RELATIVE OSCILLATION THEORY FOR JACOBI MATRICES

KERSTIN AMMANN AND GERALD TESCHL

Abstract. We develop relative oscillation theory for Jacobi matrices which, rather than counting the number of eigenvalues of one single matrix, counts the difference between the number of eigenvalues of two different matrices. This is done by replacing nodes of solutions associated with one matrix by weighted nodes of Wronskians of solutions of two different matrices.

1. Introduction

Oscillation theory for second-order differential and difference equations has a long tradition originating in the seminal work of Sturm from 1836 [9]. Since then the subject is continuously growing and many monographs have been devoted entirely to this subject. The most recent one being the monumental treatise by Agarwal, Bohner, Grace, and O'Regan [1]. One of the key results of classical oscillation theory is the fact, the k'th eigenfunction has precisely $k-1$ nodes (i.e., sign flips) and for a suitably chosen solution of the underlying difference equation, the number of nodes of this solutions equals the number of eigenvalues below a given value. Our aim is add a new wrinkle to this classical result by showing that the number of weighted nodes of the Wronskian (also known as Casoratian) of two suitable solutions of two different Jacobi difference equations can be used to count the difference between the number of eigenvalues of the two associated Jacobi matrices.

That Wronskians are related to oscillation theory is indicated by an old paper of Leighton [7], who noted that if two solutions have a non-vanishing Wronskian, then their zeros must intertwine each other. However, it seems their real power was realized only later by Gesztesy, Simon, and Teschl in [3] with the corresponding extension to Jacobi operators given by Teschl [10]. For a pedagogical discussion we refer to the survey by Simon [8]. That these results are just the tip of the iceberg was discovered only recently by Krüger and Teschl [4], [5], [6]. Our result generalizes the main result for the case of Sturm–Liouville operators from [4] to the case of Jacobi matrices.

To set the stage, let us fix some real numbers $a(j) < 0$, $b(j)$, $j = 1, \cdots, N-1$ and consider the Jacobi matrix

$$H = \begin{pmatrix} b(1) & a(1) & 0 & 0 & 0 \\ a(1) & b(2) & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & a(N-1) & b(N-2) & a(N-2) \\ 0 & 0 & 0 & a(N-2) & b(N-1) \end{pmatrix}. \tag{1.1}$$

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in the Hilbert space $\mathbb{C}^{N-1}$. Furthermore, let $s_\pm(z,n)$ be the solutions of the underlying difference equation (set $a(0) = a(N-1) = a(N) = -1$, $b(N) = 0$)

$$a(n)u(n+1) + b(n)u(n) + a(n-1)u(n-1) = zu(n), \qquad n = 1, \dots, N, \tag{1.2}$$

corresponding to the initial conditions

$$s_-(z,0) = 0, s_-(z,1) = 1, \qquad s_+(z,N) = 0, s_+(z,N+1) = 1. \tag{1.3}$$

Note that $s_-(\lambda,n)$ (resp. $s_+(\lambda,n)$) will be an eigenvector of H corresponding to the eigenvalue $\lambda \in \mathbb{R}$ if and only of $s_-(\lambda,N) = 0$ (resp. $s_+(\lambda,0) = 0$). We will abbreviate $s(z,n) = s_-(z,n)$.

We call n a node of a solution u of (1.2) if either

$$u(n) = 0 \quad \text{or} \quad u(n)u(n+1) < 0. \tag{1.4}$$

We say that a node n_0 of u *lies between* m *and* n if either

$$m < n_0 < n \quad \text{or} \quad n_0 = m \text{ but } u(m) \neq 0. \tag{1.5}$$

$\#_{(m,n)}(u)$ denotes the number of nodes of u between m and n and $\#(u) = \#_{(0,N)}(u)$.

Then we have the following classical result alluded to before (see e.g., [2], [11]):

Theorem 1.1. *Let H be a Jacobi matrix and $s(z,n)$ a corresponding solution of the underlying difference equation (1.2) corresponding to the initial condition $s(z,0) = 0$. Then for every $\lambda \in \mathbb{R}$ the number of nodes of $s(\lambda,n)$ equals the number of eigenvalues of H below λ:*

$$\#(s(\lambda)) = \#\{E \in \sigma(H) | E < \lambda\}. \tag{1.6}$$

Here $\sigma(H)$ denotes the spectrum of H, that is, the set of eigenvalues.

To generalize this result we will now consider two Jacobi matrices H_0 and H_1 associated with the coefficients $a_0(n) = a_1(n) \equiv a(n)$ and $b_0(n)$ respectively $b_1(n)$. The corresponding solutions will be denoted by $s_{j,\pm}(n)$, $j = 0,1$, in obvious notation. Given two solutions u_j of the difference equations associated with H_j we denote by

$$W_n(u_0,u_1) = a(n)(u_0(n)u_1(n+1) - u_0(n+1)u_1(n)) \tag{1.7}$$

their Wronskian. As already anticipated we will relate the number of nodes of such Wronskians to the difference between the eigenvalues of H_0 and H_1. Since this difference is a signed quantity, we will need to weight the nodes according to the sign of the difference between H_0 and H_1 as follows: Set

$$\#_n W(u_0,u_1) = \begin{cases} & \text{if } b_0(n+1) - b_1(n+1) > 0 \text{ and} \\ 1, & \text{either } W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0 \\ & \text{or } W_n(u_0,u_1) = 0 \text{ and } W_{n+1}(u_0,u_1) \neq 0, \\ & \text{if } b_0(n+1) - b_1(n+1) < 0 \text{ and} \\ -1, & \text{either } W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0 \\ & \text{or } W_n(u_0,u_1) \neq 0 \text{ and } W_{n+1}(u_0,u_1) = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.8}$$

Then we say the Wronskian has a weighted node at n if $\#_n W(u_0,u_1) \neq 0$. The weighted number of nodes of the Wronskian between 0 and N is denoted as

$$\#(W(u_0,u_1)) = \sum_{j=0}^{N-1} \#_j W(u_0,u_1) - \begin{cases} 0, & \text{if } W_0(u_0,u_1) \neq 0, \\ 1, & \text{if } W_0(u_0,u_1) = 0. \end{cases} \tag{1.9}$$

With this notation our main result reads

Theorem 1.2. *Let H_0, H_1 be two Jacobi matrices with $a_0 = a_1$ and $s_{\pm,j}(z,n)$ the corresponding solutions of the underlying difference equations. Then for every $\lambda \in \mathbb{R}$ the number of weighted nodes of $W(s_{-,0}(\lambda), s_{+,1}(\lambda))$ equals the number of eigenvalues of H_1 below λ minus the number of eigenvalues of H_0 below or equal to λ:*

$$\begin{aligned} &\#(W(s_{-,0}(\lambda), s_{+,1}(\lambda))) = \#(W(s_{+,0}(\lambda), s_{-,1}(\lambda))) = \\ &\quad = \#\{E \in \sigma(H_1) | E < \lambda\} - \#\{E \in \sigma(H_0) | E \le \lambda\}. \end{aligned} \tag{1.10}$$

Here $\sigma(H)$ denotes the spectrum of H, that is, the set of eigenvalues.

The proof is based on Prüfer angles to be investigated in Section 2. It will be given in Section 3.

An extension to Jacobi operators on $\mathbb{N}$ respectively $\mathbb{Z}$ is in preparation.

2. Prüfer angles

Since any nontrivial solution of (1.2) cannot vanish at two consecutive points we can introduce Prüfer variables $(\rho_u(n), \theta_u(n))$ in the usual way (cf., e.g., [11, Chap. 4]) via

$$u(n) = \rho_u(n)\sin(\theta_u(n)), \qquad u(n+1) = \rho_u(n)\cos(\theta_u(n)). \tag{2.1}$$

Note that $\rho_u(n) > 0$ for all $n \in \mathbb{Z}$ and $\theta_u(n)$ is only defined up to an additive integer multiple of 2π, depending on n. For our further investigations it is essential to gain unique values for the Prüfer angle and therefore we fix $\theta_u(0)$ and require

$$\lceil \theta_u(n)/\pi \rceil \le \lceil \theta_u(n+1)/\pi \rceil \le \lceil \theta_u(n)/\pi \rceil + 1, \tag{2.2}$$

where $\lceil x \rceil = \min\{n \in \mathbb{Z} \,|\, n \ge x\}$ denotes the usual ceiling function. Then the following easy result is well-known.

Lemma 2.1. *Suppose that (2.2) holds for a solution u of (1.2) and for an arbitrary n. Define k, γ, Γ via*

$$\theta_u(n) = k\pi + \gamma, \quad \theta_u(n+1) = k\pi + \Gamma, \qquad \gamma \in (0,\pi],\ \Gamma \in (0, 2\pi],\ k \in \mathbb{Z}. \tag{2.3}$$

Then

$$\gamma \in \begin{cases} (0, \frac{\pi}{2}] & \text{iff } n \text{ is not a node,} \\ (\frac{\pi}{2}, \pi] & \text{iff } n \text{ is a node,} \end{cases} \tag{2.4}$$

and

$$\Gamma \in \begin{cases} (0, \pi] & \text{iff } n \text{ is not a node,} \\ (\pi, 2\pi) & \text{iff } n \text{ is a node.} \end{cases} \tag{2.5}$$

Moreover,

$$\theta_u(n) = k\pi + \frac{\pi}{2} \quad \Leftrightarrow \quad \theta_u(n+1) = (k+1)\pi. \tag{2.6}$$

As a consequence we obtain

Corollary 2.2. *Suppose that (2.2) holds for a solution u of (1.2), then*

$$\lceil \frac{\theta_u(n+1)}{\pi} \rceil = \begin{cases} \lceil \frac{\theta_u(n)}{\pi} \rceil + 1 & \text{if } n \text{ is a node,} \\ \lceil \frac{\theta_u(n)}{\pi} \rceil & \text{otherwise.} \end{cases} \tag{2.7}$$

In particular, we obtain

$$\#(u) = \lceil \frac{\theta_u(N)}{\pi} \rceil - \lfloor \frac{\theta_u(0)}{\pi} \rfloor - 1, \tag{2.8}$$

where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \,|\, n \le x\}$ is the usual floor function.

To find the analogous formula for the number of weighted nodes of a Wronskian we observe

$$W_n(u_0, u_1) = -a(n)\rho_{u_0}(n)\rho_{u_1}(n)\sin(\Delta_{u_0,u_1}(n)), \tag{2.9}$$

where

$$\Delta_{u_0,u_1}(n) = \theta_{u_1}(n) - \theta_{u_0}(n). \tag{2.10}$$

Furthermore, note

$$W_{n+1}(u_0, u_1) - W_n(u_0, u_1) = (b_0(n+1) - b_1(n+1))u_0(n+1)u_1(n+1). \tag{2.11}$$

As a straightforward consequence of Lemma 2.1 we obtain

Lemma 2.3. *Fix some n and let $\theta_j(n) = k_j\pi + \gamma_j$ with $\gamma_j \in (0, \pi]$ and $\theta_j(n+1) = k_j\pi + \Gamma_j$ with $\Gamma_j \in (0, 2\pi]$ for $j = 0, 1$. Then we have*

$$\Delta_{u_0,u_1}(n) = (k_1-k_0)\pi+\gamma_1-\gamma_0 \quad \text{and} \quad \Delta_{u_0,u_1}(n+1) = (k_1-k_0)\pi+\Gamma_1-\Gamma_0, \tag{2.12}$$

where

(1): *either u_0 and u_1 have a node at n or both do not have a node at n, then*

$$\gamma_1 - \gamma_0 \in (-\frac{\pi}{2}, \frac{\pi}{2}) \quad \text{and} \quad \Gamma_1 - \Gamma_0 \in (-\pi, \pi). \tag{2.13}$$

(2): *u_1 has no node at n, but u_0 has a node at n, then*

$$\gamma_1 - \gamma_0 \in (-\pi, 0) \quad \text{and} \quad \Gamma_1 - \Gamma_0 \in (-2\pi, 0). \tag{2.14}$$

(3): *u_1 has a node at n, but u_0 has no node at n, then*

$$\gamma_1 - \gamma_0 \in (0, \pi) \quad \text{and} \quad \Gamma_1 - \Gamma_0 \in (0, 2\pi). \tag{2.15}$$

Now we are able to show

Lemma 2.4. *Fix some n. Then, if $b_0(n+1) \ge b_1(n+1)$, we have*

$$\lceil \Delta_{u_0,u_1}(n)/\pi \rceil \le \lceil \Delta_{u_0,u_1}(n+1)/\pi \rceil \le \lceil \Delta_{u_0,u_1}(n)/\pi \rceil + 1 \tag{2.16}$$

and if $b_0(n+1) \le b_1(n+1)$, we have

$$\lceil \Delta_{u_0,u_1}(n)/\pi \rceil - 1 \le \lceil \Delta_{u_0,u_1}(n+1)/\pi \rceil \le \lceil \Delta_{u_0,u_1}(n)/\pi \rceil. \tag{2.17}$$

Proof. We will use the notation from Lemma 2.3 where we assume $k_0 = k_1 = 0$ without loss of generality. In particular, Lemma 2.3 implies

$$\lceil \Delta_{u_0,u_1}(n)/\pi \rceil - 1 \le \lceil \Delta_{u_0,u_1}(n+1)/\pi \rceil \le \lceil \Delta_{u_0,u_1}(n)/\pi \rceil + 1.$$

Hence, to show (2.16) there are two cases to exclude. Namely, (i) $\Delta_{u_0,u_1}(n) \in (0, \frac{\pi}{2})$, $\Delta_{u_0,u_1}(n+1) \in (-\pi, 0]$ (from case (1)) and (ii) $\Delta_{u_0,u_1}(n) \in (-\pi, 0)$, $\Delta_{u_0,u_1}(n+1) \in (-2\pi, -\pi]$ (from case (2)). But in case (i) we obtain a contradiction from (2.11):

$$\underbrace{W_{n+1}(u_0, u_1)}_{\le 0} = \underbrace{W_n(u_0, u_1)}_{>0} + \underbrace{(b_0(n+1) - b_1(n+1))}_{\ge 0} \underbrace{u_0(n+1)u_1(n+1)}_{\ge 0}.$$

Similarly, in case (ii) equation (2.11) implies

$$\underbrace{W_{n+1}(u_0, u_1)}_{\ge 0} = \underbrace{W_n(u_0, u_1)}_{<0} + \underbrace{(b_0(n+1) - b_1(n+1))}_{\ge 0} \underbrace{u_0(n+1)u_1(n+1)}_{\le 0}.$$

Equation (2.17) can be established in a similar fashion. □

Lemma 2.5. *Let $n \in \mathbb{Z}$, then*

(1): $W_n(u_0,u_1) = W_{n+1}(u_0,u_1) = 0$ *or* $W_n(u_0,u_1)W_{n+1}(u_0,u_1) > 0$ *implies*

$$\lceil \frac{\Delta_{u_0,u_1}(n+1)}{\pi} \rceil = \lceil \frac{\Delta_{u_0,u_1}(n)}{\pi} \rceil. \tag{2.18}$$

(2): $W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0$ *implies*

$$\lceil \frac{\Delta_{u_0,u_1}(n+1)}{\pi} \rceil = \begin{cases} \lceil \frac{\Delta_{u_0,u_1}(n)}{\pi} \rceil + 1, & \text{if } b_0(n+1) > b_1(n+1), \\ \lceil \frac{\Delta_{u_0,u_1}(n)}{\pi} \rceil - 1, & \text{if } b_0(n+1) < b_1(n+1). \end{cases} \tag{2.19}$$

(3): $W_n(u_0,u_1) = 0$ *and* $W_{n+1}(u_0,u_1) \neq 0$ *implies*

$$\lceil \frac{\Delta_{u_0,u_1}(n+1)}{\pi} \rceil = \begin{cases} \lceil \frac{\Delta_{u_0,u_1}(n)}{\pi} \rceil + 1, & \text{if } b_0(n+1) > b_1(n+1), \\ \lceil \frac{\Delta_{u_0,u_1}(n)}{\pi} \rceil, & \text{if } b_0(n+1) < b_1(n+1). \end{cases} \tag{2.20}$$

(4): $W_n(u_0,u_1) \neq 0$ *and* $W_{n+1}(u_0,u_1) = 0$ *implies*

$$\lceil \frac{\Delta_{u_0,u_1}(n+1)}{\pi} \rceil = \begin{cases} \lceil \frac{\Delta_{u_0,u_1}(n)}{\pi} \rceil, & \text{if } b_0(n+1) > b_1(n+1), \\ \lceil \frac{\Delta_{u_0,u_1}(n)}{\pi} \rceil - 1, & \text{if } b_0(n+1) < b_1(n+1). \end{cases} \tag{2.21}$$

Note that in the cases (2)–(4) we necessarily have $b_0(n+1) \neq b_1(n+1)$.

Proof. We will use the notation from Lemma 2.3 where we assume $k_0 = k_1 = 0$ without loss of generality. Moreover, interchanging u_0 and u_1 using $\Delta_{u_1,u_0} = -\Delta_{u_0,u_1}(n)$ and

$$\lceil -x \rceil = \begin{cases} -\lceil x \rceil & \text{if } x \in \mathbb{Z}, \\ -\lceil x \rceil + 1 & \text{otherwise}, \end{cases}$$

we see that it suffices to show one case $b_0(n+1) \geq b_1(n+1)$ or $b_0(n+1) \leq b_1(n+1)$.

Suppose $W_n(u_0,u_1) = W_{n+1}(u_0,u_1) = 0$ and $W_n(u_0,u_1)W_{n+1}(u_0,u_1) > 0$ do not hold, then by (2.11) we have

$$W_{n+1}(u_0,u_1) - W_n(u_0,u_1) = (b_0(n+1) - b_1(n+1))u_0(n+1)u_1(n+1) \neq 0$$

and hence $b_0(n+1) \neq b_1(n+1)$.

(1) and (2). Suppose $W_n(u_0,u_1) = W_{n+1}(u_0,u_1) = 0$, then by (2.9) we infer

$$\sin(\Delta_{u_0,u_1}(n)) = \sin(\gamma_1 - \gamma_0) = 0, \quad \sin(\Delta_{u_0,u_1}(n+1)) = \sin(\Gamma_1 - \Gamma_0) = 0,$$

where γ_0, $\gamma_1 \in (0,\pi]$. Thus $\gamma_0 = \gamma_1$ and we have case (1) of Lemma 2.3 which implies $\Gamma_1 - \Gamma_0 \in (-\pi,\pi)$ and we conclude $\Gamma_1 - \Gamma_0 = 0$. In summary, $\Delta_{u_0,u_1}(n) = \Delta_{u_0,u_1}(n+1) = 0$ as claimed.

Next suppose $W_n(u_0,u_1)W_{n+1}(u_0,u_1) \neq 0$, then by (2.9) the sign of the Wronskian at n equals the sign of $\sin(\Delta_{u_0,u_1}(n))$ and hence (2.16) respectively (2.17) finish the proof of case (1) and (2).

(3). By (2.9) we conclude $\Delta_{u_0,u_1}(n) = \gamma_1 - \gamma_0 \equiv 0 \mod \pi$, where γ_0, $\gamma_1 \in (0,\pi]$ and thus $\gamma_1 - \gamma_0 = 0$. So we have case (1) of Lemma 2.3 and hence $\Delta_{u_0,u_1}(n+1) = \Gamma_1 - \Gamma_0 \in (-\pi,\pi)$. That is,

$$\lceil \Delta_{u_0,u_1}(n)/\pi \rceil \leq \lceil \Delta_{u_0,u_1}(n+1)/\pi \rceil \leq \lceil \Delta_{u_0,u_1}(n)/\pi \rceil + 1$$

and (2.17) finishes the proof of case (3) for $b_0(n+1) < b_1(n+1)$.

(4). By (2.9) we have $\Delta_{u_0,u_1}(n+1) = \Gamma_1 - \Gamma_0 \equiv 0 \mod \pi$ and Lemma 2.3 leaves us with the following possibilities

$$\begin{array}{llll} \text{(a)} & \Delta_{u_0,u_1}(n) \in (-\frac{\pi}{2}, \frac{\pi}{2}) & \text{and} & \Delta_{u_0,u_1}(n+1) = 0, \\ \text{(b)} & \Delta_{u_0,u_1}(n) \in (-\pi, 0) & \text{and} & \Delta_{u_0,u_1}(n+1) = -\pi, \\ \text{(c)} & \Delta_{u_0,u_1}(n) \in (0, \pi) & \text{and} & \Delta_{u_0,u_1}(n+1) = \pi. \end{array}$$

and (2.16) shows (4) if $b_0(n+1) > b_1(n+1)$. □

As a consequence we obtain the desired formula

$$\#(W(u_0, u_1)) = \lceil \Delta_{u_0,u_1}(N)/\pi \rceil - \lfloor \Delta_{u_0,u_1}(0)/\pi \rfloor - 1. \tag{2.22}$$

3. Proof of the main theorem

Our strategy will be to interpolate between H_0 and H_1 using $H_\varepsilon = (1-\varepsilon)H_0 + \varepsilon H_1$, that is, $a_\varepsilon(n) = a(n)$ and $b_\varepsilon(n) = (1-\varepsilon)b_0(n) + \varepsilon b_1(n)$. If u_ε is a solution of the difference equation corresponding to H_ε, then the corresponding Prüfer angles satisfy

$$\dot\theta_\varepsilon(n) = -\frac{W_n(u_\varepsilon, \dot u_\varepsilon)}{a(n)\rho_\varepsilon^2(n)}, \tag{3.1}$$

where the dot denotes a derivative with respect to ε.

Lemma 3.1. *We have*

$$W_n(s_{\varepsilon,\pm}(z), \dot s_{\varepsilon,\pm}(z)) = \begin{cases} -\sum_{m=n+1}^N (b_0(m) - b_1(m)) s_{\varepsilon,+}(z,m)^2, \\ \sum_{m=1}^n (b_0(m) - b_1(m)) s_{\varepsilon,-}(z,m)^2. \end{cases} \tag{3.2}$$

Proof. Summing (2.11) we obtain

$$W_n(s_{\varepsilon,\pm}(z), s_{\tilde\varepsilon,\pm}(z)) = (\tilde\varepsilon - \varepsilon) \begin{cases} -\sum_{m=n+1}^N (b_0(m) - b_1(m)) s_{\varepsilon,+}(z,m) s_{\tilde\varepsilon,+}(z,m), \\ \sum_{m=1}^n (b_0(m) - b_1(m)) s_{\varepsilon,-}(z,m) s_{\tilde\varepsilon,-}(z,m). \end{cases}$$

Now use this to evaluate the limit

$$\lim_{\tilde\varepsilon \to \varepsilon} W_n\Big(s_{\varepsilon,\pm}(z), \frac{s_{\varepsilon,\pm}(z) - s_{\tilde\varepsilon,\pm}(z)}{\varepsilon - \tilde\varepsilon}\Big).$$

□

Denoting the Prüfer angles of $s_{\varepsilon,\pm}(\lambda, n)$ by $\theta_{\varepsilon,\pm}(\lambda, n)$, this result implies for $b_0 - b_1 \ge 0$,

$$\begin{aligned} \dot\theta_{\varepsilon,+}(\lambda, n) &= \frac{\sum_{m=n+1}^N (b_0(m) - b_1(m)) s_{\varepsilon,+}(z,m)^2}{a(n)\rho_{\varepsilon,+}(\lambda,n)^2} \le 0, \\ \dot\theta_{\varepsilon,-}(\lambda, n) &= -\frac{\sum_{m=1}^n (b_0(m) - b_1(m)) s_{\varepsilon,-}(z,m)^2}{a(n)\rho_{\varepsilon,-}(\lambda,n)^2} \ge 0. \end{aligned} \tag{3.3}$$

Furthermore, we have the following result from classical perturbation theory. We add a simple direct proof for convenience of the reader.

Lemma 3.2. *Suppose $b_0 - b_1 \ge 0$ (resp. $b_0 - b_1 \le 0$). Then the eigenvalues of H_ε are analytic functions with respect to ε and they are decreasing (resp. increasing).*

Proof. First of all the Prüfer angles $\theta_{\varepsilon,\pm}(\lambda, n)$ are analytic with respect to ε since $s_{\pm,\varepsilon}(\lambda, n)$ is a polynomial with respect to ε. Moreover, $\lambda \in \sigma(H_\varepsilon)$ is equivalent to $\theta_{\varepsilon,+}(\lambda, 0) \equiv 0 \mod \pi$ (resp. $\theta_{\varepsilon,-}(\lambda, N) \equiv 0 \mod \pi$) and monotonicity follows from (3.3). □

In particular, this implies that $P(H_\varepsilon) = \#\{E \in \sigma(H_\varepsilon) | E < \lambda\}$ is continuous from below (resp. above) in ε if $b_0 - b_1 \geq 0$ (resp. $b_0 - b_1 \leq 0$).

Now we are ready for the

Proof of Theorem 1.2. It suffices to prove the result for $\#(s_{0,+}(\lambda_0), s_{1,-}(\lambda_1))$, where we can assume $\lambda_0 = \lambda_1 = 0$ without restriction and set $s_{\pm,\varepsilon}(n) = s_{\pm,\varepsilon}(0, n)$ for notational convenience. We split $b_0 - b_1$ according to

$$b_0 - b_1 = b_+ - b_-, \qquad b_+, b_- \geq 0,$$

and introduce the operator $H_- = H_0 - b_-$. Then H_- is a negative perturbation of H_0 and H_1 is a positive perturbation of H_-.

Furthermore, define H_ε by

$$H_\varepsilon = \begin{cases} H_0 + 2\varepsilon(H_- - H_0), & \varepsilon \in [0, 1/2], \\ H_- + 2(\varepsilon - 1/2)(H_1 - H_-), & \varepsilon \in [1/2, 1]. \end{cases}$$

Let us look at (using (2.22))

$$Q(\varepsilon) = \#(s_{0,+}, s_{\varepsilon,-}) = \lceil \Delta_\varepsilon(N)/\pi \rceil - \lfloor \Delta_\varepsilon(0)/\pi \rfloor - 1, \quad \Delta_\varepsilon(n) = \Delta_{s_{0,+}, s_{\varepsilon,-}}(n)$$

and consider $\varepsilon \in [0, 1/2]$. At the left boundary $\Delta_\varepsilon(0)$ remains constant whereas at the right boundary $\Delta_\varepsilon(N)$ is increasing by Lemma 3.1. Moreover, it hits a multiple of π whenever $0 \in \sigma(H_\varepsilon)$. So $Q(\varepsilon)$ is a piecewise constant function which is continuous from below and jumps by one whenever $0 \in \sigma(H_\varepsilon)$. By Lemma 3.2 the same is true for

$$P(\varepsilon) = \#\{E \in \sigma(H_\varepsilon) | E < \lambda\} - \#\{E \in \sigma(H_0) | E < \lambda\}$$

and since we have $Q(0) = P(0)$, we conclude $Q(\varepsilon) = P(\varepsilon)$ for all $\varepsilon \in [0, 1/2]$. To see the remaining case $\varepsilon = [1/2, 1]$, simply replace increasing by decreasing and continuous from below by continuous from above. □

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Faculty of Mathematics, Nordbergstrasse 15, 1090 Wien, Austria
E-mail address: Kerstin.Ammann@univie.ac.at
URL: http://www.mat.univie.ac.at/~kerstin/

Faculty of Mathematics, Nordbergstrasse 15, 1090 Wien, Austria, and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria
E-mail address: Gerald.Teschl@univie.ac.at
URL: http://www.mat.univie.ac.at/~gerald/