

$$\alpha=\begin{pmatrix}\gamma & \sigma\\ \rho & \mu\end{pmatrix},$$

and we write this norm as

$$[f]_{\left\{\substack{\Lambda\\ U}\alpha\right\}}.$$

Cauchy estimates. It follows by Cauchy estimates that

$$\begin{aligned}&[\partial_\varphi f]_{\left\{\substack{\Lambda\\ U}\alpha'\right\}}\lesssim\tfrac{1}{\rho-\rho'}[f]_{\left\{\substack{\Lambda\\ U}\alpha\right\}}\\ &[\partial_r f]_{\left\{\substack{\Lambda\\ U}\alpha'\right\}}\lesssim\tfrac{1}{\mu-\mu'}[f]_{\left\{\substack{\Lambda\\ U}\alpha\right\}}.\end{aligned}\tag{42}$$

Truncation. We obtain $\mathcal{T}_\Delta f$ from f by: 1) truncating the Taylor expansion in ζ at order 2; 2) truncating the Taylor expansion in r at order 0 for the first and the second order term in ζ and at order 1 for the zero'th order term in ζ; 3) truncating the Fourier modes at order

[11]We shall assume that all γ,σ,ρ,μ are <1, and that $0<\sigma-\sigma'\approx\sigma$, $0<\mu-\mu'\approx\mu$.

Δ; 4) truncating the space modes of the second order term in ζ at order Δ. Formally $\mathcal{T}_\Delta f$ is

$$\textstyle\sum_{|k|\le\Delta}[\hat f(0,k,0,\omega)+\partial_r\hat f(0,k,0,\omega)r+<\partial_\zeta\hat f(0,k,0,\omega),\zeta> \\ +\frac12<\zeta,\mathcal{T}_\Delta\partial_\zeta^2\hat f(0,k,0,\omega)\zeta>]e^{i<k,\varphi>}.$$

We have

$$[\mathcal{T}_\Delta f]_{\left\{\substack{\Lambda\\U}\alpha\right\}}\lesssim\Delta^{\#\mathcal{A}}[f]_{\left\{\substack{\Lambda\\U}\alpha\right\}} \tag{43}$$

and

$$[f-\mathcal{T}_\Delta f]_{\left\{\substack{\Lambda\\U}\alpha'\right\}}\lesssim A(\alpha,\alpha',\Delta)[f]_{\left\{\substack{\Lambda\\U}\alpha\right\}}, \tag{44}$$

where $A(\alpha,\alpha',\Delta)$ is

$$(\frac{\sigma'}{\sigma})^3+(\frac{\sigma'}{\sigma}+\frac{\mu'}{\mu})\frac{\mu'}{\mu}+(\frac{1}{\rho-\rho'})^{\#\mathcal{A}}e^{-\Delta(\rho-\rho')}+e^{-\Delta(\gamma-\gamma')}.$$

This follows from Proposition 3.2, from Cauchy estimates in r and φ, and from formula (8).

Poisson brackets. The Poisson bracket is defined by

$$\{f,g\}=<\partial_\zeta f,J\partial_\zeta g>+\partial_\varphi f\partial_r g-\partial_r f\partial_\varphi g.$$

If g is a quadratic polynomial in ζ, then

$$[\{f,g\}]_{\left\{\substack{\Lambda+3\\U}\alpha'\right\}}\lesssim B(\gamma-\gamma',\sigma,\rho-\rho',\mu,\Lambda)[f]_{\left\{\substack{\Lambda\\U}\alpha\right\}}[g]_{\left\{\substack{\Lambda\\U}\alpha\right\}}, \tag{45}$$

where

$$B=\Lambda^2\frac{1}{\sigma^2}(\frac{1}{\gamma-\gamma'})^{d+m_*}+\frac{1}{\rho-\rho'}\frac{1}{\mu}.$$

If also f is a quadratic polynomial in ζ and, moreover, independent of φ and of the form

$$<a,r>+\frac12<\zeta,A\zeta>,$$

then

$$[\{f,g\}]_{\left\{\substack{\Lambda+3\\U}\alpha'\right\}}\lesssim B(\bar\gamma-\gamma',\sigma_1,\bar\rho-\rho',\mu_1,\Lambda)[f]_{\left\{\substack{\Lambda\\U}\alpha_1\right\}}[g]_{\left\{\substack{\Lambda\\U}\alpha_2\right\}}, \tag{46}$$

$$\alpha_i=\begin{pmatrix}\gamma&\sigma_i\\ \rho&\mu_i\end{pmatrix},\quad i=1,2.$$

and $\bar\gamma=\min(\gamma_1,\gamma_2)$, $\bar\rho=\min(\rho_1,\rho_2)$. [12]

In both cases, the first term to the right is estimated by Proposition 3.3 and the other two terms by Cauchy estimates.

[12]In the expression for B we have assumed that $0<\sigma_j-\sigma'\approx\sigma_,\ 0<\mu_j-\mu'\approx\mu_j$, $j=1,2$.

We shall use both these estimates. Notice that (46) is much better than (45) when σ_2, μ_2 are much smaller than σ_1, μ_1.

Flow maps. Let

$$s = \mathcal{T}_\Delta s = S_0(\varphi, r, \omega) + <\zeta, S_1(\varphi, \omega)> + \frac{1}{2} <\zeta, S_2(\varphi, \omega)\zeta>$$

and notice that S_0 is first order in r. Consider the vector field

$$\frac{d}{dt}\begin{pmatrix} \zeta \\ \varphi \\ r \end{pmatrix} = \begin{pmatrix} J\partial_\zeta s \\ \partial_r s \\ -\partial_\varphi s \end{pmatrix} = \begin{pmatrix} JS_1(\varphi,\omega) + JS_2(\varphi,\omega)\zeta \\ \partial_r S_0(\varphi, 0, \omega) \\ -\partial_\varphi s(\zeta, \varphi, r, \omega) \end{pmatrix}$$

and let

$$\Phi_t = \begin{pmatrix} \zeta_t \\ \varphi_t \\ r_t \end{pmatrix} = \begin{pmatrix} \zeta + b_t(z,\omega) + B_t(z,\omega)\zeta \\ z + g_t(\zeta, z, \omega) \end{pmatrix}$$

be the flow. Here we have denoted φ and r by z.

Assume that

$$[s]_{\left\{\substack{\Lambda \\ U}\,\alpha\right\}} = \varepsilon \lesssim \min((\rho - \rho')\mu, (\gamma - \gamma')^{d+m_*}\sigma^2). \tag{47}$$

Then for $|t| \leq 1$ we have:

$$\Phi_t : \mathcal{O}^{\gamma''}(\sigma', \rho', \mu') \to \mathcal{O}^{\gamma''}(\sigma, \rho, \mu), \quad \forall \gamma'' \leq \gamma';$$

$$[g_t]_{\left\{\substack{\Lambda, \gamma', \sigma' \\ U, \rho', \mu'}\right\}} \lesssim \frac{\varepsilon}{\mu} \quad \text{or} \quad \frac{\varepsilon}{\rho - \rho'} \tag{48}$$

depending on if g is an φ-component or a r-component;

$$\|b_t + B_t\zeta\|_{\left\{\substack{\gamma' \\ U, \rho'}\right\}} \lesssim ((\frac{1}{\gamma - \gamma'})^{m_*} + (\frac{1}{\gamma - \gamma'})^{d+m_*}\frac{1}{\sigma}\|\zeta\|_{\gamma'})\frac{\varepsilon}{\sigma}; \tag{49}$$

$$< B_t >_{\left\{\substack{\Lambda+6, \gamma' \\ U, \rho'}\right\}} \lesssim \Lambda^2(\frac{1}{\gamma - \gamma'})\frac{\varepsilon}{\sigma^2}; \tag{50}$$

and Φ_t has an extension, for $1 \geq \bar\sigma \geq \sigma'$ and $1 \geq \bar\mu \geq \mu'$, to

$$\begin{gathered} \mathcal{O}^{\gamma''}(\bar\sigma, \rho', \bar\mu) \to \\ \mathcal{O}^{\gamma''}(\bar\sigma + \text{cte.}(\tfrac{1}{\gamma - \gamma'})^{d+m_*}\tfrac{\varepsilon}{\sigma}(\tfrac{\bar\sigma}{\sigma} + 1), \rho, \mu + \text{cte.}(\tfrac{1}{\rho - \rho'})\varepsilon(\tfrac{\bar\mu}{\mu} + 1)) \end{gathered} \tag{51}$$

for all $\gamma'' \leq \gamma'$.

Proof. We have $\varphi_t = \varphi + a_t(\varphi, \omega)$ and since

$$|\partial_r S_0(\varphi, 0, \omega)| \lesssim \frac{\varepsilon}{\mu}, \qquad \forall \varphi \in \mathbb{T}^{\mathcal{A}}_\rho,$$

φ_t remains in $\mathbb{T}^{\mathcal{A}}_\rho$ for $|t|\le 1$ if $\frac{\varepsilon}{\mu}\lesssim(\rho-\rho')$. The ω-derivative verifies

$$\frac{d}{dt}(\partial_\omega\varphi_t)=\partial_\omega\partial_r S_0(\varphi,0,\omega)+\partial_\varphi\partial_r S_0(\varphi,0,\omega)(\partial_\omega\varphi_t)$$

and can be solved explicitly by an integral formula. This gives (48) for $z=\varphi$ and the φ-part of (51).

For a fixed ω (49) follows from the first part of Proposition 3.4(i) if $|JS_2|_\gamma\lesssim(\gamma-\gamma')^d$, i.e. if $\varepsilon\lesssim(\gamma-\gamma')^d\sigma^2$. This also gives the ζ-part of (51). In order to get $\|\zeta_t-\zeta\|_{\gamma'}\le\sigma-\sigma'\approx\sigma$ for $\|\zeta\|_{\gamma'}\le\sigma$ we need $\varepsilon\lesssim(\gamma-\gamma')^{d+m_*}\sigma^2$. (50) follows from the second part of Proposition 3.4(ii). The ω-derivative of ζ_t satisfies

$$\frac{d}{dt}(\partial_\omega\zeta_t)=\partial_\omega JS_1(\varphi,0,\omega)+\partial_\omega JS_2(\varphi,0,\omega)\zeta_t+JS_2(\varphi,0,\omega)(\partial_\omega\zeta_t)$$

which is solved in the same way.

$r_t=r+c_t(\zeta,\varphi,\omega)+d_t(\zeta,\varphi,\omega)r$ and for a fixed ω (48) follows from Proposition 3.4(ii) if $\varepsilon\lesssim(\rho-\rho')(\mu-\mu')\approx\lesssim(\rho-\rho')\mu$. The ω-derivative satisfies a similar equation which is solved in the same way. The r-part of (51) follows from these estimates and since r_t is linear in r. $\square$

Composition. Consider now the composition $f(\Phi_t,\omega)$. If

$$\varepsilon\lesssim\min((\rho-\rho')\mu,(\gamma-\gamma')^{d+m_*+1}\sigma^2)\sqrt{\gamma-\gamma'} \tag{52}$$

then

$$[f(\Phi_t,\cdot)]_{\left\{\substack{\Lambda+18\\ U}\alpha'\right\}}\lesssim\Lambda^{14}[f]_{\left\{\substack{\Lambda\\ U}\alpha\right\}}. \tag{53}$$

Proof. Consider first a fixed ω. We have

$$\|\zeta_t(\zeta,z)-\zeta\|_{\gamma'}<\sigma-\sigma'\quad\forall(\zeta,z)\in\mathcal{O}^{\gamma'}(\sigma')\times\mathbb{T}^{\mathcal{A}}_{\rho'}\times\mathbb{D}(\mu')^{\mathcal{A}}$$

by (49)+(52) and we have

$$|g_t(\zeta,z)|<\frac12(\mu-\mu')\text{ or }\frac12(\rho-\rho')\quad\forall(\zeta,z)\in\mathcal{O}^0(\sigma')\times\mathbb{T}^{\mathcal{A}}_{\rho'}\times\mathbb{D}(\mu')^{\mathcal{A}},$$

depending on if g is an r-component or a φ-component, by (48)+(52). By Proposition 3.5 we get

$$[f(\Phi_t(\cdot,\omega),\omega)]_{\substack{\Lambda+12,\gamma'',\sigma'\\ \rho',\mu'}}\lesssim A\;[f(\cdot,\omega)]_{\substack{\Lambda+6,\gamma',\sigma,\\ \rho,\mu}}$$

where

$$A=\max(1,\alpha,\Lambda^2\frac{1}{\gamma'-\gamma''}\alpha^2)$$

and

$$\alpha = \tfrac{1}{\mu-\mu'}[\varphi_t]_{\left\{\substack{\Lambda+6,\gamma',\sigma' \\ \rho',\mu'}\right\}} + \tfrac{1}{\rho-\rho'}[r_t]_{\left\{\substack{\Lambda+6,\gamma',\sigma' \\ \rho',\mu'}\right\}}$$
$$+(\tfrac{1}{\gamma'-\gamma''})^{d+m_*} < B_t >_{\left\{\substack{\Lambda+6,\gamma' \\ \rho'}\right\}} .$$

If we choose $\gamma'-\gamma''=\gamma-\gamma'$, then (48)+(50) and the bound (52) gives $A \lesssim \Lambda^6$.

Consider now the dependence on ω. We have

$$\partial_\omega(f(\Phi_t)) = \partial_\omega f(\Phi_t) + <\partial_z f(\Phi_t), \partial_\omega g_t> + <\partial_\zeta f(\Phi_t), \partial_\omega \zeta_t> .$$

The first term is a composition and we get the same estimate as above but with f replaced by $\partial_\omega f$.

The second term is a finite sum of products, each of which is estimated by Proposition 3.1(i), i.e.

$$[<\partial_z f(\Phi_t,\omega), \partial_\omega g_t>]_{\substack{\Lambda+12,\gamma'',\sigma' \\ \rho'',\mu''}} \lesssim [\partial_z f(\Phi_t,\omega)]_{\substack{\Lambda+12,\gamma'',\sigma' \\ \rho'',\mu''}} [\partial_\omega g_t]_{\substack{\Lambda+12,\gamma'',\sigma' \\ \rho'',\mu''}}.$$

The first factor is a composition which is estimated as above: if we take $\rho'-\rho''=\rho-\rho'$ and $\mu'-\mu''=\mu-\mu'$, then we get

$$\lesssim \Lambda^6 [\partial_\zeta f(\cdot,\omega)]_{\substack{\Lambda+6,\gamma',\sigma \\ \rho',\mu'}} [\partial_\omega g_t]_{\substack{\Lambda+12,\gamma'',\sigma' \\ \rho',\mu'}}.$$

Using Cauchy estimates for the first factor and (48)+(50) for the second factor gives

$$\lesssim \Lambda^6 [f(\cdot,\omega)]_{\substack{\Lambda+6,\gamma',\sigma \\ \rho,\mu}}.$$

The third term is a composition of the function

$$\tilde f = <\partial_\zeta f, (\partial_\omega \Phi_t)\circ \Phi_{-t}>$$

with Φ_t. Evaluating $\tilde f$ we find that it has the form $<\partial_\zeta f, \tilde b_t + \tilde B_t \zeta>$ where

$$\tilde b_t = \partial_\omega b_t(\varphi_{-t}) + \partial_\omega B_t(\varphi_{-t}) b_{-t}$$
$$\tilde B_t = \partial_\omega B_t(\varphi_{-t}) + \partial_\omega B_t(\varphi_{-t}) B_{-t}.$$

For $\varphi \in \mathbb{T}^{\mathcal{A}}_{\rho''}$ we get by (48)+(52) that

$$|\varphi_{-t}-\varphi| \le \rho'-\rho'' = \rho-\rho',$$

so $\tilde b_t$ and $\tilde B_t$ are defined on $\mathbb{T}^{\mathcal{A}}_{\rho''}$. By (49)+(52)

$$\left\| \tilde b_t \right\|_{\gamma'} \le \sigma-\sigma',$$

and by (50)+(52) and the product formula (10)

$$< \tilde B_t >_{\substack{\Lambda+9,\gamma' \\ \rho''}} \lesssim \Lambda^6 (\frac{1}{\gamma-\gamma'}) \frac{\varepsilon}{\sigma^2},$$

so by Proposition 3.1(ii-iii) and (52) we obtain

$$[\tilde f]_{\substack{\Lambda+9,\gamma',\sigma'\\ \rho'',\mu'}} \lesssim \Lambda^8 [f]_{\substack{\Lambda+6,\gamma,\sigma\\ \rho'',\mu'}}.$$

Finally by the same argument as above we get

$$[\tilde f(\Phi_t(\cdot,\omega),\omega)]_{\substack{\Lambda+15,\gamma'',\sigma''\\ \rho''',\mu''}} \lesssim \Lambda^6 [\tilde f(\cdot,\omega)]_{\substack{\Lambda+9,\gamma',\sigma'\\ \rho'',\mu'}},$$

if we choose $\rho''-\rho'''=\rho'-\rho''$, $\sigma'-\sigma''=\sigma-\sigma'$ and $\mu'-\mu''=\mu-\mu'$. This completes the proof. $\square$

8.2. **A finite induction.**
Let

$$h(\zeta,r,\omega)=<\omega,r>+\frac12<\zeta,(\Omega(\omega)+H(\omega))\zeta>$$

satisfy (13-15)+(38-39) and let $H(\omega)$ and $\partial_\omega H(\omega)$ be $\mathcal{NF}_\Delta$. Let

$$[f]_{\left\{\substack{\Lambda\\ U}\alpha\right\}}=\varepsilon,\quad \alpha=\begin{pmatrix}\gamma & \sigma\\ \rho & \mu\end{pmatrix}.$$

Besides the assumption that all constants γ,σ,ρ,μ are <1 and that Δ,Λ are ≥ 3 and we shall also assume that

$$\mu=\sigma^2 \quad\text{and}\quad d_\Delta\gamma\leq 1.$$

The first assumption is just for convenience, but the second is forced upon us by the occurrence of a factor $e^{d_\Delta\gamma}$ in the estimates which we must control.

Fix $\rho'<\rho$ and $\gamma'<\gamma$ and let

$$\Delta'=(\log(\frac1\varepsilon))^2\frac{1}{\min(\gamma-\gamma',\rho-\rho')},\qquad n=[\log(\frac1\varepsilon)].$$

Define for $1\leq j\leq n$

$$\begin{array}{ll}
\varepsilon_{j+1}=(\frac{\varepsilon}{\sigma^2\kappa^3})\varepsilon_j & \varepsilon_1=\varepsilon,\\
\Lambda_{j+1}=\Lambda_j+d_\Delta+23, & \Lambda_1=\text{cte.}\max(\Lambda,d_\Delta^2,(d_{\Delta'})^2)\\
\gamma_j=\gamma-(j-1)\frac{\gamma-\gamma'}{n}, & \rho_j=\rho-(j-1)\frac{\rho-\rho'}{n}\\
\sigma_{j+1}=(\frac{\varepsilon}{\sigma^2\kappa^3})^{\frac13}\sigma_j & \sigma_1=\sigma\\
\mu_{j+1}=(\frac{\varepsilon}{\sigma^2\kappa^3})^{\frac23}\mu_j & \mu_1=\mu.
\end{array}$$

[13]

We have the following proposition.

[13] The constant in the definition of Λ_1 is the one in Proposition 6.7.

Proposition 8.1. *Under the above assumptions there exist a constant* $\mathcal{C}$ *and an exponent* $\exp_1$ *such that if*

$$\varepsilon \ \le\ \kappa^3 \mathcal{C} \min(\gamma-\gamma', \rho-\rho', \frac{1}{\Delta}, \frac{1}{\Lambda}, \frac{1}{\log(\frac{1}{\varepsilon})})^{\exp_1} \min(\sigma^2, \mu),$$

then there is a subset $U' \subset U$,

$$\mathrm{Leb}(U \setminus U') \le \mathrm{cte}.\varepsilon^{\exp_2},$$

such that for all $\omega \in U'$ *the following holds for* $1 \le j \le n$: *there is an analytic symplectic diffeomorphism*

$$\Phi_j : \mathcal{O}^{\gamma''}(\sigma_{j+1}, \rho_{j+1}, \mu_{j+1}) \to \mathcal{O}^{\gamma''}(\sigma_j, \rho_j, \mu_j), \quad \forall \gamma'' \le \gamma_{j+1},$$

such that

$$(h + h_1 + \ldots + h_{j-1} + f_j) \circ \Phi_j = h + h_1 + \ldots + h_j + f_{j+1}$$

$(f_1 = f)$ *with*

(i)

$$h_j = c + <\chi_j(\omega), r> + \frac{1}{2} <\zeta, H_j(\omega)\zeta>$$

with $H_j(\omega)$ *and* $\partial_\omega H_j(\omega)$ *in* $\mathcal{NF}_{\Delta'}$ *and*

$$[h_j]_{\left\{ {\Lambda_j \atop U'} \alpha_j \right\}} \le \beta^{j-1} \varepsilon_j$$

(ii)

$$[f_{j+1}]_{\left\{ {\Lambda_{j+1} \atop U'} \alpha_{j+1} \right\}} \le \beta^j \varepsilon_{j+1},$$

where

$$\beta \lesssim \mathrm{cte.} \max(\frac{1}{\gamma-\gamma'}, \frac{1}{\rho-\rho'}, \Lambda, \Delta, \log(\frac{1}{\varepsilon}))^{\exp_3}.$$

Moreover, Φ_j *extends to an analytic symplectic diffeomorphism*

$$\mathcal{O}^{\gamma''}(\bar\sigma, \rho_{j+1}, \bar\mu) \to \mathcal{O}^{\gamma''}(\bar\sigma + \beta^j \frac{\varepsilon_j}{\sigma_j^2} \bar\sigma, \rho_j, \bar\mu + \beta^j \frac{\varepsilon_j}{\mu_j} \bar\mu)$$

for all $\gamma'' \le \gamma_{j+1}$.

The exponents $\exp_1, \exp_2, \exp_3$ *only depend on* $d, \#\mathcal{A}, m_*$ *while the constants* $\mathcal{C}$ *and* cte. *also depend on all* $C_1, \ldots, C_5$.

Proof. We start by solving inductively

$$\{h, s_j\} = -\mathcal{T}_{\Delta'} f_j + h_j$$

using Propositions 6.6 and 6.7. To see how this works, write

$$\begin{array}{l} s_j = S_0 + <\zeta, S_1> + \frac{1}{2} <\zeta, S_2\zeta> \\ \mathcal{T}_{\Delta'} f_j = F_0 + <\zeta, F_1> + \frac{1}{2} <\zeta, F_2\zeta> \\ h_j = c + <\chi_j(\omega), r> + \frac{1}{2} <\zeta, H_2\zeta> . \end{array}$$

The equation written in Fourier modes becomes

$$\begin{aligned}
&-i <k,\omega> \hat{S}_0(k) = -\hat{F}_0(k) + \delta_0^k(c+ <\chi_j(\omega), r>)\\
&-i <k,\omega> \hat{S}_1(k) + J(\Omega(\omega) + H(\omega))\hat{S}_1(k) = -\hat{F}_1(k)\\
&-i <k,\omega> \hat{S}_2(k) + (\Omega(\omega) + H(\omega))J\hat{S}_2(k) - \hat{S}_2(k)J(\Omega(\omega) + H(\omega))\\
&\qquad = -\hat{F}_2(k) + \delta_0^k H_2.
\end{aligned}$$

Using Propositions 6.6 and 6.7 and (43) these equations can be solved for ω in a set U_j with

$$\mathrm{Leb}(U_{j-1} \setminus U_j) \le \mathrm{cte}.\varepsilon^{\exp} \quad (U_0 = U).$$

This gives, after summing up the (finite) Fourier series,

$$\begin{aligned}
&[s_j]_{\left\{\substack{\Lambda_j+d_\Delta+2 \\ U_j}\ \alpha_j\right\}} \le \mathrm{cte}.(\Delta'\Delta)^{\exp}\tfrac{1}{\kappa^3}\beta^{j-1}\varepsilon_j = \tilde{\varepsilon}_j\\
&[h_j]_{\left\{\substack{\Lambda_j+d_\Delta+2 \\ U_j}\ \alpha_j\right\}} \le \mathrm{cte}.(\Delta'\Delta)^{\exp}\beta^{j-1}\varepsilon_j
\end{aligned}$$

If the solutions s_j and h_j were non-real (they are not because the construction gives really real functions) then their real parts would give real solutions.

In a second step, for $0 \le t \le 1$ we estimate

$$f_j - h_j + \{h + h_1 + \ldots + h_{j-1} + (1-t)h_j + tf_j, s_j\}$$

which is equal

$$(f_j - \mathcal{T}_{\Delta'} f_j) + t\{f_j, s_j\} + \{h_1 + \ldots + h_{j-1} + (1-t)h_j, s_j\} =: g_1 + g_2 + g_3.$$

According to (44) we have

$$[g_1]_{\left\{\substack{\Lambda_j+d_\Delta+2 \\ U_j}\ \tilde{\alpha}_{j+1}\right\}} \lesssim A(\alpha_j, \tilde{\alpha}_{j+1}, \Delta')\beta^{j-1}\varepsilon_j,$$

where

$$\tilde{\alpha}_{j+1} = \begin{pmatrix} \gamma_j - \frac{\gamma_j - \gamma_{j+1}}{2} & 2\sigma_{j+1} \\ \rho_j - \frac{\rho_j - \rho_{j+1}}{2} & 2\mu_{j+1}. \end{pmatrix}$$

By our choice of constants and the assumption on ε we have

$$A \lesssim (\frac{1}{\sigma^2\kappa^3} + (\frac{1}{\rho - \rho'})^{\#\mathcal{A}})\varepsilon \lesssim \frac{1}{\Lambda_j^{14}}\beta\frac{\varepsilon}{\sigma^2\kappa^3}.$$

According to (45) we have

$$[g_2]_{\left\{\substack{\Lambda_j+d_\Delta+5 \\ U_j}\ \tilde{\alpha}_{j+1}\right\}} \lesssim B_j(\Delta'\Delta)^{\exp}\frac{1}{\kappa^3}\beta^{2j-2}\varepsilon_j^2,$$

where

$$B_j = B(\gamma_j - \gamma_{j+1}, \sigma_j, \rho_j - \rho_{j+1}, \mu_j, \Lambda_j).$$

β takes care of this when $j = 1$ and when $j \geq 2$ we have the factor $\frac{\varepsilon_j}{\varepsilon_1}$ that controls everything, and we get the bound

$$\lesssim \frac{1}{\Lambda_j^{14}} \beta^j \frac{\varepsilon}{\sigma^2 \kappa^3} \varepsilon_j.$$

According to (46) we have

$$[g_3]_{\left\{ \begin{smallmatrix} \Lambda_j + d_\Delta + 5 & \\ U_j & \tilde{\alpha}_{j+1} \end{smallmatrix} \right\}} \lesssim \sum_{1 \leq i \leq n} B_i (\Delta' \Delta)^{\exp} \beta^{i-1} \varepsilon_i \text{cte.} (\Delta' \Delta)^{\exp} \frac{1}{\kappa^3} \beta^{j-1} \varepsilon_j,$$

where

$$B_i = B(\gamma_j - \gamma_{j+1}, \sigma_i, \rho_j - \rho_{j+1}, \mu_i, \Lambda_j).$$

The same argument applies again: β takes care of this when $i = 1$ and when $i \geq 2$ we have the factor$\frac{\varepsilon_i}{\varepsilon_1}$ that controls everything. We get as before the bound

$$\lesssim \frac{1}{\Lambda_j^{14}} \beta^j \frac{\varepsilon}{\sigma^2 \kappa^3} \varepsilon_j.$$

In a third step we construct the time-t-map, $|t| \leq 1$, Φ_t for the vector field $J\partial s$. Condition (47),

$$\tilde{\varepsilon}_j \lesssim \min((\tilde{\rho}_{j+1} - \rho_{j+1}) \tilde{\mu}_{j+1}, (\tilde{\gamma}_{j+1} - \gamma_{j+1})^{d+m_*} \tilde{\sigma}_{j+1}^2),$$

is fulfilled for all j by assumption on ε, so

$$\Phi_t : \mathcal{O}^{\gamma''}(\sigma_{j+1}, \rho_{j+1}, \mu_{j+1}) \to \mathcal{O}^{\gamma''}(\tilde{\sigma}_{j+1}, \tilde{\rho}_{j+1}, \tilde{\mu}_{j+1})$$

for all $\gamma'' < \gamma_{j+1}$, and it will verify conditions (48-51) with α, α', Λ replaced by $\tilde{\alpha}_{j+1}, \alpha_{j+1}, \Lambda_j + d_\Delta + 2$.

Finally we define

$$f_{j+1} = \int_0^1 (g_1 + g_2 + g_3) \circ \Phi_t dt.$$

Then the time-1-map Φ_t will be our Φ_j and do what we want – this is a well-known relation. It only remains to verify the estimate for f_{j+1}. Condition (52),

$$\tilde{\varepsilon}_j \lesssim \min((\tilde{\rho}_{j+1} - \rho_{j+1}) \tilde{\mu}_{j+1}, (\tilde{\gamma}_{j+1} - \gamma_{j+1})^{d+m_*+1} \tilde{\sigma}_{j+1}^2) \sqrt{\tilde{\gamma}_{j+1} - \gamma_{j+1}},$$

is fulfilled for all j by assumption on ε, so we get by (53)

$$[f_{j+1}]_{\left\{ \begin{smallmatrix} \Lambda_{j+1} & \\ U_j & \alpha_{j+1} \end{smallmatrix} \right\}} \lesssim \Lambda_j^{14} [g]_{\left\{ \begin{smallmatrix} \Lambda_j + d_\Delta + 5 & \\ U_j & \tilde{\alpha}_{j+1} \end{smallmatrix} \right\}},$$

and we are done. $\square$

Corollary 8.2. *There exist a constant* $\mathcal{C}$ *and an exponent* exp *such that, if*

$$\varepsilon \ \leq \ \mathcal{C}\min(\gamma-\gamma',\rho-\rho',\frac{1}{\Delta},\frac{1}{\Lambda})^{\exp}\min(\sigma^2,\mu)^{\frac{1}{1-3\tau}}, \quad \tau=\frac{1}{33},$$

[14] *then there is a subset* $U'\subset U$,

$$\mathrm{Leb}(U\setminus U')\leq \mathrm{cte}.\varepsilon^{\exp_1},$$

such that for all $\omega\in U'$ *the following hold: there is an analytic symplectic diffeomorphism*

$$\Phi:\mathcal{O}^{\gamma''}(\sigma',\rho',\mu')\to\mathcal{O}^{\gamma''}(\sigma,\rho,\mu), \quad \forall\gamma''\leq\gamma',$$

and a vector ω' *such that*

$$(h_{\omega'}+f)\circ\Phi=h'+f'$$

with

(i)

$$h'=<\omega,r>+\frac{1}{2}<\zeta,(\Omega(\omega)+H'(\omega))\zeta>$$

with $H'(\omega)$ *and* $\partial_\omega H'(\omega)$ *in* $\mathcal{NF}_{\Delta'}$ *and*

$$[h'-h]_{\left\{\substack{\Lambda'\\U'}, \alpha'\right\}}\leq \mathrm{cte}.\varepsilon$$

and

$$|\omega'-\omega|\leq \mathrm{cte}.\frac{\varepsilon}{\mu},$$

(ii)

$$[f']_{\left\{\substack{\Lambda'\\U'}, \alpha'\right\}}\leq\varepsilon'\leq e^{-\tau(\log(\frac{1}{\varepsilon}))^2}$$

where

$$\begin{aligned}
&\Delta'=(\log(\tfrac{1}{\varepsilon}))^2\tfrac{1}{\min(\gamma-\gamma',\rho-\rho')},\\
&\Lambda'=\mathrm{cte.}\max(\Lambda,d_\Delta^2,(d_{\Delta'})^2)+\log(\tfrac{1}{\varepsilon})(d_\Delta+23)\\
&\sigma'=(\varepsilon')^{\frac{1}{3}+\tau}\\
&\mu'=(\varepsilon')^{\frac{2}{3}+2\tau}.
\end{aligned}$$

Moreover, Φ *extends to an analytic symplectic diffeomorphism*

$$\mathcal{O}^{\gamma''}(\bar\sigma,\rho',\bar\mu)\to\mathcal{O}^{\gamma''}(\bar\sigma+\beta\frac{\varepsilon}{\sigma^2}\bar\sigma,\rho,\bar\mu+\beta\frac{\varepsilon}{\mu}\bar\mu).$$

for all $\gamma''\leq\gamma'$, *where*

$$\beta\lesssim \mathrm{cte.}\max(\frac{1}{\gamma-\gamma'},\frac{1}{\rho-\rho'},\Lambda,\Delta,\log(\frac{1}{\varepsilon}))^{\exp_2}.$$

[14] The bound on ε in Proposition 8.1 is implicit due to κ and $\log(\frac{1}{\varepsilon})$). Here we have an explicit bound, but the price for taking κ equal to a fractional power of ε is that the bound must depend on $\max(\sigma^2,\mu)$ to a power larger than 1.

The exponents exp, $\exp_1, \exp_2$ *only depend on* $d, \#\mathcal{A}, m_*$ *while the constants* $\mathcal{C}$ *and* cte. *also depend on all* $C_1, \dots, C_5$.

Proof. Take $\kappa^3 = \varepsilon^\tau$. Then

$$\varepsilon_{n+1} = \varepsilon', \quad \sigma_{n+1} \geq (\varepsilon')^{\frac{1}{3}+\tau}, \quad \mu_{n+1} \geq (\varepsilon')^{\frac{2}{3}+2\tau},$$

and

$$\varepsilon' \leq e^{-\tau(\log(\frac{1}{\varepsilon}))^2}$$

if

$$\varepsilon^{1-2\tau} \lesssim \frac{1}{\beta}\sigma.$$

The result is an immediate consequence of Proposition 8.1 if we take $\omega' = \omega$ and

$$h' = <\omega + \chi(\omega), r> + \frac{1}{2}<\zeta, (\Omega(\omega) + H'(\omega))\zeta> .$$

By the bound on the derivative of χ, which is part of (ii), the image of U' under the mapping $\omega \to \omega + \chi(\omega)$ covers a subset U'' of U of the same complementary Lebesgue measure, and we can replace $\omega + \chi(\omega)$ by ω if we take $\omega' = (Id + \chi)^{-1}(\omega)$.

Since $|\chi(\omega)| \leq \text{cte.}\frac{\varepsilon}{\mu}$ we get the estimate for $|\omega' - \omega|$. □

8.3. **The infinite induction.**

Let

$$f : \mathcal{O}^\gamma(\sigma, \rho, \mu) \times U \to \mathbb{C}$$

be real analytic in ζ, φ, r and $\mathcal{C}^1$ in $\omega \in U$ and consider

$$[f]_{\left\{ {\Lambda \atop U} \alpha \right\}} = \varepsilon, \quad \alpha = \begin{pmatrix} \gamma & \sigma \\ \rho & \mu \end{pmatrix}.$$

Choice of constants. We define

$$\begin{array}{ll}
\varepsilon_{j+1} = e^{-\tau(\log(\frac{1}{\varepsilon_j}))^2} \ (\tau = \frac{1}{33}), & \varepsilon_1 = \varepsilon \\
\gamma_j = (d_{\Delta_j})^{-1}, & \gamma_1 = \min(d_\Delta, \gamma) \\
\sigma_j = \varepsilon_j^{\frac{1}{3}+\tau} \ \ j \geq 2 & \sigma_1 = \sigma \\
\mu_j = \varepsilon_j^{\frac{2}{3}+2\tau} \ \ j \geq 2 & \mu_1 = \mu \\
\rho_j = (\frac{1}{2} + \frac{1}{2^j})\rho & \\
\Delta_{j+1} = (\log(\frac{1}{\varepsilon_j}))^2 \frac{1}{\min(\gamma_j, \rho_j - \rho_{j+1})}, & \Delta_1 = \Delta \\
\Lambda_j = \text{cte.}(d_{\Delta_j})^2).
\end{array}$$

[15]

With this choice of constants we prove

[15] The constant in the definition of Λ_j is the one in Proposition 6.7.

Lemma 8.3. *There exist a constant* $\mathcal{C}'$ *and an exponent* $\exp'$ *such that if*

$$\varepsilon \le \mathcal{C}' \min(\gamma, \rho, \frac{1}{\Delta}, \frac{1}{\Lambda})^{\exp'} \min(\sigma^2, \mu)^{\frac{1}{1-3\tau}},$$

then for all $j \ge 1$

$$\varepsilon_j \le \mathcal{C} \min(\gamma_j - \gamma_{j+1}, \rho_j - \rho_{j+1}, \frac{1}{\Delta_j}, \frac{1}{\Lambda_j})^{\exp} \min(\sigma_j^2, \mu_j)^{\frac{1}{1-3\tau}}$$

and

$$\sum_{1 \le i \le j} (d_{\Delta_i})^2 \varepsilon_i \le \frac{1}{4} \min(C_4, C_5),$$

where $\mathcal{C}, \exp$ *are those of Corollary 8.2.*

The exponents $\exp'$ *only depend on* $d, \#\mathcal{A}, m_*$ *while the constant* $\mathcal{C}'$ *also depend on all* $C_1, \dots, C_5$.

Remark. Notice that Δ_j increases much faster than quadratically at each step — $\Delta_{j+1} \ge \Delta_j^{\frac{(d+1)!}{2}}$ due to its coupling with γ_j. This is the reason why we cannot grant the convergence by a quadratic iteration but need a much faster iteration scheme, as the one provided by Proposition 8.1 and Corollary 8.2.

The proof is an exercise on the theme "superexponential growth beats (almost) everything".

Proposition 8.4. *There exist a constant* $\mathcal{C}$ *and an exponent* $\exp$ *such that if*

$$\varepsilon \ \le \ \mathcal{C} \min(\gamma - \gamma', \rho - \rho', \frac{1}{\Delta}, \frac{1}{\Lambda})^{\exp} \min(\sigma^2, \mu)^{\frac{1}{1-3\tau}},$$

then there is a subset $U' \subset U$,

$$\mathrm{Leb}(U \setminus U') \le \mathrm{cte}.\varepsilon^{\exp_1},$$

such that for all $\omega \in U'$ *the following hold: for all* $1 \le j$ *there is an analytic symplectic diffeomorphism*

$$\Phi_j : \mathcal{O}^{\gamma''}(\sigma_{j+1}, \rho_{j+1}, \mu_{j+1}) \to \mathcal{O}^{\gamma''}(\sigma_j, \rho_j, \mu_j), \quad \forall \gamma'' \le \gamma_{j+1},$$

and a vector ω_j *such that*

$$(h_{j-1} + f_j) \circ \Phi_j = h_j + f_{j+1} \quad (h_0 = h_{\omega_j},\ f_1 = f)$$

and satisfying:

(i)

$$h_j = c + <\omega, r> + \frac{1}{2} <\zeta, (\Omega(\omega) + H_j(\omega))\zeta>$$

with $H_j(\omega)$ *and* $\partial_\omega H_j(\omega)$ *in* $\mathcal{NF}_{\Delta_{j+1}}$ *and*

$$[h_{j+1}-h_j]_{\left\{ {\Lambda_j \atop U'} \alpha_{j+1} \right\}} \le \mathrm{cte.}\varepsilon_j$$

and

$$|\omega_{j+1}-\omega_j| \le \mathrm{cte.}\frac{\varepsilon_j}{\mu_j},$$

(ii)

$$[f_{j+1}]_{\left\{ {\Lambda_{j+1} \atop U'} \alpha_{j+1} \right\}} \le \varepsilon_{j+1}.$$

Moreover, $\Phi_1 \circ \cdots \circ \Phi_j$ *converges to an analytic symplectic diffeomorphism* Φ

$$\mathcal{O}^0(\frac{\sigma}{2},\frac{\rho}{2},\frac{\mu}{2}) \to \mathcal{O}^0(\frac{\sigma}{2}+\varepsilon^{\frac12},\frac{\rho}{2}+\varepsilon^{\frac12},\frac{\mu}{2}+\varepsilon^{\frac12}).$$

The exponents exp, exp_1 *only depend on* $d, \#\mathcal{A}, m_*$ *while the constants* $\mathcal{C}$ *and* cte. *also depend on all* $C_1,\dots,C_5$.

Proof. The proof is an immediate consequence of Corollary 8.2 and Lemma 8.3. The first part of the lemma implies that the smallness assumption in the proposition is fulfilled for every $j\ge 1$, and the second part implies that assumption (39) for every $j \ge 1$. The remaining assumptions are only on Ω. □

Theorem 7.1 now follows from this proposition. Indeed,

$$\omega_j \to \omega'$$

and we have

$$(h_{\omega'}+f)\circ\Phi = \lim_{t\to\infty}(h_{\omega_j}+f)\circ\Phi_1\circ\cdots\circ\Phi_j = \lim_{t\to\infty}(h_j+f_{j+1}),$$

and since the sequence h_j clearly converges on $\mathcal{O}^0(\frac{\sigma}{2},\frac{\rho}{2},\frac{\mu}{2})$, also f_j converges on this set – to a function f'.

Moreover, for $\zeta = r = 0$ and $|\Im\varphi| < \frac{\rho}{2}$ we have, as $j\to\infty$,

$$|f_j|\,,\ |\partial_r f_j|\,, \|\partial_\zeta f_j\|_0 \to 0$$

and, by Young's inequality,

$$\left\|\partial_\zeta^2 f_j \hat\zeta\right\|_0 \lesssim (\frac{1}{\gamma_j})^d \left|\partial_\zeta^2 f_j\right|_0 \left\|\hat\zeta\right\|_0 \to 0.$$

Therefore

$$\partial_\zeta f' = \partial_r f' = \partial_\zeta^2 f' = 0 \ \text{ for } \ \zeta = r = 0.$$

9. Appendix A - Some estimates

Lemma 9.1. *Let* $f : I =]-1,1[\to \mathbb{R}$ *be of class* $\mathcal{C}^n$ *and*

$$\left|f^{(n)}(t)\right| \geq 1, \quad \forall t \in I.$$

Then, $\forall \varepsilon > 0$, *the Lebesgue measure of* $\{t \in I : |f(t)| < \varepsilon\}$ *is*

$$\leq \text{cte}.\varepsilon^{\frac{1}{n}},$$

where the constant only depends on n.

Proof. We have $\left|f^{(n)}(t)\right| \geq \varepsilon^{\frac{0}{n}}$ for all $t \in I$. Since

$$f^{(n-1)}(t) - f^{(n-1)}(t_0) = \int_{t_0}^{t} f^{(n)}(s)ds,$$

we get that $\left|f^{(n-1)}(t)\right| \geq \varepsilon^{\frac{1}{n}}$ for all t outside an interval of length $\leq 2e^{\frac{1}{n}}$. By induction we get that $\left|f^{(n-j)}(t)\right| \geq \varepsilon^{\frac{j}{n}}$ for all t outside 2^{j-1} intervals of length $\leq 2\varepsilon^{\frac{1}{n}}$. $j = n$ gives the result. $\square$

Remark. The same is true if

$$\max_{0 \leq j \leq n} \left|f^{(j)}(t)\right| \geq 1, \quad \forall t \in I$$

and $f \in \mathcal{C}^{n+1}$. In this case the constant will depend on $|f|_{\mathcal{C}^{n+1}}$.

Let $B(t)$ be a Hermitian $N \times N$-matrix of class $\mathcal{C}^1$ in $I =]-1,1[$ with

$$\|B'(t)\| \leq \frac{1}{2}, \quad \forall t \in I.$$

Lemma 9.2. *The Lebesgue measure of the set*

$$\{t \in I : \min_{\lambda(t) \in \sigma(B(t))} |t + \lambda(t)| < \varepsilon\}$$

is

$$\leq \text{cte}.N\varepsilon,$$

where the constant is independent of N.

Proof. Assume first that $B(t)$ is analytic in t. Then the eigenvalue $\lambda_j(t)$, $j = 1, \ldots, N$, are analytic in t with

$$|\lambda_j(t)| \leq \|B(t)\|$$
$$\left|\lambda_j'(t)\right| \leq \|B'(t)\|.$$

Lemma 9.1 applied to each $f(t) = t + \lambda_j(t)$ gives the result.

If B is non-analytic we get the same result by analytic approximation. $\square$

Proposition 9.3.
$$\left\|(tI+B(t))^{-1}\right\| \le \frac{1}{\varepsilon}$$
outside a set of $t \in I$ of Lebesgue measure
$$\le \text{cte.} N\varepsilon.$$

Proof. The exists an orthogonal matrix $U(t)$ such that
$$U(t)^*(tI+B(t))U(t) = \begin{pmatrix} t+\lambda_1(t) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & t+\lambda_N(t) \end{pmatrix}$$
Now
$$\left\|(tI+B(t))^{-1}\right\| = \max_{0\le j\le N} \left|\frac{1}{t+\lambda_j(t)}\right|.$$
□

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