ON THE SPECTRUM OF TWO DIFFERENT FRACTIONAL OPERATORS

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Abstract. In this paper we deal with two nonlocal operators, that are both well known and widely studied in the literature in connection with elliptic problems of fractional type. Precisely, for a fixed $s \in (0,1)$ we consider the *integral* definition of the fractional Laplacian given by

$$(-\Delta)^s u(x) := \frac{c(n,s)}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \, dy\,, \quad x \in \mathbb{R}^n\,,$$

where $c(n,s)$ is a positive normalizing constant, and another fractional operator obtained via a *spectral* definition, that is

$$A_s u = \sum_{i \in \mathbb{N}} a_i \, \lambda_i^s \, e_i\,,$$

where $e_i\,, \lambda_i$ are the eigenfunctions and the eigenvalues of the Laplace operator $-\Delta$ in Ω with homogeneous Dirichlet boundary data, while a_i represents the projection of u on the direction e_i.

Aim of this paper is to compare these two operators, with particular reference to their spectrum, in order to emphasize their differences.

Contents

1. Introduction

Recently in the literature a great attention has been devoted to the study of nonlocal problems driven by fractional Laplace type operators, not only for a pure academic interest, but also for the various applications in different fields. Indeed, many different problems driven by the fractional Laplacian were considered in order to get existence, non-existence and regularity results and, also, to obtain qualitative properties of the solutions.

In particular, two notions of fractional operators were considered in the literature, namely the *integral* one (which reduces to the *classical fractional Laplacian*, see, for instance, [7, 8, 9, 10, 14, 15, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33] and references therein) and the

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spectral one (that is sometimes called the *regional, or local, fractional Laplacian*, see, e.g. [2, 4, 5, 6, 35] and references therein).

For any fixed $s \in (0,1)$ the fractional Laplace operator $(-\Delta)^s$ at the point x is defined by

$$(-\Delta)^s u(x) := \frac{c(n,s)}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy\,, \tag{1.1}$$

where $c(n,s)$ is a positive normalizing constant[1] depending only on n and s.

A different operator, which is sometimes denoted by A_s, is defined as the power of the Laplace operator $-\Delta$, obtained by using the spectral decomposition of the Laplacian. Namely, let Ω be a smooth bounded domain of $\mathbb{R}^n$, and let λ_k and e_k, $k \in \mathbb{N}$, be the eigenvalues and the corresponding eigenfunctions of the Laplacian operator $-\Delta$ in Ω with zero Dirichlet boundary data on $\partial\Omega$, that is

$$\begin{cases} -\Delta e_k = \lambda_k e_k & \text{in } \Omega \\ e_k = 0 & \text{on } \partial\Omega\,, \end{cases}$$

normalized in such a way that $\|e_k\|_{L^2(\Omega)} = 1$. For any $s \in (0,1)$ and any $u \in H^1_0(\Omega)$ with

$$u(x) = \sum_{i \in \mathbb{N}} a_i\, e_i(x)\,, \quad x \in \Omega\,,$$

one considers the operator

$$A_s u = \sum_{i \in \mathbb{N}} a_i\, \lambda_i^s\, e_i\,. \tag{1.2}$$

Aim of this paper is to compare the two previous definitions of fractional Laplace operators. First of all, we would like to note that these two fractional operators (i.e. the 'integral' one and the 'spectral' one) are *different* (in spite of some confusion that it is possible to find in some of the existent literature in which the two operators are somehow freely interchanged). Indeed, the spectral operator A_s depends on the domain Ω considered (since its eigenfunctions and eigenvalues depend on Ω), while the integral one $(-\Delta)^s$ evaluated at some point is independent on the domain in which the equation is set.[2]

Of course, by definition of A_s, it is easily seen that *the eigenvalues and the eigenfunctions of A_s are respectively λ_k^s and e_k, $k \in \mathbb{N}$*, that is the s-power of the eigenvalues of the Laplacian and the very same eigenfunctions of the Laplacian, respectively.

On the other hand, the spectrum of $(-\Delta)^s$ may be less explicit to describe. We refer to [28, Proposition 9 and Appendix A], [23, 24], [25, Proposition 5] and [30, Proposition 4] for the variational characterization of the eigenvalues and for some basic properties.

A natural question is whether or not there is a relation between the spectrum of A_s and $(-\Delta)^s$ and, of course, between the respective eigenfunctions. In the present paper, by using the classical regularity theory for the eigenfunctions of the Laplace operator $-\Delta$ and some recent regularity results for the fractional Laplace equation (see [22, 23, 24, 32]), we will show that the eigenfunctions of A_s and $(-\Delta)^s$ are different (for more details see Section 2). In particular, we will show that *the eigenfunctions of $(-\Delta)^s$ are, in general, no better than Hölder continuous up to the boundary*, differently from the eigenfunctions of A_s (i.e. of the classical Laplacian) that are smooth up to the boundary (if so is the domain).

[1]Different definitions of the fractional Laplacian consider different normalizing constants. The constant $c(n,s)$ chosen here is the one coming from the equivalence of the integral definition of $(-\Delta)^s$ and the one by Fourier transform (see, e.g., [7] and [10, (3.1)–(3.3) and (3.8)]) and it has the additional properties that $\lim_{s\to 1^-} (-\Delta)^s u = -\Delta u$ and $\lim_{s\to 0^+} (-\Delta)^s u = u$ (see [10, Proposition 4.4]).

[2]Also, the natural functional domains for the operators $(-\Delta)^s$ and A_s are different, but this is a minor distinction, since one could consider both the operators as acting on a very restricted class of functions for which they both make sense - e.g., $C^\infty_0(\Omega)$.

Furthermore, with respect to the eigenvalues of A_s and $(-\Delta)^s$, we will prove that *the first eigenvalue of* $(-\Delta)^s$ *is strictly less than the first one of* A_s. To this purpose we will use some extension results for the fractional operators A_s and $(-\Delta)^s$ (see [7, 34]).

Summarizing, the results given in this paper are the following:

Theorem 1. *The operators* $(-\Delta)^s$ *and* A_s *are not the same, since they have different eigenvalues and eigenfunctions. In particular:*

- *the first eigenvalues of* $(-\Delta)^s$ *is strictly less than the one of* A_s*;*
- *the eigenfunctions of* $(-\Delta)^s$ *are only Hölder continuous up to the boundary, differently from the ones of* A_s *that are as smooth up the boundary as the boundary allows.*

For further comments on similarities and differences between the operators A_s and $(-\Delta)^s$ for $s = 1/2$ see [13, Remark 0.4].

The paper is organized as follows. Section 2 is devoted to a comparison between the eigenfunctions of A_s and $(-\Delta)^s$. In Section 3 we deal with the spectrum of the two fractional operators we are considering. Section 4 is devoted to the extension of the operator A_s, while in Section 5 we discuss the relation between the first eigenvalues of A_s and $(-\Delta)^s$.

2. A comparison between the eigenfunctions of A_s and $(-\Delta)^s$

This section is devoted to some remarks about the eigenfunctions of the operators A_s and $(-\Delta)^s$. Precisely, we will consider the following eigenvalue problems in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, with Dirichlet homogeneous boundary data, driven, respectively, by A_s and $(-\Delta)^s$,

$$\begin{cases} A_s u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

and

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases} \tag{2.2}$$

Note that in (2.2) the boundary condition is given in $\mathbb{R}^n \setminus \Omega$ and not simply on $\partial\Omega$, due to the nonlocal character of the operator $(-\Delta)^s$.

In what follows we will denote by $e_{k,\,A_s}$ and $e_{k,\,s}$, $k \in \mathbb{N}$, the k–th eigenfunction of A_s and $(-\Delta)^s$, respectively.

Taking into account the definition of A_s, it is easily seen that its eigenfunctions $e_{k,\,A_s}$, $k \in \mathbb{N}$, are exactly the eigenfunctions of the Laplace operator $-\Delta$, i.e.

$$e_{k,\,A_s} = e_k\,.$$

Also, since $e_k \in C^\infty(\Omega) \cap C^m(\overline{\Omega})$ for any $m \in \mathbb{N}$ (see, for instance, [11]), then

$$e_{k,\,A_s} \in C^\infty(\Omega) \cap C^m(\overline{\Omega})\,. \tag{2.3}$$

Of course, constructing the eigenfunctions of $(-\Delta)^s$ is more difficult. In spite of this, we have some regularity results for them. Precisely, denoting by $\delta(x) = dist(x, \partial\Omega)$, $x \in \mathbb{R}^n$, by [22, Theorems 1.1 and 1.3] and [30, Proposition 4], we have that

$$e_{k,\,s}/\delta^s_{|\Omega} \in C^{0,\alpha}(\overline{\Omega}) \ \text{ for some } \ \alpha \in (0,1)\,,$$

namely $e_{k,\,s}/\delta^s_{|\Omega}$ has a continuous extension to $\overline{\Omega}$ which is $C^{0,\alpha}(\overline{\Omega})$. In particular, $e_{k,\,s}$ is Hölder continuous up to the boundary.

Aim of this section will be to show that *the Hölder regularity is optimal* for the eigenfunctions $e_{k,\,s}$ of $(-\Delta)^s$. To this purpose, first of all we recall the notion of Poisson kernel of fractional type and, then, we discuss the optimal regularity of the eigenfunctions $e_{k,\,s}$.

2.1. **Poisson kernel of fractional type.** Here we recall the notion of Poisson kernels of fractional type and their relation with the Dirichlet problem (see [20, Chapter I]).

First of all, for any $r > 0$, $x \in B_r$ (that is the ball of radius r centered at the origin) and $y \in \mathbb{R}^n \setminus B_r$, we define

$$P_r(x, y) := c_o(n, s) \left(\frac{r^2 - |x|^2}{|y|^2 - r^2}\right)^s \frac{1}{|x - y|^n},$$

with $c_o(n, s) > 0$. It is known (see [20, Appendix]) that, for any fixed $x \in B_r$ the function

$$I(x) := \int_{\mathbb{R}^n \setminus B_r} P_r(x, y)\, dy$$

is constant in x. Therefore, we normalize $c_o(n, s)$ in such a way that[3]

$$\int_{\mathbb{R}^n \setminus B_r} P_r(x, y)\, dy = 1. \tag{2.4}$$

The function P_r plays the role of a fractional Poisson kernel, namely if $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and

$$u_g(x) := \begin{cases} \displaystyle\int_{\mathbb{R}^n \setminus B_r} P_r(x, y)\, g(y)\, dy & \text{if } x \in B_r \\ g(x) & \text{if } x \in \mathbb{R}^n \setminus B_r, \end{cases} \tag{2.5}$$

then u_g is the unique solution of

$$\begin{cases} (-\Delta)^s u_g = 0 & \text{in } B_r \\ u_g = g & \text{outside } B_r. \end{cases} \tag{2.6}$$

For this, see [20, 33].

2.2. **Optimal regularity for the eigenfunctions of $(-\Delta)^s$.** In this subsection we prove that the $C^{0,\alpha}$-regularity of the eigenfunctions $e_{k,s}$ is optimal. Precisely, we show that, in general, *the eigenfunctions of $(-\Delta)^s$ need not to be Lipschitz continuous up to the boundary (i.e. the Hölder regularity is optimal).*

For concreteness, we consider the case

$$n > 2s, \tag{2.7}$$

the domain $\Omega := B_r$ and the first eigenfunction $e_{1,s}$ (normalized in such a way that $\|e_{1,s}\|_{L^2(\mathbb{R}^n)} = 1$ and $e_{1,s} \geqslant 0$ in $\mathbb{R}^n$, see [28, Proposition 9 and Appendix A]) of $(-\Delta)^s$ in B_r, i.e.

$$\begin{cases} (-\Delta)^s e_{1,s} = \lambda_{1,s} e_{1,s} & \text{in } B_r \\ e_{1,s} = 0 & \text{in } \mathbb{R}^n \setminus B_r. \end{cases} \tag{2.8}$$

We prove that

Proposition 2. *The function $e_{1,s}$ given in (2.8) is such that*

$$e_{1,s} \notin W^{1,\infty}(B_r).$$

Proof. The proof is by contradiction. We suppose that $e_{1,s} \in W^{1,\infty}(B_r)$ and so $e_{1,s} \in W^{1,\infty}(\mathbb{R}^n)$, that is

$$|e_{1,s}(x)| + |\nabla e_{1,s}(x)| \leqslant M, \quad x \in \mathbb{R}^n \tag{2.9}$$

for some $M > 0$.

From now on, we proceed by steps.

Step 1. *The function $e_{1,s}$ is spherically symmetric and radially decreasing in $\mathbb{R}^n$.*

[3]More explicitly, one can choose $c_o(n, s) := \Gamma(n/2)\sin(\pi s)/\pi^{(n/2)+1}$, see [20, pages 399–400].

Proof. For this, since $e_{1,s} \geqslant 0$ in $\mathbb{R}^n$, we consider its symmetric radially decreasing rearrangement $e^\star_{1,s}$ (see, e.g., [19, Chapter 2] for the basics of such a rearrangement). We observe that $e^\star_{1,s}$ vanishes outside B_r, since so does $e_{1,s}$. Moreover, we recall that the L^2-norm is preserved by the rearrangement, while the fractional Gagliardo seminorm decreases, see, e.g. [1, 3, 21]. Then, by this and since $\lambda_{1,s}$ is obtained by minimizing the fractional Gagliardo seminorm under constraint on the L^2-norm for functions that vanish outside B_r (see [28, Proposition 9]), we conclude that the minimum is attained by $e^\star_{1,s}$ (as well as by $e_{1,s}$).

Since $\lambda_{1,s}$ is a simple eigenvalue (see [28, Proposition 9 and Appendix A]), it follows that $e^\star_{1,s} = e_{1,s}$ and Step 1 is proved. □

Now, let Q be the fractional fundamental solution given by

$$Q(x) := c_1(n,s)|x|^{2s-n}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Here the constant $c_1(n,s) > 0$ is chosen in such a way that $(-\Delta)^s Q$ is the Dirac's delta δ_0 centered at the origin (see, e.g., [20, page 44] for the basic properties of fractional fundamental solutions).

We define

$$\tilde{v}(x) := \lambda_{1,s} Q * e_{1,s}(x) = \lambda_{1,s} c_1(n,s) \int_{\mathbb{R}^n} |y|^{2s-n} e_{1,s}(x-y)\,dy, \quad x \in \mathbb{R}^n \tag{2.10}$$

and

$$v(x) := e_{1,s}(x) - \tilde{v}(x), \quad x \in \mathbb{R}^n. \tag{2.11}$$

First of all, notice that $\tilde{v} \geqslant 0$ in $\mathbb{R}^n$, since $\lambda_{1,s} > 0$, $Q > 0$ and $e_{1,s} \geqslant 0$ in $\mathbb{R}^n$.

Step 2. *The function $\tilde{v}$ is spherically symmetric and radially decreasing in $\mathbb{R}^n$.*

Proof. Indeed, if $\mathcal{R}$ is a rotation, we use Step 1 and the substitution $\tilde{y} := \mathcal{R}y$ to obtain for any $x \in \mathbb{R}^n$

$$\begin{aligned}\tilde{v}(x) &= \lambda_{1,s} c_1(n,s) \int_{\mathbb{R}^n} |y|^{2s-n} e_{1,s}(x-y)\,dy = \\ &= \lambda_{1,s} c_1(n,s) \int_{\mathbb{R}^n} |y|^{2s-n} e_{1,s}\big(\mathcal{R}(x-y)\big)\,dy \\ &= \lambda_{1,s} c_1(n,s) \int_{\mathbb{R}^n} |\tilde{y}|^{2s-n} e_{1,s}(\mathcal{R}x - \tilde{y})\,d\tilde{y} = \tilde{v}(\mathcal{R}x),\end{aligned}$$

that shows the spherical symmetry of $\tilde{v}$.

As for the fact that $\tilde{v}$ is radially decreasing in $\mathbb{R}^n$, we take $\rho > 0$ and define

$$v_\star(\rho) := -\big(\lambda_{1,s} c_1(n,s)\big)^{-1} \tilde{v}(0,\dots,0,\rho) = -\int_{\mathbb{R}^n} |y|^{2s-n} e_{1,s}(-y', \rho - y_n)\,dy, \tag{2.12}$$

where we used the notation $y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ for the coordinates in $\mathbb{R}^n$.

The goal is to show that for any $\rho > 0$

$$v'_\star(\rho) \geqslant 0. \tag{2.13}$$

For this, first note that

$$\begin{aligned}v_\star(\rho) = &-\int_{\mathbb{R}^n \cap \{|\rho - y_n| \leqslant r\}} |y|^{2s-n} e_{1,s}(-y', \rho - y_n)\,dy \\ &\qquad - \int_{\mathbb{R}^n \cap \{|\rho - y_n| > r\}} |y|^{2s-n} e_{1,s}(-y', \rho - y_n)\,dy \\ = &-\int_{\mathbb{R}^n \cap \{|\rho - y_n| \leqslant r\}} |y|^{2s-n} e_{1,s}(-y', \rho - y_n)\,dy,\end{aligned}$$

since $\big\{|\rho - y_n| > r\big\} \subseteq \big\{|(-y', \rho - y_n)| > r\big\}$ and $e_{1,s}$ vanishes outside B_r. Also, since the function $e_{1,s}$ is spherically symmetric and radially decreasing in $\mathbb{R}^n$ by Step 1, we write $e_{1,s}(x) = -E(|x|)$ with $E' \geqslant 0$ in $\mathbb{R}^+$. Thus,

$$v_\star(\rho) = \int_{\mathbb{R}^n \cap \{|\rho - y_n| \leqslant r\}} |y|^{2s-n} E(|(-y', \rho - y_n)|)\, dy$$

and so

$$v_\star'(\rho) = \int_{\mathbb{R}^n \cap \{|\rho - y_n| \leqslant r\}} |y|^{2s-n} E'(|(-y', \rho - y_n)|) \frac{\rho - y_n}{|(-y', \rho - y_n)|}\, dy\,. \tag{2.14}$$

Now, let us consider the following change of variables

$$\begin{cases} \tilde y' := y' \\ \tilde y_n := 2\rho - y_n\,. \end{cases} \tag{2.15}$$

First of all, note that if $\tilde y_n - \rho \geqslant 0$, then $-\tilde y_n \leqslant 2\rho - \tilde y_n \leqslant \tilde y_n$, so that

$$(2\rho - \tilde y_n)^2 \leqslant \tilde y_n^2$$

and

$$\big|(\tilde y', 2\rho - \tilde y_n)\big| = \sqrt{|\tilde y'|^2 + (2\rho - \tilde y_n)^2} \leqslant \sqrt{|\tilde y'|^2 + \tilde y_n^2} = |\tilde y|\,.$$

As a consequence of this and recalling that $n \geqslant 2s$, we obtain that

$$\big|(\tilde y', 2\rho - \tilde y_n)\big|^{2s-n} \geqslant |\tilde y|^{2s-n}\,. \tag{2.16}$$

Therefore, by (2.15) and (2.16) we get

$$\begin{aligned}
&\int_{\mathbb{R}^n \cap \{0 \leqslant \rho - y_n \leqslant r\}} |y|^{2s-n} E'(|(-y', \rho - y_n)|) \frac{\rho - y_n}{|(-y', \rho - y_n)|}\, dy \\
= &\int_{\mathbb{R}^n \cap \{0 \leqslant \tilde y_n - \rho \leqslant r\}} \big|(\tilde y', 2\rho - \tilde y_n)\big|^{2s-n} E'(|(\tilde y', \rho - \tilde y_n)|) \frac{\tilde y_n - \rho}{(|(\tilde y', \rho - \tilde y_n)|}\, d\tilde y \\
\geqslant &\int_{\mathbb{R}^n \cap \{0 \leqslant \tilde y_n - \rho \leqslant r\}} |\tilde y|^{2s-n} E'(|(\tilde y', \rho - \tilde y_n)|) \frac{\tilde y_n - \rho}{(|(\tilde y', \rho - \tilde y_n)|}\, d\tilde y\,,
\end{aligned}$$

due to the fact that $E' \geqslant 0$ in $\mathbb{R}^+$.

Hence, recalling (2.14), we get

$$\begin{aligned}
v_\star'(\rho) &= \int_{\mathbb{R}^n \cap \{|\rho - y_n| \leqslant r\}} |y|^{2s-n} E'(|(-y', \rho - y_n)|) \frac{\rho - y_n}{|(-y', \rho - y_n)|}\, dy \\
&= \int_{\mathbb{R}^n \cap \{0 \leqslant \rho - y_n \leqslant r\}} |y|^{2s-n} E'(|(-y', \rho - y_n)|) \frac{\rho - y_n}{|(-y', \rho - y_n)|}\, dy \\
&\qquad + \int_{\mathbb{R}^n \cap \{0 \leqslant y_n - \rho \leqslant r\}} |y|^{2s-n} E'(|(-y', \rho - y_n)|) \frac{\rho - y_n}{|(-y', \rho - y_n)|}\, dy \\
&\geqslant \int_{\mathbb{R}^n \cap \{0 \leqslant \tilde y_n - \rho \leqslant r\}} |\tilde y|^{2s-n} E'(|(\tilde y', \rho - \tilde y_n)|) \frac{\tilde y_n - \rho}{(|(\tilde y', \rho - \tilde y_n)|}\, d\tilde y \\
&\qquad + \int_{\mathbb{R}^n \cap \{0 \leqslant \tilde y_n - \rho \leqslant r\}} |\tilde y|^{2s-n} E'(|(-\tilde y', \rho - \tilde y_n)|) \frac{\rho - \tilde y_n}{|(-\tilde y', \rho - \tilde y_n)|}\, dy \\
&= 0\,,
\end{aligned}$$

due to the fact that $|(\tilde y', \rho - \tilde y_n)| = |(-\tilde y', \rho - \tilde y_n)|$. Hence, (2.13) is proved.

Then, by (2.12), the spherical symmetry of $\tilde v$ and the fact that $\lambda_{1,s}$ and $c_1(n,s)$ are positive constants, we get that $\tilde v$ is radially decreasing in $\mathbb{R}^n$. This concludes the proof of Step 2. □

Next step will exploit assumption (2.9) taken for the argument by contradiction.

Step 3. *The function $\tilde{v}$ is such that*

$$\tilde{v} \in W^{1,\infty}(B_{2r})\,.$$

Proof. To check this, we observe that for any $x \in \mathbb{R}^n$

$$\tilde{v}(x) = \lambda_{1,s} c_1(n,s) \int_{\mathbb{R}^n} |y|^{2s-n} e_{1,s}(x-y)\,dy = \lambda_{1,s} c_1(n,s) \int_{B_r(x)} |y|^{2s-n} e_{1,s}(x-y)\,dy\,,$$

since $e_{1,s}$ vanishes outside B_r by (2.8). Here, $B_r(x)$ denotes the ball of radius r centered at x.

Now, we notice that if $x \in B_{2r}$ then $B_r(x) \subset B_{3r}$. As a consequence, recalling also (2.9), we obtain that for any $x \in B_{2r}$

$$\begin{aligned} |\tilde{v}(x)| + |\nabla \tilde{v}(x)| &\leqslant \lambda_{1,s} c_1(n,s) \int_{B_r(x)} |y|^{2s-n} \Big(|e_{1,s}(x-y)| + |\nabla e_{1,s}(x-y)| \Big)\,dy \\ &\leqslant \lambda_{1,s} c_1(n,s) M \int_{B_{3r}} |y|^{2s-n}\,dy, \end{aligned}$$

which is finite (being $s > 0$). Hence, Step 3 is established. □

Now we can conclude the proof of Proposition 2.

For this, note that, from (2.9) and Step 3, we get

$$v = e_{1,s} - \tilde{v} \in W^{1,\infty}(B_{2r})\,,$$

i.e. there exists $\tilde{M} > 0$ such that

$$|v(x) - v(y)| \leqslant \tilde{M} |x-y| \tag{2.17}$$

for any $x, y \in B_{2r}$.

Also, by (2.10) and the choice of Q

$$(-\Delta)^s \tilde{v} = \lambda_{1,s} e_{1,s} * (-\Delta)^s Q = \lambda_{1,s} e_{1,s} * \delta_0 = \lambda_{1,s} e_{1,s}$$

and so, by (2.8) and (2.11)

$$(-\Delta)^s v = (-\Delta)^s e_{1,s} - (-\Delta)^s \tilde{v} = \lambda_{1,s} e_{1,s} - \lambda_{1,s} e_{1,s} = 0$$

in B_r. Therefore, we can reconstruct v by its values outside B_r via the fractional Poisson kernel, that is, for any $x \in B_r$,

$$v(x) = \int_{\mathbb{R}^n \setminus B_r} P_r(x,y) v(y)\,dy\,, \tag{2.18}$$

for this see (2.5) and (2.6).

Since (2.11) holds true and $e_{1,s} = 0$ outside B_r, by (2.18) we deduce

$$\begin{aligned} v(x) &= \int_{\mathbb{R}^n \setminus B_r} P_r(x,y) v(y)\,dy \\ &= \int_{\mathbb{R}^n \setminus B_r} P_r(x,y) e_{1,s}(y)\,dy - \int_{\mathbb{R}^n \setminus B_r} P_r(x,y) \tilde{v}(y)\,dy \\ &= -\int_{\mathbb{R}^n \setminus B_r} P_r(x,y) \tilde{v}(y)\,dy. \end{aligned} \tag{2.19}$$

By (2.11), (2.17), (2.18) and (2.19) we get

$$
\begin{aligned}
&\Big| \int_{\mathbb{R}^n\setminus B_r} P_r(x,y)\tilde v(y)\,dy - \tilde v(0,\dots,0,r)\Big| \\
&\qquad\qquad = \left| -\int_{\mathbb{R}^n\setminus B_r} P_r(x,y) v(y)\,dy + v(0,\dots,0,r) - e_{1,s}(0,\dots,0,r)\right| \\
&\qquad\qquad = \left| \int_{\mathbb{R}^n\setminus B_r} P_r(x,y) v(y)\,dy - v(0,\dots,0,r)\right| \\
&\qquad\qquad = |v(0,\dots,0,r) - v(x)| \\
&\qquad\qquad \leqslant \tilde M |(0,\dots,0,r) - x|
\end{aligned}
\tag{2.20}
$$

for any $x \in B_r$.

If in (2.20) we take $x := (0,\dots,0,r-\varepsilon) \in B_r$ for a small $\varepsilon \in (0,r)$, recalling (2.4), we deduce that

$$
\begin{aligned}
\tilde M \varepsilon &= \tilde M |(0,\dots,0,r) - x| \\
&\geqslant \left| \int_{\mathbb{R}^n\setminus B_r} P_r(x,y)\tilde v(y)\,dy - \tilde v(0,\dots,0,r)\right| \\
&= \left| \int_{\mathbb{R}^n\setminus B_r} P_r(x,y)\Big(\tilde v(y) - \tilde v(0,\dots,0,r)\Big)\,dy\right| \\
&= c_o(n,s)\int_{|y|>r} \left(\frac{r^2-|x|^2}{|y|^2-r^2}\right)^s \frac{\tilde v(0,\dots,0,r)-\tilde v(y)}{|x-y|^n}\,dy \\
&= c_o(n,s)\big(r^2-|x|^2\big)^s \int_{|y|>r} \frac{\tilde v(0,\dots,0,r)-\tilde v(y)}{\big(|y|^2-r^2\big)^s |x-y|^n}\,dy \\
&\geqslant c_o(n,s)\, r^s\, (r-|x|)^s \int_{|y|>r} \frac{\tilde v(0,\dots,0,r)-\tilde v(y)}{\big(|y|^2-r^2\big)^s \big(|y'|^2+|y_n-r+\varepsilon|^2\big)^{n/2}}\,dy \\
&= \varepsilon^s \int_{|y|>r} f_\varepsilon(y)\,dy\,,
\end{aligned}
\tag{2.21}
$$

where

$$
f_\varepsilon(y) := c_o(n,s)\, r^s \frac{\tilde v(0,\dots,0,r)-\tilde v(y)}{\big(|y|^2-r^2\big)^s \big(|y'|^2+|y_n-r+\varepsilon|^2\big)^{n/2}}\,.
$$

We remark that $f_\varepsilon(y) \geqslant 0$ for any $|y|>r$, since

$$
\tilde v(0,\dots,r) \geqslant \tilde v(y) \text{ for any } |y|>r\,, \tag{2.22}
$$

thanks to Step 1. Moreover

$$
\lim_{\varepsilon\to 0^+} f_\varepsilon(y) = c_o(n,s)\, r^s \frac{\tilde v(0,\dots,0,r)-\tilde v(y)}{\big(|y|^2-r^2\big)^s \big(|y'|^2+|y_n-r|^2\big)^{n/2}}\,.
$$

So, we divide by ε^s the inequality obtained in (2.21) and we use Fatou's Lemma: we conclude that

$$
\begin{aligned}
0 = \liminf_{\varepsilon\to 0^+} \tilde M \varepsilon^{1-s} &\geqslant \liminf_{\varepsilon\to 0^+} \int_{|y|>r} f_\varepsilon(y)\,dy \\
&= c_o(n,s)\, r^s \int_{|y|>r} \frac{\tilde v(0,\dots,0,r)-\tilde v(y)}{\big(|y|^2-r^2\big)^s \big(|y'|^2+|y_n-r|^2\big)^{n/2}}\,dy\,.
\end{aligned}
$$

This and (2.22) yield that $\tilde v(y)$ is constantly equal to $\tilde v(0,\dots,0,r)$ for any $|y|>r$, so that, in particular, if $x^\star := (0,\dots,2r)$ we have that

$$
\partial_n \tilde v(x^\star) = 0\,. \tag{2.23}
$$

On the other hand, by (2.10),

$$\begin{aligned}
\frac{1}{\lambda_{1,s}c_1(n,s)}\partial_n\tilde{v}(x^\star) &= \frac{\partial}{\partial x_n}\int_{B_r}|x-z|^{2s-n}e_{1,s}(z)\,dz\Big|_{x=x^\star} \\
&= (2s-n)\int_{B_r}|x^\star-z|^{2s-n-2}(x_n^\star-z_n)e_{1,s}(z)\,dz \\
&= (2s-n)\int_{B_r}\big(|z'|^2+|2r-z_n|^2\big)^{(2s-n-2)/2}(2r-z_n)e_{1,s}(z)\,dz\,,
\end{aligned}$$

which is strictly negative, by (2.7). This is a contradiction with (2.23) and hence Proposition 2 is proved. □

3. The spectrum of A_s and $(-\Delta)^s$

In this section we focus on the spectrum of the operators A_s and $(-\Delta)^s$. In what follows, we will denote by

$$0<\lambda_1<\lambda_2\leqslant\ldots\leqslant\lambda_k\leqslant\ldots$$

the divergent sequence of the eigenvalues of the Laplace operator $-\Delta$ in Ω with Dirichlet homogeneous boundary data, while by λ_{k,A_s} the sequence of eigenvalues of problem (2.1) and, finally, by $\lambda_{k,\,s}$ the eigenvalues of (2.2).

By definition of A_s, it easily follows that the eigenvalues λ_{k,A_s} are exactly the s–power of the ones of the Laplacian, that is

$$\lambda_{k,A_s}=\lambda_k^s\,,\quad k\in\mathbb{N}\,. \tag{3.1}$$

As for $\lambda_{k,\,s}$, we refer to [28, Proposition 9 and Appendix A], [25, Proposition 5] and [30, Proposition 4] for their variational characterizations and some basic properties. In particular, we recall that for $k\in\mathbb{N}$

$$\lambda_{k,\,s}=\frac{c(n,s)}{2}\min_{u\in\mathbb{P}_{k,\,s}\setminus\{0\}}\frac{\displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy}{\displaystyle\int_\Omega|u(x)|^2\,dx}\,, \tag{3.2}$$

where

$$\mathbb{P}_{1,\,s}=X_0(\Omega):=\big\{u\in H^s(\mathbb{R}^n)\text{ s.t. }u=0\text{ a.e. in }\mathbb{R}^n\setminus\Omega\big\}$$

and

$$\mathbb{P}_{k,\,s}:=\big\{u\in X_0(\Omega)\text{ s.t. }\langle u,e_{j,s}\rangle_{X_0(\Omega)}=0\quad\forall j=1,\ldots,k-1\big\}\,,\quad k\geqslant 2 \tag{3.3}$$

with

$$\langle u,v\rangle_{X_0(\Omega)}=\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dx\,dy\,.$$

In what follows we will show that A_s and $(-\Delta)^s$ have different eigenvalues. Of course, at this purpose we will use properties (3.1) and (3.2), but the main ingredient will be the extension of the operator A_s, carried on in the forthcoming Section 4.

4. One-dimensional analysis

In this section we perform an ODE analysis related to the extension of the operator A_s, as it will be clear in the forthcoming Section 5.

This analysis is not new in itself (see also [7, Section 3.2] and [34, Section 3.1]): similar results were obtained, for instance, in [34] by using a conjugate equation and suitable special functions such as different kinds of Bessel and Hankel functions. Here, we use an elementary and self-contained approach.

Given $a\in(-1,1)$ in what follows we denote by $W^{1,2}_a(\mathbb{R}^+)$ the following Sobolev space

$$W^{1,2}_a(\mathbb{R}^+):=\left\{g\in W^{1,1}_{\rm loc}(\mathbb{R}^+)\ :\ \int_{\mathbb{R}^+}t^a|g(t)|^2\,dt<+\infty\ \text{ and }\ \int_{\mathbb{R}^+}t^a|\dot g(t)|^2\,dt<+\infty\right\}$$

endowed with the norm

$$\|g\|_{W^{1,2}_a(\mathbb{R}^+)} := \left(\int_{\mathbb{R}^+} t^a|g(t)|^2\,dt + \int_{\mathbb{R}^+} t^a|\dot g(t)|^2\,dt\right)^{1/2}. \tag{4.1}$$

Here, as usual, we used the notation $\mathbb{R}^+ := (0,+\infty)$. We also denote by $W^{1,2}_{1,a}(\mathbb{R}^+)$ the closure, with respect to the norm in (4.1), of the set of all functions $g \in C^\infty(\mathbb{R}^+) \cap C^0(\overline{\mathbb{R}^+})$ with bounded support and $g(0) = 1$.

It is useful to point out that $W^{1,2}_a(\mathbb{R}^+)$ and $W^{1,2}_{1,a}(\mathbb{R}^+)$ are contained in a classical Sobolev space. Precisely, denoting by $W^{1,p}((0,\kappa))$, $p \geqslant 1$ and $\kappa > 0$, the classical Sobolev space endowed with the norm

$$\|g\|_{W^{1,p}((0,\kappa))} = \left(\|g\|^p_{L^p((0,\kappa))} + \|\dot g\|^p_{L^p((0,\kappa))}\right)^{1/p},$$

the following result holds true:

Lemma 3. *Fix $a \in (-1,1)$ and $\kappa > 0$. Then,*

$$W^{1,2}_a(\mathbb{R}^+) \subseteq W^{1,p}((0,\kappa))$$

for any $p \in \big[1, a^*\big)$, with

$$a^* = \begin{cases} 2/(a+1) & \text{if } a \in (0,1) \\ 2 & \text{if } a \in (-1,0]. \end{cases}$$

Moreover, there exists $C_\kappa > 0$ such that

$$\|g\|_{W^{1,p}((0,\kappa))} \leqslant C_\kappa \|g\|_{W^{1,2}_a(\mathbb{R}^+)}$$

for any $g \in W^{1,2}_a(\mathbb{R}^+)$.

Proof. Let $a \in (-1,1)$, $g \in W^{1,2}_a(\mathbb{R}^+)$ and $p \in \big[1, a^*\big)$. We use the Hölder Inequality with exponents $2/(2-p)$ and $2/p$ (note that both these exponents are greater than 1, thanks to the choice of p) to see that

$$\begin{aligned} \|g\|^p_{L^p((0,\kappa))} &= \int_0^\kappa t^{-pa/2} t^{pa/2} |g(t)|^p\,dt \\ &\leqslant \left[\int_0^\kappa t^{-pa/(2-p)}\,dt\right]^{(2-p)/2} \left[\int_0^\kappa t^a |g(t)|^2\,dt\right]^{p/2} \\ &= \left[\frac{2-p}{2-p(1+a)} \kappa^{(2-p(1+a))/(2-p)}\right]^{(2-p)/2} \left[\int_{\mathbb{R}^+} t^a|g(t)|^2\,dt\right]^{p/2} < +\infty, \end{aligned}$$

again since $p < a^*$.

A similar inequality holds if we replace g with $\dot g$, and this proves the desired result. □

Hence, as a consequence of Lemma 3, the functions in $W^{1,2}_{1,a}(\mathbb{R}^+)$ are uniformly continuous in any interval, by the standard Sobolev embedding, and have a distributional derivative which is well-defined a.e.

Now, for any $\lambda > 0$ and any $g \in W^{1,2}_{1,a}(\mathbb{R}^+)$, we consider the functional

$$G_\lambda(g) := \int_{\mathbb{R}^+} t^a \Big(|g(t)|^2\,dt + \lambda|\dot g(t)|^2\Big)\,dt.$$

The minimization problem of G_λ is described in detail by the following result:

Theorem 4. *There exists a unique $g_\lambda \in W^{1,2}_{1,a}(\mathbb{R}^+)$ such that*

$$\mathrm{m}_\lambda := \inf_{g \in W^{1,2}_{1,a}(\mathbb{R}^+)} G_\lambda(g) = G_\lambda(g_\lambda), \tag{4.2}$$

that is, the above infimum is attained.

Moreover, $g_\lambda \in C^\infty(\mathbb{R}^+) \cap C^0(\overline{\mathbb{R}^+})$ *and it satisfies*

$$\begin{cases} \ddot{g}_\lambda(t) + \dfrac{a}{t}\,\dot{g}_\lambda(t) - \lambda g_\lambda(t) = 0 & \text{for any } t \in \mathbb{R}^+ \\ \\ g_\lambda(0) = 1, \end{cases} \tag{4.3}$$

and

$$\lim_{t\to 0^+} t^a \dot{g}_\lambda(t) = -\mathrm{m}_1 \lambda^{(a+1)/2}. \tag{4.4}$$

Finally,

$$g_\lambda(t) \in [0,1] \text{ and } \dot{g}_\lambda(t) \leqslant 0 \text{ for all } t \in \mathbb{R}^+ \text{ and } \lim_{t\to+\infty} g_\lambda(t) = 0. \tag{4.5}$$

Proof. By plugging a smooth and compactly supported function in G_λ, we see that $\mathrm{m}_\lambda \in [0, +\infty)$, so we can take a minimizing sequence g_j in $W^{1,2}_{1,a}(\mathbb{R}^+)$, that is a sequence g_j such that

$$G_\lambda(g_j) \to \mathrm{m}_\lambda$$

as $j \to +\infty$.

In particular, $G_\lambda(g_j) \leqslant \mathrm{m}_\lambda + 1$. As a consequence of this, $\|g_j\|_{W^{1,2}_a(\mathbb{R}^+)}$ is bounded uniformly in j. Hence, there exists $g_\lambda \in W^{1,2}_{1,a}(\mathbb{R}^+)$ such that $g_j \to g_\lambda$ weakly in $W^{1,2}_{1,a}(\mathbb{R}^+)$ as $j \to +\infty$. Also, for any $k \in \mathbb{N}$, $k \geqslant 2$, we have that

$$\tilde{C}_k \int_{1/k}^{k} \Big(|g_j(t)|^2 + |\dot{g}_j(t)|^2 \Big)\, dt \leqslant \int_{1/k}^{k} t^a \Big(|g_j(t)|^2 + |\dot{g}_j(t)|^2 \Big)\, dt \leqslant \mathrm{m}_\lambda + 1\,,$$

where $\tilde{C}_k = (1/k)^a$ if $a \geqslant 0$, while $\tilde{C}_k = k^a$ if $a < 0$. Namely, $\|g_j\|_{W^{1,2}([1/k,k])}$ is bounded uniformly in j.

Now, we perform a diagonal compactness argument over the index k. Namely, we take an increasing function $\phi_k : \mathbb{N} \to \mathbb{N}$ and we use it to extract subsequences. We have a subsequence $g_{\phi_2(j)}$ that converges a.e. in $[1/2, 2]$ to g_λ with $\dot{g}_{\phi_2(j)}$ converging to $\dot{g}_\lambda$ weakly in $L^2([1/2, 2])$ as $j \to +\infty$. Then, we take a further subsequence $g_{\phi_3(\phi_2(j))}$ that converges a.e. in $[1/3, 3]$ to g_λ with $\dot{g}_{\phi_3(\phi_2(j))}$ converging to $\dot{g}_\lambda$ weakly in $L^2([1/3, 3])$ as $j \to +\infty$. Iteratively, for any $k \in \mathbb{N}$, we get a subsequence $g_{\phi_k\circ\ldots\phi_2(j)}$ that converges a.e. in $[1/k, k]$ to g_λ with $\dot{g}_{\phi_k\circ\cdots\circ\phi_2(j)}$ converging to $\dot{g}_\lambda$ weakly in $L^2([1/k, k])$ as $j \to +\infty$.

Hence we look at the diagonal sequence $\overline{g}_j := g_{\phi_j\circ\cdots\circ\phi_2(j)}$. By construction $\overline{g}_j$ converges to g_λ a.e. in $\mathbb{R}^+$ as $j \to +\infty$ and therefore, by Fatou Lemma,

$$\liminf_{j\to+\infty} \int_{\mathbb{R}^+} t^a |\overline{g}_j(t)|^2\, dt \geqslant \int_{\mathbb{R}^+} t^a |g_\lambda(t)|^2\, dt. \tag{4.6}$$

On the other hand, by the weak convergence of $\dot{\overline{g}}_j$ to $\dot{g}_\lambda$ in $L^2([1/k, k])$ as $j \to +\infty$, we have that $\dot{g}_\lambda \in L^2([1/k, k])$ and so $\psi(t) := t^a \dot{g}_\lambda(t)$ is also in $L^2([1/k, k])$, which gives

$$\lim_{j\to+\infty} \int_{1/k}^{k} \dot{\overline{g}}_j(t)\psi(t)\, dt = \int_{1/k}^{k} \dot{g}_\lambda(t)\psi(t)\, dt\,,$$

that is

$$\lim_{j\to+\infty} \int_{1/k}^{k} t^a \dot{\overline{g}}_j(t)\dot{g}_\lambda(t)\, dt = \int_{1/k}^{k} t^a |\dot{g}_\lambda(t)|^2\, dt$$

for any $k \in \mathbb{N}$, $k \geqslant 2$. As a consequence of this, we obtain that

$$\begin{aligned} 0 &\leqslant \liminf_{j\to+\infty} \int_{1/k}^{k} t^a |\dot{\overline{g}}_j(t) - \dot{g}_\lambda(t)|^2\,dt \\ &= \liminf_{j\to+\infty} \Big(\int_{1/k}^{k} t^a |\dot{\overline{g}}_j(t)|^2\,dt + \int_{1/k}^{k} t^a |\dot{g}_\lambda(t)|^2\,dt - 2\int_{1/k}^{k} t^a \dot{\overline{g}}_j(t)\cdot \dot{g}_\lambda(t)\Big)\,dt \qquad (4.7)\\ &= \liminf_{j\to+\infty} \int_{1/k}^{k} t^a |\dot{\overline{g}}_j(t)|^2\,dt - \int_{1/k}^{k} t^a |\dot{g}_\lambda(t)|^2\,dt \end{aligned}$$

for any $k \in \mathbb{N}$, $k \geqslant 2$.

By (4.6), (4.7) and the positivity of λ we get

$$\begin{aligned} \mathrm{m}_\lambda &= \lim_{j\to+\infty} G_\lambda(g_j) \\ &= \lim_{j\to+\infty} \left(\int_{\mathbb{R}^+} t^a |\overline{g}_j(t)|^2\,dt + \lambda \int_{\mathbb{R}^+} t^a |\dot{\overline{g}}_j(t)|^2\,dt \right) \\ &\geqslant \liminf_{j\to+\infty} \left(\int_{\mathbb{R}^+} t^a |\overline{g}_j(t)|^2\,dt + \lambda \int_{1/k}^{k} t^a |\dot{\overline{g}}_j(t)|^2\,dt \right) \\ &\geqslant \int_{\mathbb{R}^+} t^a |g_\lambda(t)|^2\,dt + \lambda \int_{1/k}^{k} t^a |\dot{g}_\lambda(t)|^2\,dt \end{aligned}$$

for any $k \in \mathbb{N}$, $k \geqslant 2$. By taking $k \to +\infty$, we deduce that $\mathrm{m}_\lambda \geqslant G_\lambda(g_\lambda)$. This proves that the infimum in (4.2) is attained at g_λ. The uniqueness of the minimizer is due to the convexity of the functional G_λ. This completes the proof of (4.2).

Now, notice that, since $g_\lambda \in W^{1,2}_{1,a}(\mathbb{R}^+)$, then $g_\lambda(0) = 1$ and $g_\lambda \in W^{1,p}((0,\kappa))$ for any $p \in [1, a^*)$ and any $\kappa > 0$, by Lemma 3. Hence, it is uniformly continuous on $(0,\kappa)$ for any $\kappa > 0$, by the standard Sobolev embedding, and so it can be extended with continuity at 0, that is the function $g_\lambda \in C^0(\overline{\mathbb{R}^+})$.

Moreover, by taking standard perturbation of the functional G_λ at $g_\lambda + \varepsilon\phi$, with $\phi \in C^\infty_0(\mathbb{R}^+)$ and $\varepsilon \in \mathbb{R}$ small, one obtains that

$$\int_{\mathbb{R}^+} t^a \Big(g_\lambda(t)\phi(t) + \lambda \dot{g}_\lambda(t)\dot{\phi}(t) \Big)\,dt = 0\,. \qquad (4.8)$$

Hence, g_λ satisfies weakly an ODE and so $g_\lambda \in C^\infty(\mathbb{R}^+)$ by uniformly elliptic regularity theory (see for instance[4] [16, Theorem 8.10]). Moreover, integrating by parts in (4.8) it easily follows that g_λ solves problem (4.3).

Now, we prove (4.4). For this, it is convenient to reduce to the case $\lambda = 1$, by noticing that if $g^{(\lambda)}(t) := g(t/\sqrt{\lambda})$, we have that

$$G_\lambda(g^{(\lambda)}) = \lambda^{(a+1)/2} G_1(g)$$

and therefore

$$\mathrm{m}_\lambda = \lambda^{(a+1)/2}\mathrm{m}_1 \quad \text{and} \quad g_\lambda(t) = g_1(t/\sqrt{\lambda}). \qquad (4.9)$$

Let us fix $\phi \in C^\infty_0([-1,1])$ with $\phi(0) = 1$ and let

$$\gamma(t) := t^a \big(g_1(t)\phi(t) + \dot{g}_1(t)\dot{\phi}(t) \big).$$

By the Cauchy–Schwarz Inequality, we have that

$$\int_{\mathbb{R}^+} \gamma(t)\,dt \leqslant G_1(g_1)\,G_1(\phi) < +\infty,$$

[4]In further detail, g_λ satisfies [16, Equation (8.2)] with $n = 1$, $a^{ij} = a^{11} = \lambda t^a$, $b^i = b^1 = 0$, $c^i = c^1 = 0$, $d = t^a$ and this equation is uniformly elliptic in bounded domains separated from 0: so we can apply [16, Theorem 8.10] with $f = 0$ and obtain that $g_\lambda \in W^{k,2}(b_1, b_2)$ for any $b_2 > b_1 > 0$ and any $k \in \mathbb{N}$.

so that $\gamma \in L^1(\mathbb{R}^+)$. Therefore, by the absolute continuity of the Lebesgue integral, for any fixed $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $0 < t_1 < t_2 < \delta_\varepsilon$ then

$$\int_{t_1}^{t_2} \gamma(\tau)\, d\tau < \varepsilon.$$

As a consequence, the function

$$\Gamma(t) := \int_t^{+\infty} \gamma(\tau)\, d\tau$$

is uniformly continuous in $(0,1)$ and therefore it may be extended with continuity at 0 as follows

$$\Gamma(0) = \int_0^{+\infty} \gamma(\tau)\, d\tau = \int_0^{+\infty} \tau^a \Big(g_1(\tau)\phi(\tau) + \dot g_1(\tau)\dot\phi(\tau) \Big)\, d\tau. \tag{4.10}$$

By (4.3) with $\lambda = 1$ it is easy to see that for any $t \in \mathbb{R}^+$

$$t^a g_1(t) = \frac{d}{dt}\left(t^a \dot g_1(t) \right).$$

So, by this and recalling that $\phi(0) = 1$ and $\phi(t) = 0$ if $t \geqslant 1$, we get

$$\begin{aligned}
\Gamma(0) &= \int_0^1 \tau^a \Big(g_1(\tau)\phi(\tau) + \dot g_1(\tau)\dot\phi(\tau) \Big)\, d\tau \\
&= \int_0^1 \left[\frac{d}{d\tau}\Big(\tau^a \dot g_1(\tau) \Big)\phi(\tau) + \tau^a \dot g_1(\tau)\dot\phi(\tau) \right] d\tau \\
&= \lim_{t\to 0^+} \int_t^1 \frac{d}{d\tau}\Big(\tau^a \dot g_1(\tau)\phi(\tau) \Big)\, d\tau \\
&= -\lim_{t\to 0^+} t^a \dot g_1(t)\phi(t) \\
&= -\lim_{t\to 0^+} t^a \dot g_1(t).
\end{aligned} \tag{4.11}$$

Note that the computation carried on in (4.11) has also shown that the above limit exists.

Now, to compute explicitly such limit, we consider the perturbation

$$g_{1,\varepsilon} := (g_1 + \varepsilon\phi)/(1+\varepsilon)$$

with $\varepsilon \in \mathbb{R}$ small. First of all, notice that $g_{1,\varepsilon} = g_1 + \varepsilon\phi - \varepsilon g_1 + o(\varepsilon)$ and so

$$|g_{1,\varepsilon}|^2 = |g_1|^2 + 2\varepsilon g_1\phi - 2\varepsilon |g_1|^2 + o(\varepsilon),$$

and similarly if we replace $g_{1,\varepsilon}$ with $\dot g_{1,\varepsilon}$. It follows that

$$G_1(g_{1,\varepsilon}) = G_1(g_1) + 2\varepsilon \int_{\mathbb{R}^+} \tau^a \Big(g_1(\tau)\phi(\tau) - |g_1(\tau)|^2 + \dot g_1(\tau)\dot\phi(\tau) - |\dot g_1(t\tau)|^2 \Big)\, dt + o(\varepsilon).$$

Then, the minimality condition implies that

$$\int_{\mathbb{R}^+} \tau^a \Big(g_1(\tau)\phi(\tau) - |g_1(\tau)|^2 + \dot g_1(\tau)\dot\phi(\tau) - |\dot g_1(\tau)|^2 \Big)\, d\tau = 0.$$

Hence, by this, (4.10) and the definition of m_1 we deduce

$$\begin{aligned}
0 &= \int_{\mathbb{R}^+} \tau^a \Big(g_1(\tau)\phi(\tau) - |g_1(\tau)|^2 + \dot g_1(\tau)\dot\phi(\tau) - |\dot g_1(\tau)|^2 \Big)\, d\tau \\
&= \int_{\mathbb{R}^+} \tau^a \Big(g_1(\tau)\phi(\tau) + \dot g_1(\tau)\dot\phi(\tau) \Big)\, d\tau - \int_{\mathbb{R}^+} \tau^a \Big(|g_1(\tau)|^2 + |\dot g_1(\tau)|^2 \Big)\, d\tau \\
&= \Gamma(0) - \mathrm{m}_1.
\end{aligned}$$

This and (4.11) prove (4.4) for $\lambda = 1$. In general, recalling (4.9), we obtain

$$\lim_{t\to 0^+} t^a \dot g_\lambda(t) = \lambda^{-1/2} \lim_{t\to 0^+} t^a \dot g_1(t\lambda^{-1/2}) = \lambda^{(a+1)/2} \lim_{t\to 0^+} (t\lambda^{-1/2})^a \dot g_1(t\lambda^{-1/2}) = -\mathrm{m}_1 \lambda^{(a+1)/2},$$

thus establishing (4.4) for any $\lambda > 0$.

Now, let us prove (4.5) For this we first observe that $G_1(|g_1|) = G_1(g_1)$, which implies, by the uniqueness of the minimizer, that $g_1 = |g_1|$ and so $g_1 \geqslant 0$ in $\mathbb{R}^+$.

We start showing that

$$\dot{g}_1 \leqslant 0 \text{ in the whole of } \mathbb{R}^+. \tag{4.12}$$

By contradiction, if g_1 was increasing somewhere, there would exist $t_2 > t_1 \geqslant 0$ such that $0 \leqslant g_1(t_1) < g_1(t_2)$. Let $b := \big(g_1(t_1) + g_1(t_2)\big)/2 \in \big(g_1(t_1),\, g_1(t_2)\big)$. Notice that there exists $t_3 > t_2$ such that $g(t_3) = b$: otherwise, by continuity, we would have that $g(t) > b > 0$ for any $t > t_2$ and so, using that $a \in (-1, 1)$,

$$G_1(g_1) \geqslant \int_{t_2}^{+\infty} t^a |g_1(t)|^2\,dt \geqslant b^2 \int_{t_2}^{+\infty} t^a\,dt = +\infty,$$

which is against our contraction.

Having established the existence of the desired t_3, we use the Weierstrass Theorem to obtain $t_\star \in [t_1, t_3]$ in such a way that

$$g_1(t_\star) = \max_{t\in[t_1,t_3]} g_1(t)\,.$$

Note that, by definition of b,

$$g_1(t_\star) \geqslant g_1(t_2) > b > g_1(t_1)\,.$$

Hence, $t_\star \neq t_1$ and also $t_\star \neq t_3$, being $g_1(t_3) = b$. Thus, $t_\star$ is an interior maximum for g_1. Accordingly $\dot{g}_1(t_\star) = 0$ and $\ddot{g}_1(t_\star) \leqslant 0$. Thus, by (4.3),

$$0 = \ddot{g}_1(t_\star) + \frac{a}{t_\star}\,\dot{g}_1(t_\star) - g_1(t_\star) \leqslant 0 + 0 - b = -b < 0.$$

This is a contradiction and it proves (4.12).

A consequence of (4.12) is also that $g_1(t) \leqslant g_1(0) = 1$ for any $t \in \mathbb{R}^+$. Moreover, it implies that the limit

$$\ell := \lim_{t\to+\infty} g_1(t) \in [0, 1]$$

exists. Necessarily, it must be

$$\ell = 0\,.$$

Otherwise, if $\ell > 0$, it would follow that $g(t) \geqslant \ell/2$ for any $t \geqslant t_o$, for a suitable $t_o > 0$. This yields that (using also that $a \in (-1, 1)$)

$$G_1(g_1) \geqslant \int_{t_o}^{+\infty} t^a |g_1(t)|^2\,dt \geqslant (\ell/2)^2 \int_{t_o}^{+\infty} t^a\,dt = +\infty,$$

which is against our contraction. All these considerations imply (4.5) for $\lambda = 1$, and thus for any $\lambda > 0$, thanks to (4.9). □

5. A relation between the first eigenvalue of A_s and that of $(-\Delta)^s$

This section is devoted to the study of the relation between the first eigenvalue of A_s and of $(-\Delta)^s$, that is between $\lambda_{1,\,A_s}$ and $\lambda_{1,\,s}$. Precisely, in this framework our main result is the following:

Proposition 5. *The relation between the first eigenvalue of $(-\Delta)^s$ and the one of A_s is given by*

$$\lambda_{1,\,s} < \lambda_{1,\,A_s}\,.$$

Proof. Let us take $a := 2s - 1 \in (-1, 1)$ and for any $(x, t) \in \Omega \times \mathbb{R}^+$, set

$$E_1(x, t) := g_{\lambda_1}(t)e_1(x),$$

where the setting of Theorem 4 is in use, λ_1 is the first eigenvalue of the Laplacian $-\Delta$ and $e_1 = e_{1,\,A_s}$ is the first eigenfunction of the operator A_s (see Section 2).

Notice that E_1 may be thought as an extension of e_1 in the half-space $\mathbb{R}^n \times \mathbb{R}^+$ that vanishes in $(\mathbb{R}^n \setminus \Omega) \times \mathbb{R}^+$. However, we point out that E_1 does not verify $\operatorname{div}(\nabla E_1) = 0$ in the whole of $\mathbb{R}^n \times \mathbb{R}^+$.

Also, note that the function $E_1 \in C^\infty(\Omega \times \mathbb{R}^+) \cap C(\overline{\Omega \times \mathbb{R}^+})$, since $e_1 \in C^\infty(\Omega) \cap C^m(\overline{\Omega})$ for any $m \in \mathbb{N}$ (see formula (2.3)) and $g_{\lambda_1} \in C^\infty(\mathbb{R}^+) \cap C^0(\overline{\mathbb{R}^+})$ by Theorem 4. Also,

$$\lim_{t \to 0^+} t^a \partial_t E_1(x,t) = \lim_{t \to 0^+} t^a \dot{g}_{\lambda_1}(t) e_1(x) = -\mathrm{m}_1 \lambda_1^{(a+1)/2}\, e_1(x)\,,$$

thanks to (4.4).

Furthermore, since $G_{\lambda_1}(g_{\lambda_1})$ is finite by Theorem 4, we have that

$$L^1(\mathbb{R}^+) \ni t^a |g_{\lambda_1}(t)|^2 + t^a |\dot{g}_{\lambda_1}(t)|^2 \geqslant 2t^a |g_{\lambda_1}(t) \dot{g}_{\lambda_1}(t)|$$

and, therefore, there exists a diverging sequence of R for which

$$\lim_{R \to +\infty} R^a |g_{\lambda_1}(R) \dot{g}_{\lambda_1}(R)| = 0\,. \tag{5.1}$$

Now, note that, using[5] the definition of E_1, the fact that e_1 is the first eigenfunction of $-\Delta$ (for this see Section 2), for any $(x,t) \in \Omega \times \mathbb{R}^+$ we have

$$\begin{aligned}
t^a |\nabla E_1(x,t)|^2 &= \operatorname{div}\Big(t^a E_1(x,t) \nabla E_1(x,t)\Big) - a t^{a-1} E_1(x,t) \partial_t E_1(x,t) \\
&\quad - t^a E_1(x,t) \Delta E_1(x,t) \\
&= \operatorname{div}\Big(t^a E_1(x,t) \nabla E_1(x,t)\Big) - a t^{a-1} E_1(x,t) \dot{g}_{\lambda_1}(t) e_1(x) \\
&\quad - t^a E_1(x,t) g_{\lambda_1}(t) \Delta_x e_1(x) - t^a E_1(x,t) \ddot{g}_{\lambda_1}(t) e_1(x) \\
&= \operatorname{div}\Big(t^a E_1(x,t) \nabla E_1(x,t)\Big) - a t^{a-1} E_1(x,t) \dot{g}_{\lambda_1}(t) e_1(x) \\
&\quad + \lambda_1 t^a E_1(x,t) g_{\lambda_1}(t) e_1(x) - t^a E_1(x,t) \ddot{g}_{\lambda_1}(t) e_1(x) \\
&= \operatorname{div}\Big(t^a E_1(x,t) \nabla E_1(x,t)\Big) \\
&\quad + t^a E_1(x,t) e_1(x) \Big(- a t^{-1} \dot{g}_{\lambda_1}(t) + \lambda_1 g_{\lambda_1}(t) - \ddot{g}_{\lambda_1}(t) \Big) \\
&= \operatorname{div}\Big(t^a E_1(x,t) \nabla E_1(x,t)\Big)\,,
\end{aligned} \tag{5.2}$$

thanks to (4.3).

By (5.2) and the Divergence Theorem, we have that

$$\begin{aligned}
\iint_{\Omega \times \mathbb{R}^+} t^a |\nabla E_1(x,t)|^2 \, dx\, dt &= \lim_{R \to +\infty} \iint_{\Omega \times (0,R)} t^a |\nabla E_1(x,t)|^2 \, dx\, dt \\
&= \lim_{R \to +\infty} \iint_{\Omega \times (0,R)} \operatorname{div}\Big(t^a E_1(x,t) \nabla E_1(x,t)\Big)\, dx\, dt \\
&= \lim_{R \to +\infty} \int_\Omega \Big(t^a E_1(x,t) \partial_t E_1(x,t)\Big)\Big|_{t=R} - \Big(t^a E_1(x,t) \partial_t E_1(x,t)\Big)\Big|_{t=0}\, dx\,,
\end{aligned} \tag{5.3}$$

[5]We remark that here ∇ is the vector collecting all the derivatives, both in x and it t. Similarly, $\Delta = \Delta_x + \partial_t^2$.

since for any $t \in \mathbb{R}^+$, $E_1(\cdot, t) = 0$ on $\partial\Omega$, being $e_1 = 0$ outside Ω. Hence, by (5.3) and using again the definition of E_1 we deduce that

$$\begin{aligned}
&\iint_{\Omega\times\mathbb{R}^+} t^a |\nabla E_1(x,t)|^2\,dx\,dt \\
&\qquad = \lim_{R\to+\infty} \int_\Omega \Big[\Big(t^a E_1(x,t)\partial_t E_1(x,t)\Big)|_{t=R} - \Big(t^a E_1(x,t)\partial_t E_1(x,t)\Big)|_{t=0}\Big]\,dx \\
&\qquad = \lim_{R\to+\infty} \int_\Omega \Big[R^a E_1(x,R)\partial_t E_1(x,R) - \Big(t^a E_1(x,t)\partial_t E_1(x,t)\Big)|_{t=0}\Big]\,dx \\
&\qquad = \lim_{R\to+\infty} \int_\Omega \Big[R^a g_{\lambda_1}(R)\dot g_{\lambda_1}(R)|e_1(x)|^2 - \Big(t^a g_{\lambda_1}(t)\dot g_{\lambda_1}(t)|e_1(x)|^2\Big)|_{t=0}\Big]\,dx \\
&\qquad = \lim_{R\to+\infty} \int_\Omega \Big(R^a g_{\lambda_1}(R)\dot g_{\lambda_1}(R)|e_1(x)|^2 + \mathrm{m}_1\lambda_1^{(a+1)/2}|e_1(x)|^2\Big)\,dx \\
&\qquad = \lim_{R\to+\infty} \Big(R^a g_{\lambda_1}(R)\dot g_{\lambda_1}(R) + \mathrm{m}_1\lambda_1^{(a+1)/2}\Big)\,\|e_1\|^2_{L^2(\Omega)} \\
&\qquad = \mathrm{m}_1\lambda_1^{(a+1)/2} \\
&\qquad = \mathrm{m}_1\lambda_1^s\,,
\end{aligned}$$

thanks to (4.4), the fact that $g_\lambda(0) = 1$, (5.1) and the choice of a. As a consequence,

$$\begin{aligned}
\inf_{\substack{U\in C(\mathbb{R}^n\times\mathbb{R}^+)\\ \|U(\cdot,0)\|_{L^2(\Omega)}=1\\ U(\cdot,0)=0 \text{ outside } \Omega}} \iint_{\mathbb{R}^n\times\mathbb{R}^+} t^a|\nabla U(x,t)|^2\,dx\,dt &\leqslant \iint_{\mathbb{R}^n\times\mathbb{R}^+} t^a|\nabla E_1(x,t)|^2\,dx\,dt \\
&= \iint_{\Omega\times\mathbb{R}^+} t^a|\nabla E_1(x,t)|^2\,dx\,dt \\
&= \mathrm{m}_1\lambda_1^s\,,
\end{aligned} \tag{5.4}$$

since $E_1(\cdot, t) = e_1(\cdot)g_\lambda(t) = 0$ in $\mathbb{R}^n \setminus \Omega$ for any $t \in \mathbb{R}^+$.

Now, we use a result in [7] to relate the first term in (5.4) to $\lambda_{1,s}$ (which, roughly speaking, says the optimal U is realized by the so called a-harmonic extension of $u := U(\cdot, 0)$). Namely, by [7, formula (3.7) and its proof at page 1250] and [10, Proposition 3.4], we get

$$\begin{aligned}
\inf_{\substack{U\in C(\mathbb{R}^n\times\mathbb{R}^+)\\ \|U(\cdot,0)\|_{L^2(\Omega)}=1\\ U(\cdot,0)=0 \text{ outside } \Omega}} \iint_{\mathbb{R}^n\times\mathbb{R}^+} t^a|\nabla U(x,t)|^2\,dx\,dt &= \mathrm{m}_1 \inf_{\substack{u\in C(\mathbb{R}^n)\\ \|u\|_{L^2(\Omega)}=1\\ u=0 \text{ outside } \Omega}} \int_{\mathbb{R}^n} |\xi|^{2s}|\hat u(\xi)|^2\,d\xi \\
&= \mathrm{m}_1\frac{c(n,s)}{2} \inf_{\substack{u\in C(\mathbb{R}^n)\\ \|u\|_{L^2(\Omega)}=1\\ u=0 \text{ outside } \Omega}} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy, \\
&\geqslant \mathrm{m}_1\frac{c(n,s)}{2} \min_{\substack{u\in X_0(\Omega)\\ \|u\|_{L^2(\Omega)}=1}} \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&= \mathrm{m}_1\,\lambda_{1,s}\,,
\end{aligned}$$

thanks also to the variational characterization of $\lambda_{1,s}$ given in (3.2). Here $\hat u$ denotes the Fourier transform of u.

Thus,

$$\inf_{\substack{U\in C(\mathbb{R}^n\times\mathbb{R}^+)\\ \|U(\cdot,0)\|_{L^2(\Omega)}=1\\ U(\cdot,0)=0 \text{ outside } \Omega}} \iint_{\mathbb{R}^n\times\mathbb{R}^+} t^a|\nabla U(x,t)|^2\,dx\,dt \geqslant \mathrm{m}_1\lambda_{1,s}.$$

This and (5.4) give that

$$\lambda_{1,s} \leqslant \lambda_1^s.$$

We claim that the strict inequality occurs. If, by contradiction, equality holds here, then it does in (5.4), namely

$$E_1 \in \arg\min \left\{ \inf_{\substack{U \in C(\overline{\mathbb{R}^n \times \mathbb{R}^+}) \\ \|U(\cdot,0)\|_{L^2(\Omega)}=1 \\ U(\cdot,0)=0 \text{ outside } \Omega}} \iint_{\mathbb{R}^n \times \mathbb{R}^+} t^a |\nabla U(x,t)|^2 \, dx \, dt \right\}.$$

We remark that such minimizers are continuous up to $\overline{\mathbb{R}^n \times \mathbb{R}^+}$, and they solve the associated elliptic partial differential equation in $\mathbb{R}^n \times \mathbb{R}^+$, see [12]: in particular E_1 would solve an elliptic partial differential equation in $\mathbb{R}^n \times \mathbb{R}^+$ and it vanishes in a nontrivial open set (just take a ball B outside Ω and consider $B \times (1,2)$).

As a consequence of this and of the Unique Continuation Principle (see [18]), E_1 has to vanish identically in $\Omega \times \mathbb{R}^+$ and so, by taking $t \to 0^+$, we would have that $e_1(x) = 0$ for any $x \in \Omega$ (here we use also the fact that $g_{\lambda_1}(0) = 1$ by (4.3)). This is a contradiction and it establishes that $\lambda_{1,s} < \lambda_1^s = \lambda_{1,A_s}$. □

Our main result, i.e. Theorem 1, is now a direct consequences of Propositions 2 and 5.

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