ON COLISSIONS IN NONHOLONOMIC SYSTEMS

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Abstract. We consider nonholonomic systems with elastic collisions and propose a concept of generalised solutions to Lagrange-d'Alembert equations. In the light of this concept we describe dynamics of the collisions. Several applications have been investigated.

1. The Description of the Problem

Let us start from the following model example. There is a solid ball B of radius r and of mass m and let its centre of mass coincide with the geometric centre S. The moment of inertia relative to any axis passing trough the point S is equal to J.

Give an informal description of the problem. Being undergone with some potential forces the ball rolls on the floor and sometimes it collides with a vertical wall. After the collision it jumps aside the wall. The wall and the floor are rough: the ball can not slide on the floor and along the wall.

We wish to construct a theory of such a motion in the Lagrangian frame. Particularly, we wish to give sense to the term "elastic collision" in nonholonomic context.

In physical space introduce a Cartesian coordinate system $Oxyz$. Let (x_S, y_S, z_S) be the coordinates of the point S.

Suppose that the plane Oxy is a solid and rough floor and the plane Ozy is a solid and rough wall. For all the time $t \geq 0$ we have $z_S = r, \quad x_S \geq r$.

By $C \in B$ denote the contact point of the ball and the floor. The ball can not slide on the floor:

$$\overline{v}_C = \overline{v}_S + [\overline{\omega}, \overline{SC}] = 0, \tag{1.1}$$

here $\overline{v}_C$ is the velocity of the point C, and $\overline{\omega}$ is the angle velocity of the ball.

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When the ball reaches the wall (say by its point $G \in B$) then we also have

$$\overline{v}_G = \overline{v}_S + [\overline{\omega}, \overline{SG}] = 0, \qquad (1.2)$$

The configuration manifold of the system is $M = \mathbb{R}^2 \times SO(3)$, where $(x_S, y_S) \in \mathbb{R}^2$ and an element of $SO(3)$ determines the orientation of the ball. We use the Euler angles for the local coordinates in $SO(3)$.

Consequently the position of the ball is determined by the vector

$$x = (x_S, y_S, -\varphi, \theta, \psi)^T.$$

Why do we write φ with negative sign will be clear below.

The wall is a 4-dimensional manifold $N = \{x_S = r\} \subset M$.

Thus the general construction is as follows. We have a configuration space $M, \quad \dim M = m$ and a submanifold $N \subset M, \quad \dim N = m - 1$ (the wall). The manifolds M and N carry the distributions.

In the example under consideration the manifold M carries the nonholonomic constraint given by (1.1) and the manifold N carries the nonholonomic constraint given by (1.1) and (1.2).

Let

$$x = (x_1, \ldots, x_m)^T \in M$$

be local coordinates in M.

To determine the distribution at each point $x \in M$ introduce a linear operator

$$A(x) : T_x M \to \mathbb{R}^{m-l}, \quad \dim \operatorname{im} A(x) = m - l, \quad x \in M$$

and the mapping $x \mapsto A(x)$ is smooth. The subspaces $\ker A(x) \subseteq T_x M$ define an $l-$dimensional distribution in M.

To define an $s-$dimensional distribution in N introduce a linear operator

$$B(x) : T_x M \to \mathbb{R}^{m-s}, \quad \dim \operatorname{im} A(x) = m - s, \quad x \in N.$$

The distribution on N consists of the subspaces $\ker B(x) \subseteq T_x N$.

The operators A, B are not uniquely defined: the same distributions can be generated by the different operators A, B but we use them because they naturally arise in the applications.

Assume also that

$$\ker B(x) \subseteq \ker A(x), \quad \ker A(x) \not\subseteq T_x N \qquad (1.3)$$

for each $x \in N$.

The dynamics of the system is described by the smooth Lagrangian $L(x, \dot{x})$.

From the configuration manifold's geometry viewpoint the collisions of the rigid bodies was considered in [2].

One of the results of this article is as follows. The manifold M is endowed with the Riemann metric generated by the kinetic energy of the system. The evolution of the system is expressed by the function $x(t) \in M$. When the point with coordinates $x(t)$ collides with the submanifold N i.e. $x(\tau) \in N$ it then jumps aside obeying the law "the angle of incidence is equal to the angle of reflection". The authors obtained this law from a system with non-solid consraint (wall) by means of some natural limit process.

These results have been obtained in the absence of nonholonomic constraints. We generalize them to the nonholonomic case.

2. The Lagrange-d'Alembert Principle

2.1. **The Generalized Solutions to the Lagrange-d'Alembert Equation.** In the absence of one-sided constraint N the standard nonholonomic Lagrange-d'Alembert principle [1],[3] is read as follows.

Theorem 2.1. *The smooth function $x(t)\in M,\quad t\in[t_1,t_2]$ is a solution to the system of Lagrange-d'Alembert equations and the equations of constraints:*

$$\Big(\frac{\partial L}{\partial x}(x(t),\dot x(t))-\frac{d}{dt}\frac{\partial L}{\partial \dot x}(x(t),\dot x(t))\Big)u=0,\quad \dot x(t),u\in\ker A(x(t))$$

if and only if for any smooth function $\delta x(t)$,

$$\delta x(t_i)=0,\quad \delta x(t)\in\ker A(x(t)),\quad i=1,2,\quad t\in[t_1,t_2]$$

the following equations hold

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\int_{t_1}^{t_2}L\big(x(t)+\epsilon\delta x(t),\dot x(t)+\epsilon\delta\dot x(t)\big)dt=0,\quad \dot x(t)\in\ker A(x(t)).\tag{2.1}$$

The idea is as follows. If the motion $x(t)$ contains collisions it is point-wise differentiable: at the moment of collision its first derivative is not continuous and the second one does not exist.

Equations (2.1) demand the function $x(t)$ to be piecewise differentiable. These equations do not contain the second derivative of $x(t)$. Therefore formulas (2.1) fit to be a definition of generalized solution to the system of the Lagrange-d'Alembert equations.

Our main hypothesis is as follows. *The collision is described by such a generalised solution.*

Let us turn to the details.

Consider a solution $x(t)$ that collides the wall at the moment $\tau\in(t_1,t_2)$ i.e. $x(\tau)\in N$. Since we expect that the ball jumps aside the wall it is reasonable to assume that the points $x(t_i)=\tilde x_i,\quad i=1,2$ are situated from the same side of the manifold N.

We suppose that $x(t)\in C[t_1,t_2]$ and

$$x(t)=\begin{cases}x^-(t), & t\in[t_1,\tau],\\ x^+(t), & t\in[\tau,t_2]\end{cases}$$

and $x^-(t)\in C^2[t_1,\tau],\quad x^+(t)\in C^2[\tau,t_2]$.

The solution $x(t)$ must obey nonholonomic constraint that is

$$\dot x^{\pm}(t)\in\ker A(x^{\pm}(t)).$$

Fix such a solution $x(t)$ and consider various families of functions

$$x_\epsilon(t) = \begin{cases} x_\epsilon^-(t), & t \in [t_1, \tau_\epsilon], \\ x_\epsilon^+(t), & t \in [\tau_\epsilon, t_2] \end{cases}, \quad \epsilon \in (-a, a), \quad \tau_\epsilon \in (t_1, t_2)$$

such that $x(t) = x_0(t), \quad \tau = \tau_0$. For any $t \in [t_1, t_2]$ the function $x_\epsilon(t)$ is differentiable in $\epsilon, \quad \epsilon \in (-a, a)$, and

$$x_\epsilon(t), \frac{\partial x_\epsilon(t)}{\partial \epsilon} \in C([t_1, t_2] \times (-a, a)).$$

For any $\epsilon \in (-a, a)$ the functions $x_\epsilon^\pm(t)$ are smooth in t on the corresponding intervals. The function τ_ϵ is smooth in $(-a, a)$.

Moreover, for all admissible t, ϵ one has

$$x_\epsilon(t_i) = \tilde{x}_i, \quad i = 1, 2, \quad x_\epsilon(\tau_\epsilon) \in N$$

and

$$\frac{\partial x_\epsilon}{\partial \epsilon}(\tau_\epsilon) \in \ker B(x_\epsilon(\tau_\epsilon)), \quad \frac{\partial x_\epsilon}{\partial \epsilon}(t) \in \ker A(x_\epsilon(t)). \tag{2.2}$$

The collided solution $x(t)$ satisfies to the following Lagrange-d'Alembert principle:

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \Big(\int_{t_1}^{\tau_\epsilon} L(x_\epsilon^-(t), \dot{x}_\epsilon^-(t))dt + \int_{\tau_\epsilon}^{t_2} L(x_\epsilon^+(t), \dot{x}_\epsilon^+(t))dt \Big) = 0. \tag{2.3}$$

This equality holds for any family $\{x_\epsilon\}$ described above.

2.2. **Lemma from Vector Algebra.** The following lemma is mainly used in Section 3. But we put it here because it provides a good introduction to the geometry of the next Section.

Lemma 1. *Let $L = \mathbb{R}^m$ be a Euclidean vector space with scalar product given by its Gramian matrix G. And let B be the matrix of a linear operator (we denote operators and their matrices by the same letters)*

$$B : L \to \mathbb{R}^{m-s}, \quad \operatorname{rang} B = m - s.$$

Let

$$L = \ker B \oplus W, \quad W \perp \ker B$$

be the orthogonal decomposition of the space.

Then the square matrix of orthogonal projector $P : L \to L, \quad P(L) = W$ is

$$P = G^{-1}B^T\big(BG^{-1}B^T\big)^{-1}B. \tag{2.4}$$

If an operator

$$A : L \to \mathbb{R}^{m-l}$$

is such that $\ker B \subseteq \ker A$ then one has

$$AP = A. \tag{2.5}$$

Particularly, this implies that $P(\ker A) \subseteq \ker A$.

Proof. To obtain formula (2.4) fix an arbitrary vector $x \in L$ and introduce a linear function $f(\xi) = (Px)^T G\xi$. It is clear

$$\ker B \subseteq \ker f.$$

This implies that there is an operator $\lambda : \mathbb{R}^{m-s} \to \mathbb{R}$ such that $(Px)^T G = \lambda B$ and $Px = G^{-1}B^T\lambda^T$. It remains to find λ^T from the equation $B(x-Px) = 0$.

To obtain formula (2.5) note that there exists an operator

$$\gamma : \mathbb{R}^{m-s} \to \mathbb{R}^{m-l}$$

such that $A = \gamma B$. Consequently, formula (2.5) follows from (2.4).

The Lemma is proved.

2.3. **The Equations of Collision.** Introduce the notation

$$v^{\pm} = \dot{x}^{\pm}(\tau) \in \ker A(x(\tau)).$$

By integrating by parts the left side of formula (2.3) equals

$$\begin{aligned}
&\big(L(x(\tau), v^-) - L(x(\tau), v^+)\big)\frac{d\tau_\epsilon}{d\epsilon}\Big|_{\epsilon=0}\\
&\quad + \int_{t_1}^{\tau}\Big(\frac{\partial L}{\partial x}(x^-(t), \dot{x}^-(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x^-(t), \dot{x}^-(t))\Big)\frac{\partial x_\epsilon^-(t)}{\partial \epsilon}\Big|_{\epsilon=0}dt\\
&\quad + \int_{\tau}^{t_2}\Big(\frac{\partial L}{\partial x}(x^+(t), \dot{x}^+(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x^+(t), \dot{x}^+(t))\Big)\frac{\partial x_\epsilon^+(t)}{\partial \epsilon}\Big|_{\epsilon=0}dt\\
&\quad + \Big(\frac{\partial L}{\partial \dot{x}}(x(\tau), v^+) - \frac{\partial L}{\partial \dot{x}}(x(\tau), v^-)\Big)\frac{\partial x_\epsilon(\tau)}{\partial \epsilon}\Big|_{\epsilon=0}.
\end{aligned}$$

Recall that this formula holds for any family $\{x_\epsilon\}$ with described above properties. Thus by the standard argument we obtain

$$L(x(\tau), v^-) - L(x(\tau), v^+) = 0, \tag{2.6}$$

$$\int_{t_1}^{\tau}\Big(\frac{\partial L}{\partial x}(x(t), \dot{x}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x(t), \dot{x}(t))\Big)\frac{\partial x_\epsilon(t)}{\partial \epsilon}\Big|_{\epsilon=0}dt = 0, \tag{2.7}$$

$$\int_{\tau}^{t_2}\Big(\frac{\partial L}{\partial x}(x(t), \dot{x}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x(t), \dot{x}(t))\Big)\frac{\partial x_\epsilon(t)}{\partial \epsilon}\Big|_{\epsilon=0}dt = 0, \tag{2.8}$$

$$\Big(\frac{\partial L}{\partial \dot{x}}(x(\tau), v^+) - \frac{\partial L}{\partial \dot{x}}(x(\tau), v^-)\Big)\frac{\partial x_\epsilon(\tau)}{\partial \epsilon}\Big|_{\epsilon=0} = 0. \tag{2.9}$$

Equations (2.7), (2.8) express that the functions $x^{\pm}(t)$ satisfy to the Lagrange-d'Alembert equations. By the assumption they also satisfy the equations of constraint: $\dot{x}^{\pm}(t) \in \ker A(x^{\pm}(t))$. That is before and after the collision the system obeys to the Lagrange-d'Alembert equations and the equations of constraint.

Particularly, if the system is holonomic outside N (i.e. $A(x) = 0$) then in their domains the functions $x^{\pm}(t)$ satisfy the Lagrange equations

$$\frac{\partial L}{\partial x}(x^{\pm}(t), \dot{x}^{\pm}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x^{\pm}(t), \dot{x}^{\pm}(t)) = 0.$$

Equations (2.6) and (2.9) describe the behaviour of the system at the moment of collision. These equations are of main importance for us.

To proceed with our analysis put

$$L = T(x, \dot{x}, \dot{x}) - V(x).$$

The form

$$T(x, \xi, \eta) = \frac{1}{2}\xi^T G(x)\eta, \quad \xi = (\xi_1, \dots, \xi_m)^T, \quad \eta = (\eta_1, \dots, \eta_m)^T$$

is the kinetic energy of the system, the matrix $G(x) \equiv G^T(x)$ is positive definite. It defines a Riemann metric in M. The potential energy V is a smooth function in M.

Equation (2.6) takes the form

$$T(x(\tau), v^-, v^-) = T(x(\tau), v^+, v^+). \tag{2.10}$$

This means that the norm of the velocity after and before the collision is the same.

By (2.2) equation (2.9) is reduced to

$$T(x(\tau), v^+ - v^-, u) = 0, \quad u \in \ker B(x(\tau)). \tag{2.11}$$

From formula (2.11) it follows that the difference $v^+ - v^-$ is perpendicular to $\ker B(x(\tau))$.

It is reasonable to consider the following decomposition

$$T_{x(\tau)}M = \ker B(x(\tau)) \oplus W(x(\tau)),$$

here $W(x(\tau))$ is the orthogonal complement for $\ker B(x(\tau))$, and let

$$P : T_{x(\tau)}M \to W(x(\tau))$$

is the orthogonal projection.

Introduce notations $Pv = v_\perp$, $(I - P)v = v_\parallel$ and the norm $|\xi|^2 = T(x(\tau), \xi, \xi)$. Then write

$$v^\pm = v_\perp^\pm + v_\parallel^\pm.$$

Actually we deal with vectors $v \in \ker A(x(\tau))$ only. By Lemma 1 one has $P(\ker A(x(\tau)) \subseteq \ker A(x(\tau))$. Observe also that the inclusion $v_\perp^- \in T_{x(\tau)}N$ implies $v_\perp^- = 0$.

Since the difference $v^+ - v^- = (v_\parallel^+ - v_\parallel^-) + (v_\perp^+ - v_\perp^-)$ is perpendicular to $\ker B(x(\tau))$ we have $v_\parallel^+ = v_\parallel^-$ and by formula (2.10) this implies that $|v_\perp^+| = |v_\perp^-|$.

Consequently one has

$$v^+ = Qv_\perp^- + v_\parallel^-,$$

here

$$Q : F(x(\tau)) \to F(x(\tau)), \quad F(x(\tau)) = W(x(\tau)) \bigcap \ker A(x(\tau))$$

is an isometric mapping.

Formulas (1.3) imply

$$F(x(\tau)) \nsubseteq T_{x(\tau)}N. \tag{2.12}$$

Assume that Q is a linear operator that depends only on the point

$$x(\tau) \in N.$$

Then since the whole trajectory $x(t)$ is situated from the one side of the manifold N and by (2.12) it follows that

$$Qv_\perp^- = -v_\perp^-, \quad v^+ = -v_\perp^- + v_\parallel^-.$$

In terms of the matrix P the same is written as

$$v^+ = -Pv^- + (I-P)v^- = (I-2P)v^-. \tag{2.13}$$

We have got the law "the angle of incidence is equal to the angle of reflection". Particularly, the energy before the collision is equal to energy after the collision.

3. Applications

Introduce the following notations $J' = J + r^2 m, \quad \tilde{J} = J + r^2 m/2$.

3.1. **The Ball Rolls on the Floor and Meets the Wall.** In this section we solve the problem we started with.

Let $\overline{v}_S^\pm, \quad \overline{\omega}^\pm$ stand for velocity of the point S and for the ball's angular velocity respectively. Superscripts + and − mark the states after the collision and before the collision respectively.

In the coordinates $Oxyz$ one has :

$$\overline{v}_S^\pm = v_1^\pm \overline{e}_x + v_2^\pm \overline{e}_y, \quad \overline{\omega}^\pm = \omega_1^\pm \overline{e}_x + \omega_2^\pm \overline{e}_y + \omega_3^\pm \overline{e}_z.$$

From formula (1.1) one has

$$v_1^\pm = \omega_2^\pm r, \quad v_2^\pm = -\omega_1^\pm r. \tag{3.1}$$

Therefore the velocity of any point of the ball is completely defined by quantities $v_1^\pm, v_2^\pm, \omega_3^\pm$.

Theorem 3.1. *The following formalas hold*

$$\begin{aligned} v_1^+ &= -\, v_1^-, \\ v_2^+ &= \frac{r^2 m}{2\tilde{J}} v_2^- + \frac{rJ}{\tilde{J}}\omega_3^-, \\ \omega_3^+ &= \frac{J'}{r\tilde{J}} v_2^- - \frac{r^2 m}{2\tilde{J}}\omega_3^-. \end{aligned}$$

Proof. Introduce the Euler angles so that at the moment of collision one has $\varphi = \psi = 0, \quad \theta = \pi/2$. Then it follows that $\overline{\omega} = \dot{\theta}\overline{e}_x + \dot{\psi}\overline{e}_z - \dot{\varphi}\overline{e}_y$. Thus at the moment of collision we have

$$v^\pm = (v_1^\pm, v_2^\pm, \omega_1^\pm, \omega_2^\pm, \omega_3^\pm)^T.$$

The formula $T = \frac{1}{2} m\overline{v}_S^2 + \frac{1}{2} J\overline{\omega}^2$ implies $G = \mathrm{diag}(m, m, J, J, J)$.

Combining formulas (3.1) and (1.2) we obtain

$$A = \begin{pmatrix} 1 & 0 & 0 & -r & 0 \\ 0 & 1 & r & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -r \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & r & 0 & 0 \end{pmatrix}.$$

The matrix P is calculated with the help of Lemma 1:

$$P = \frac{1}{2\tilde{J}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2J & Jr & 0 & -Jr \\ 0 & rm & J' & 0 & J \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -rm & J & 0 & J' \end{pmatrix}.$$

Now the Theorem follows by direct calculation from formula (2.13).
The Theorem is proved.

3.2. **The Ball is Thrown to the Floor.** In this section we consider another problem with the ball. Now we have only the floor Oxy and there is no wall.

Being undergone with some potential forces the ball can move in the half-space $\{z_S > r\}$ and sometimes it can collide with the floor.

After the ball meets the floor ($z_S = r$) it then jumps aside. The point of contact $C \in B$ has the zero velocity (1.1).

Introduce the configuration manifold of our system as $M = \mathbb{R}^3 \times SO(3)$, where $(x_S, y_S, z_S) \in \mathbb{R}^3$ and an element of $SO(3)$ states the orientation of the ball. As a local coordinates in $SO(3)$ we use the Euler angles. The floor is a fifth dimensional submanifold $N \subset M$ which is given by the equation $z_S = r$.

Let $\overline{v}_S^+$, $\overline{\omega}^+$ stand for velocity of the point S and for the ball's angular velocity before the collision respectively. Let $\overline{v}_S^-$, $\overline{\omega}^-$ stand for velocity of the point S and for the ball's angular velocity after the collision respectively.

In the coordinates $Oxyz$ one has :

$$\overline{v}_S^\pm = (v_1^\pm, v_2^\pm, v_3^\pm), \quad \overline{\omega}^\pm = (\omega_1^\pm, \omega_2^\pm, \omega_3^\pm).$$

Theorem 3.2. *The following formulas hold true*

$$\begin{aligned} v_1^+ &= \frac{mr^2 - J}{J'} v_1^- + \frac{2Jr}{J'} \omega_2^-, \\ v_2^+ &= \frac{mr^2 - J}{J'} v_2^- - \frac{2Jr}{J'} \omega_1^-, \\ v_3^+ &= -v_3^-, \\ \omega_1^+ &= -\frac{2rm}{J'} v_2^- + \frac{J - mr^2}{J'} \omega_1^-, \\ \omega_2^+ &= \frac{2rm}{J'} v_1^- + \frac{J - mr^2}{J'} \omega_2^-, \\ \omega_3^+ &= \omega_3^-. \end{aligned}$$

Proof. Introduce the Euler angles so that at the moment of collision one has $\varphi = \psi = 0, \quad \theta = \pi/2$. Then it follows that $\overline{\omega} = \dot{\theta}\overline{e}_x + \dot{\psi}\overline{e}_z - \dot{\varphi}\overline{e}_y$. Thus at the moment of collision we have

$$v^{\pm} = (v_1^{\pm}, v_2^{\pm}, v_3^{\pm}, \omega_1^{\pm}, \omega_2^{\pm}, \omega_3^{\pm})^T.$$

The formula $T = \frac{1}{2}m\overline{v}_S^2 + \frac{1}{2}J\overline{\omega}^2$ implies $G = \mathrm{diag}(m, m, m, J, J, J)$. From formula (1.1) one obtains

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & -r & 0 \\ 0 & 1 & 0 & r & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix of the operator P is computed with the help of Lemma 1:

$$P = \frac{1}{J'} \begin{pmatrix} J & 0 & 0 & 0 & -Jr & 0 \\ 0 & J & 0 & Jr & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & rm & 0 & r^2m & 0 & 0 \\ -rm & 0 & 0 & 0 & r^2m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now Theorem 3.2 follows from formula (2.13).
The Theorem is proved.

3.2.1. *Nonholonomic Pendulum.* Suppose that the ball moves in the standard gravity field $\overline{g} = -g\overline{e}_z$.

Throw our ball to the floor so that

$$\overline{v}_S^- = -v\overline{e}_x - u\overline{e}_z, \quad \overline{\omega}^- = \frac{rmv}{J}\overline{e}_y, \quad u, v > 0.$$

From Theorem 3.2 it follows that

$$\overline{v}_S^+ = -\overline{v}_S^-, \quad \overline{\omega}^+ = -\overline{\omega}^-.$$

Thus after the ball jumped up from the floor its centre S moves along the same parabola just in the opposite direction. Since the angle velocity also changes its direction the periodic motion begins. The ball knocks the floor jumps up and go down along the parabola, knocks the floor at another point then flies along the same parabola to the initial point and so on.

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