

ON THE SPECTRUM OF NARROW NEUMANN WAVEGUIDE WITH PERIODICALLY DISTRIBUTED δ' TRAPS

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ABSTRACT. We analyze a family of singular Schrödinger operators describing a Neumann waveguide with a periodic array of singular traps of a δ' type. We show that in the limit when perpendicular size of the guide tends to zero and the δ' interactions are appropriately scaled, the first spectral gap is determined exclusively by geometric properties of the traps.

1. INTRODUCTION

The problem addressed in this paper concerns the limiting behaviour of a particular class of Schrödinger operators with singular coefficients. They can be characterized as Neumann Laplacians on a cylindrical region in \mathbb{R}^n perturbed by a array a singular “traps” consisting of a δ' interaction [1] supported by the boundary of fixed subsets of the cylinder. We will be interested in the situation when the cylinder shrinks in the perpendicular direction and the parameter of the δ' interaction simultaneously changes making the latter weaker. We are going to show that the limiting behaviour of the first spectral gap is determined exclusively by geometric properties of the traps, namely their volume and surface area.

The motivation to study such an asymptotic behaviour is twofold. On one hand it is an interesting spectral problem falling within one of the traditional mathematical-physics categories, asking about relations between the geometric and spectral properties. On the other hand, it is of some practical interest, especially in the light of the recently growing interest to metamaterials and engineering of spectral properties. True, the δ' interactions with their peculiar scattering properties are rather a mathematical construct, however, they can be approximated by regular or singular potentials following a seminal idea put forward in [5] and

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made mathematically rigorous in [2, 7]. Hence at least in the principal sense the result of this paper provides a way to achieve a prescribed spectral filtering.

2. SETTING OF THE PROBLEM AND MAIN RESULT

In what follows $\varepsilon > 0$ will be a small parameter. For a fixed $n \in \mathbb{N} \setminus \{1\}$ we denote by $x' = (x_1, \dots, x_{n-1})$ and $x = (x', x_n)$ the Cartesian coordinates in \mathbb{R}^{n-1} and \mathbb{R}^n , correspondingly. Let ω be an open domain in \mathbb{R}^{n-1} with a Lipschitz boundary. By Ω^ε we denote a straight cylinder in \mathbb{R}^n with a cross-section $\varepsilon\omega$, i.e.

$$\Omega^\varepsilon = \{x = (x', x_n) \in \mathbb{R}^n : \varepsilon^{-1}x' \in \omega\}.$$

Furthermore, we introduce the set

$$Y = \{x = (x', x_n) \in \mathbb{R}^n : |x_n| < 1/2, x' \in \omega\},$$

which will play role of the period cell of the problem before scaling. Let B be an arbitrary domain with a Lipschitz boundary $S = \partial B$ and such that $\overline{B} \subset Y$. For any $i \in \mathbb{Z}$ we denote

$$S_i^\varepsilon = \varepsilon(S + ie_n), \quad B_i^\varepsilon = \varepsilon(B + ie_n), \quad Y_i^\varepsilon = \varepsilon(Y + ie_n),$$

where $e_n = (0, 0, \dots, 0, 1)$ is the unit vector along the cylinder axis.

Next we describe the family of operators \mathcal{A}^ε which will be the main object of our interest in this paper. We denote

$$\Gamma^\varepsilon = \bigcup_{i \in \mathbb{Z}} S_i^\varepsilon$$

and introduce the sesquilinear form in the Hilbert space $L_2(\Omega^\varepsilon)$ by

$$\eta^\varepsilon[u, v] = \int_{\Omega^\varepsilon \setminus \Gamma^\varepsilon} \nabla u \cdot \nabla \bar{v} dx + a^\varepsilon \sum_{i \in \mathbb{Z}} \int_{S_i^\varepsilon} (u_+ - u_-) \overline{(v_+ - v_-)} ds, \quad a^\varepsilon > 0, \quad (2.1)$$

here and in the following we denote by u_+ (respectively, u_-) the traces of the function u taken from the exterior (respectively, interior) side of S_i^ε . The form domain is supposed to be

$$\begin{aligned} \text{dom}(\eta^\varepsilon) &= H^1(\Omega^\varepsilon \setminus \Gamma^\varepsilon) \\ &:= \left\{ u \in L_2(\Omega^\varepsilon) : u \in H^1(\Omega^\varepsilon \setminus \bigcup_{i \in \mathbb{Z}} \overline{B_i^\varepsilon}), u \in H^1(B_i^\varepsilon) \text{ for all } i \in \mathbb{Z}, \sum_{i \in \mathbb{Z}} \|\nabla u\|_{L_2(B_i^\varepsilon)}^2 < \infty \right\}. \end{aligned}$$

The definition of $\eta^\varepsilon[u, v]$ makes sense: the second sum in (2.1) is finite because of the standard trace inequalities, namely

$$\begin{aligned} \sum_{i \in \mathbb{Z}} \int_{S_i^\varepsilon} |u_+ - u_-|^2 ds &\leq 2 \sum_{i \in \mathbb{Z}} \int_{S_i^\varepsilon} (|u_+|^2 + |u_-|^2) ds \\ &\leq C(\varepsilon) \sum_{i \in \mathbb{Z}} \left(\|u\|_{H^1(Y_i^\varepsilon \setminus \overline{B_i^\varepsilon})}^2 + \|u\|_{H^1(B_i^\varepsilon)}^2 \right) = C(\varepsilon) \|u\|_{H^1(\Omega^\varepsilon \setminus \Gamma^\varepsilon)}^2, \end{aligned}$$

where the constant $C(\varepsilon)$ is independent of u . Furthermore, it is straightforward to check that the form $\eta^\varepsilon[u, v]$ is densely defined, closed and positive. Then (see, e.g., [10, Chapter 6, Theorem 2.1]) there exists the unique self-adjoint and positive operator \mathcal{A}^ε associated with the form η^ε , i.e.

$$(\mathcal{A}^\varepsilon u, v)_{L_2(\Omega^\varepsilon)} = \eta^\varepsilon[u, v], \quad \forall u \in \text{dom}(\mathcal{A}^\varepsilon), \quad \forall v \in \text{dom}(\eta^\varepsilon). \quad (2.2)$$

If $u \in \text{dom}(\mathcal{A}^\varepsilon)$ and $u \in C^2(\Omega^\varepsilon \setminus \Gamma^\varepsilon)$ then via the integration by parts one can show easily that $(\mathcal{A}^\varepsilon u)(x) = -\Delta u(x)$ for $x \in \Omega^\varepsilon \setminus \Gamma^\varepsilon$ and on the boundary S_i^ε one has

$$\left(\frac{\partial u}{\partial n} \right)_+ = \left(\frac{\partial u}{\partial n} \right)_- =: \frac{\partial u}{\partial n}, \quad \frac{\partial u}{\partial n} = a^\varepsilon (u_+ - u_-), \quad (2.3)$$

where n is the outward-pointing unit normal to S_i^ε . This makes it clear that the operators \mathcal{A}^ε have the meaning of Hamiltonians describing a waveguide with the Neumann outer boundary and an array of periodically spaced obstacles or traps given by a δ' interaction supported by S_i^ε . Note that analogous Schrödinger operators in \mathbb{R}^n with a δ' interaction supported by surfaces have been discussed recently in [3].

By $\sigma(\mathcal{A}^\varepsilon)$ we denote the spectrum of \mathcal{A}^ε . Our goal in this paper is to describe its behavior as $\varepsilon \rightarrow 0$ under the assumption that the coupling constant a^ε satisfies

$$\lim_{\varepsilon \rightarrow 0} \frac{a^\varepsilon}{\varepsilon} = a > 0. \quad (2.4)$$

Remark 2.1. A comment is due at this point to explain why we spoke in the introduction about a *weak* δ' interaction. Comparing (2.3) with the standard definition of such an interaction [1, Sec. I.4] we should regard rather the inverse $(a^\varepsilon)^{-1}$ as the coupling parameter. On the other hand, due to peculiar properties of the interaction [1, Theorem I.4.3] the δ' coupling is *weak* if this quantity is *large* satisfying, for instance, the asymptotic relation (2.4).

To state the result we shall use the notation $|\cdot|$ both for the volume of domain in \mathbb{R}^n and for the area of $(n-1)$ -dimensional surface in \mathbb{R}^n . Furthermore, we denote

$$\alpha = \frac{a|S|}{|B|}, \quad \beta = \frac{a|S|}{|B|} \frac{|Y|}{|Y| - |B|};$$

it is clear that $\alpha < \beta$. Now we are in position to formulate the main result of this work.

Theorem 2.1. *Let $L > 0$ be an arbitrary number. Then the spectrum of the operator \mathcal{A}^ε in $[0, L]$ has under the assumptions stated above the following structure for ε small enough:*

$$\sigma(\mathcal{A}^\varepsilon) \cap [0, L] = [0, L] \setminus (\alpha^\varepsilon, \beta^\varepsilon), \quad (2.5)$$

where the endpoints of the interval $(\alpha^\varepsilon, \beta^\varepsilon)$ satisfy the relations

$$\lim_{\varepsilon \rightarrow 0} \alpha^\varepsilon = \alpha, \quad \lim_{\varepsilon \rightarrow 0} \beta^\varepsilon = \beta. \quad (2.6)$$

The theorem will be proven in the next section. We postpone the outline of the proof to the remark preceding Lemma 3.1 because we need to introduce first some more notations.

3. PROOF OF THEOREM 2.1.

In what follows C, C_1, \dots will be generic constants that do not depend on ε . Let D be an open domain in \mathbb{R}^n ; by $\langle u \rangle_D$ we denote the normalized mean value of the function $u(x)$ in the domain D ,

$$\langle u \rangle_D = \frac{1}{|D|} \int_D u(x) \, dx.$$

Furthermore, if $\Gamma \subset \mathbb{R}^n$ is an $(n-1)$ -dimensional surface then the Euclidean metrics in \mathbb{R}^n induces on Γ the Riemannian metrics and measure. We denote by ds the density of this measure. Again by $\langle u \rangle_\Gamma$ we denote the normalized mean value of the function u over Γ , i.e

$$\langle u \rangle_\Gamma = \frac{1}{|\Gamma|} \int_\Gamma u \, ds.$$

Next we introduce the following sets:

- $Y^\varepsilon = \varepsilon Y$, the period cell,
- $B^\varepsilon = \varepsilon B$, the trap,
- $S^\varepsilon = \varepsilon S$, the trap boundary,
- $F^\varepsilon = Y^\varepsilon \setminus \overline{B^\varepsilon}$, the trap complement to the period cell,
- $S_\pm^\varepsilon = \{x = (x', x_n) \in \partial Y^\varepsilon : x_n = \pm \varepsilon/2\}$, the period cell “lids”.

The Floquet-Bloch theory — see, e.g., [4, 11, 12] — establishes a relationship between the spectrum of \mathcal{A}^ε and the spectra of appropriate operators on Y^ε . Specifically, for $\varphi \in [0, 2\pi)$ we introduce the functional space $H_\varphi^1(Y^\varepsilon \setminus S^\varepsilon)$ consisting of functions from $H^1(Y^\varepsilon \setminus S^\varepsilon)$ that satisfy the following condition on the lateral parts of ∂Y^ε :

$$u|_{S_+^\varepsilon} = \exp(i\varphi) T^\varepsilon u|_{S_-^\varepsilon}, \quad (3.1)$$

where $T^\varepsilon : L_2(S_-^\varepsilon) \rightarrow L_2(S_+^\varepsilon)$, $(T^\varepsilon f)(x) = f(x - \varepsilon e_n)$, $u|_{S_\pm^\varepsilon}$ are the traces of u on S_\pm^ε .

By η_φ^ε we denote the sesquilinear form defined by formula

$$\eta_\varphi^\varepsilon[u, v] = \int_{Y^\varepsilon \setminus S^\varepsilon} \nabla u \cdot \nabla \bar{v} dx + a^\varepsilon \int_{S^\varepsilon} (u_+ - u_-) \overline{(v_+ - v_-)} ds \quad (3.2)$$

with the domain $H_\varphi^1(Y^\varepsilon \setminus S^\varepsilon)$. We define $\mathcal{A}_\varphi^\varepsilon$ as the operator acting in $L_2(Y^\varepsilon)$ being associated with the form η_φ^ε :

$$(\mathcal{A}_\varphi^\varepsilon u, v)_{L_2(Y^\varepsilon)} = \eta_\varphi^\varepsilon[u, v], \quad \forall u \in \text{dom}(\mathcal{A}_\varphi^\varepsilon), \quad \forall v \in \text{dom}(\eta_\varphi^\varepsilon).$$

Since $Y^\varepsilon \setminus S^\varepsilon$ is compact, the operator $\mathcal{A}_\varphi^\varepsilon$ has a purely discrete spectrum. We denote by $\{\lambda_k^\varphi(\varepsilon)\}_{k \in \mathbb{N}}$ the sequence of eigenvalues of $\mathcal{A}_\varphi^\varepsilon$ arranged in the increasing order and repeated according to their multiplicity.

According to the Floquet-Bloch theory one has the following representation:

$$\sigma(\mathcal{A}^\varepsilon) = \bigcup_{k=1}^{\infty} \bigcup_{\varphi \in [0, 2\pi)} \{\lambda_k^\varphi(\varepsilon)\}. \quad (3.3)$$

Moreover, for any fixed $k \in \mathbb{N}$ the set $\bigcup_{\varphi \in [0, 2\pi)} \{\lambda_k^\varphi(\varepsilon)\}$, in other words, the k th spectral band, is a compact interval.

We also introduce the operators $\mathcal{A}_N^\varepsilon$ and $\mathcal{A}_D^\varepsilon$, which are defined in a similar way as $\mathcal{A}_\varphi^\varepsilon$, however, with (3.1) replaced by Neumann and Dirichlet boundary conditions of S_\pm^ε , respectively. More precisely, we denote by η_N^ε (correspondingly, η_D^ε) the sesquilinear forms in $L_2(Y^\varepsilon)$ defined by (3.2) and the domain $H^1(Y^\varepsilon \setminus S^\varepsilon)$ (correspondingly, $\widehat{H}_0^1(Y^\varepsilon \setminus S^\varepsilon) = \{u \in H^1(Y^\varepsilon \setminus S^\varepsilon) : u = 0 \text{ on } S_+^\varepsilon \cup S_-^\varepsilon\}$). The above indicated operators are then associated with these forms, η_N^ε and η_D^ε , respectively, i.e.

$$(\mathcal{A}_*^\varepsilon u, v)_{L_2(Y^\varepsilon)} = \eta_*^\varepsilon[u, v], \quad \forall u \in \text{dom}(\mathcal{A}_*^\varepsilon), \quad \forall v \in \text{dom}(\eta_*^\varepsilon),$$

where $*$ is N (correspondingly, D).

As with $\mathcal{A}_\varphi^\varepsilon$, the spectra of the operators $\mathcal{A}_N^\varepsilon$ and $\mathcal{A}_D^\varepsilon$ are purely discrete. We denote by $\{\lambda_k^N(\varepsilon)\}_{k \in \mathbb{N}}$ (correspondingly, $\{\lambda_k^D(\varepsilon)\}_{k \in \mathbb{N}}$) the sequence of eigenvalues of $\mathcal{A}_N^\varepsilon$ (correspondingly, of $\mathcal{A}_D^\varepsilon$) arranged in the ascending order and repeated according to their multiplicity.

From the min-max principle — see, e.g., [12, Chapter XIII] — and the inclusions

$$H^1(Y^\varepsilon \setminus S^\varepsilon) \supset H_\varphi^1(Y^\varepsilon \setminus S^\varepsilon) \supset \widehat{H}_0^1(Y^\varepsilon \setminus S^\varepsilon)$$

we infer that

$$\forall k \in \mathbb{N}, \quad \forall \varphi \in [0, 2\pi) : \quad \lambda_k^N(\varepsilon) \leq \lambda_k^\varphi(\varepsilon) \leq \lambda_k^D(\varepsilon). \quad (3.4)$$

Remarks 3.1. (a) With these preliminaries, we are able to provide the promised brief description of the proof of Theorem 2.1. It is clear that the left edge of the first spectral band of \mathcal{A}^ε coincides with zero, while the right one is situated between the first antiperiodic eigenvalue $\lambda_1^\varphi(\varepsilon)$, $\varphi = \pi$, and the first Dirichlet

eigenvalue $\lambda_1^D(\varepsilon)$. We are going to prove (see Lemmata 3.2 and 3.4 below) that they both converge to α as $\varepsilon \rightarrow 0$. Similarly we can localize the left edge of the second spectral band between the second Neumann eigenvalue $\lambda_2^N(\varepsilon)$ and the second periodic eigenvalue $\lambda_2^\varphi(\varepsilon)$, $\varphi = 0$, of which we will prove (see Lemmata 3.3 and 3.4 below) that they both converge to β as $\varepsilon \rightarrow 0$. Finally, we intend to prove that $\lambda_2^\varphi(\varepsilon)$, $\varphi \neq 0$, converges to infinity as $\varepsilon \rightarrow 0$ which means that the right edge of the second spectral band exceeds any fixed L provided ε is small enough. These results taken together constitute the claim of Theorem 2.1.

(b) We stress that the band edges need not in general coincide with the corresponding periodic (antiperiodic) solutions even if the system exhibits periodicity in one direction only [6, 8]. What matters is that we can squeeze them between two values which converge to the same limit as $\varepsilon \rightarrow 0$.

Lemma 3.1. *Let $\varphi \neq 0$. In the limit $\varepsilon \rightarrow 0$ one has*

$$\lambda_2^\varphi(\varepsilon) \rightarrow \infty. \quad (3.5)$$

Proof. We denote

- $F = \varepsilon^{-1}F^\varepsilon$, the scaled trap complement to the scaled period cell,
- $S_\pm = \varepsilon^{-1}S_\pm^\varepsilon$, the scaled period cell ‘‘lids’’.

We introduce the sesquilinear form $\tilde{\eta}_\varphi^\varepsilon$ in the space $L_2(Y)$ defined by the formula

$$\tilde{\eta}_\varphi^\varepsilon[u, v] = \int_{Y \setminus S} \nabla u \cdot \nabla \bar{v} \, dx + a^\varepsilon \varepsilon \int_S (u_+ - u_-) \overline{(v_+ - v_-)} \, ds,$$

with

$$\text{dom}(\tilde{\eta}_\varphi^\varepsilon) = \left\{ u \in H^1(Y \setminus S) : u|_{S_+} = \exp(i\varphi) T u|_{S_-} \right\},$$

where $T : L_2(S_-) \rightarrow L_2(S_+)$, $(Tf)(x) = f(x - e_n)$, and $u|_{S_\pm}$ are the traces of u on S_\pm . Let $\tilde{\mathcal{A}}_\varphi^\varepsilon$ be the operator associated with this form and let $\{\tilde{\lambda}_k^\varphi(\varepsilon)\}_{k \in \mathbb{N}}$ be the sequence of its eigenvalues arranged in the ascending order and repeated according to their multiplicity. It is easy to see that for all $k \in \mathbb{N}$

$$\tilde{\lambda}_k^\varphi(\varepsilon) = \varepsilon^2 \lambda_k^\varphi(\varepsilon). \quad (3.6)$$

In the same Hilbert space $L_2(Y)$ we introduce the sesquilinear form $\tilde{\eta}_\varphi$ by the formula

$$\tilde{\eta}_\varphi[u, v] = \int_{Y \setminus S} \nabla u \cdot \nabla \bar{v} \, dx$$

with $\text{dom}(\tilde{\eta}_\varphi) = \text{dom}(\tilde{\eta}_\varphi^\varepsilon)$, and by $\tilde{\mathcal{A}}_\varphi$ we denote the operator generated by this form, with $\{\tilde{\lambda}_k^\varphi\}_{k \in \mathbb{N}}$ being the sequence of eigenvalues of $\tilde{\mathcal{A}}_\varphi$ written in the increasing order and repeated according to their multiplicity. It is clear from the definition that

$$\tilde{\mathcal{A}}_\varphi = -\Delta_\varphi(F) \oplus -\Delta(B),$$

where $\Delta_\varphi(F)$ (respectively, $\Delta(B)$) is the Laplacian in $L_2(F)$ (respectively, $L_2(B)$) with the Neumann boundary conditions on $\partial F \setminus (S_- \cup S_+)$ and φ -periodic boundary conditions on $S_- \cup S_+$ (respectively, with the Neumann boundary condition on S). One can check easily that for any $\varphi \neq 0$ we have

$$\tilde{\lambda}_2^\varphi > 0. \quad (3.7)$$

To conclude the argument we are going to demonstrate that

$$\forall k \in \mathbb{N} : \tilde{\lambda}_k^\varphi(\varepsilon) \rightarrow \tilde{\lambda}_k^\varphi \quad (3.8)$$

holds, then (3.5) will follow directly from (3.6)–(3.8). It remains therefore to prove (3.8). We denote

$$\tilde{L}_\varphi^\varepsilon = (\tilde{\mathcal{A}}_\varphi^\varepsilon + I)^{-1}, \quad \tilde{L}_\varphi = (\tilde{\mathcal{A}}_\varphi + I)^{-1},$$

where I is the identity operator. The operators $\tilde{L}_\varphi^\varepsilon, \tilde{L}_\varphi$ are compact and positive (thus self-adjoint), and furthermore

$$\|\tilde{L}_\varphi^\varepsilon\| \leq 1. \quad (3.9)$$

Next we want to prove that

$$\forall f \in L_2(Y) : \tilde{L}_\varphi^\varepsilon f \rightarrow \tilde{L}_\varphi f \text{ in } L_2(Y) \text{ as } \varepsilon \rightarrow 0. \quad (3.10)$$

We take an arbitrary $f \in L_2(Y)$ and set $u^\varepsilon = \tilde{L}_\varphi^\varepsilon f$. It is clear that

$$\tilde{\eta}_\varphi^\varepsilon[u^\varepsilon, u^\varepsilon] + \|u^\varepsilon\|_{L_2(Y)} \leq C,$$

which, in particular, implies that the functions u^ε are bounded in $H^1(Y \setminus S)$ uniformly in ε . Therefore by the Rellich-Kondrachov embedding theorem there exist $u \in H^1(Y \setminus S)$ and a subsequence $\varepsilon_k, k = 1, 2, 3, \dots$, satisfying $\varepsilon_k \searrow 0$ as $\varepsilon \rightarrow 0$ such that

$$u^\varepsilon \rightharpoonup u \text{ in } H^1(Y \setminus S) \text{ as } \varepsilon = \varepsilon_k \rightarrow 0, \quad (3.11)$$

$$u^\varepsilon \rightarrow u \text{ in } L_2(Y) \text{ as } \varepsilon = \varepsilon_k \rightarrow 0. \quad (3.12)$$

Furthermore, in view of the trace theorem

$$u^\varepsilon \rightarrow u \text{ in } L_2(\partial Y \cup S) \text{ as } \varepsilon = \varepsilon_k \rightarrow 0, \quad (3.13)$$

and consequently, $u \in \text{dom}(\tilde{\eta}_\varphi)$. Given an arbitrary $v \in \text{dom}(\tilde{\eta}_\varphi^\varepsilon)$, we have the following identity:

$$\int_{Y \setminus S} \nabla u^\varepsilon \cdot \nabla \bar{v} \, dx + a^\varepsilon \varepsilon \int_S (u_+^\varepsilon - u_-^\varepsilon) \overline{(v_+ - v_-)} \, ds + \int_Y u^\varepsilon \bar{v} \, dx = \int_Y f \bar{v} \, dx. \quad (3.14)$$

Using (3.11)–(3.13) and keeping in mind that $a^\varepsilon \varepsilon \rightarrow 0$ holds as $\varepsilon \rightarrow 0$ we pass to the limit in (3.14) as $\varepsilon = \varepsilon_k \rightarrow 0$ and obtain

$$\int_{Y \setminus S} \nabla u \cdot \nabla \bar{v} \, dx + \int_Y u \bar{v} \, dx = \int_Y f \bar{v} \, dx, \quad \forall v \in \text{dom}(\tilde{\eta}_\varphi). \quad (3.15)$$

It follows from (3.15) that $u = \tilde{L}_\varphi f$. Since u is independent of the subsequence ε_k , the whole sequence u^ε converges to u as $\varepsilon \rightarrow 0$, in other words, (3.10) holds true.

Finally, again using the Rellich-Kondrachev embedding theorem, we conclude that for an arbitrary sequence f^ε , which is bounded in $L_2(Y)$ uniformly in ε , there exist $w \in L_2(Y)$ and a subsequence ε_k , $k = 1, 2, 3, \dots$, with $\varepsilon_k \searrow_{k \rightarrow \infty} 0$ such that

$$\tilde{L}_\varphi^\varepsilon f^\varepsilon \rightarrow w \text{ in } L_2(Y) \text{ as } \varepsilon = \varepsilon_k \rightarrow 0. \quad (3.16)$$

We denote by $\{\tilde{\mu}_k^\varphi(\varepsilon)\}_{k \in \mathbb{N}}$ and $\{\tilde{\mu}_k^\varphi\}_{k \in \mathbb{N}}$ the sequences of eigenvalues of $\tilde{L}_\varphi^\varepsilon$ and \tilde{L}_φ , respectively, written in the decreasing order and repeated according to their multiplicity. According to Lemma 1 from [9] it follows from (3.9), (3.10), and (3.16) that

$$\forall k \in \mathbb{N} : \tilde{\mu}_k^\varphi(\varepsilon) \rightarrow \tilde{\mu}_k^\varphi \text{ as } \varepsilon \rightarrow 0. \quad (3.17)$$

However, since $\tilde{\lambda}_k^\varphi(\varepsilon) = \frac{1}{\tilde{\mu}_k^\varphi(\varepsilon)} - 1$ and $\tilde{\lambda}_k^\varphi = \frac{1}{\tilde{\mu}_k^\varphi} - 1$, (3.17) implies the sought relation (3.8) concluding thus the proof of the lemma. \square

Now we have to inspect the behaviour of the first Dirichlet eigenvalue, which we use to estimate the left gap edge from above, in the the limit $\varepsilon \rightarrow 0$.

Lemma 3.2. *One has*

$$\lambda_1^D(\varepsilon) \rightarrow \alpha \text{ as } \varepsilon \rightarrow 0. \quad (3.18)$$

Proof. Let u_D^ε be the eigenfunction of $\mathcal{A}_D^\varepsilon$, which corresponds to $\lambda_1^D(\varepsilon)$, determined uniquely by the following requirements:

$$\|u_D^\varepsilon\|_{L_2(Y^\varepsilon)} = 1, \quad (3.19)$$

$$u_D^\varepsilon \text{ is real-valued, } \int_{B^\varepsilon} u_D^\varepsilon dx \geq 0. \quad (3.20)$$

From the min-max principle we get

$$\lambda_1^D(\varepsilon) = \eta_D^\varepsilon[u_D^\varepsilon, u_D^\varepsilon] = \min_{u \in \mathcal{H}_D^\varepsilon} \eta_D^\varepsilon[u, u], \quad (3.21)$$

where $\mathcal{H}_D^\varepsilon = \{u \in \text{dom}(\eta_D^\varepsilon) : \|u\|_{L_2(Y^\varepsilon)} = 1\}$. We construct an approximation, denoted as v_D^ε , to the eigenfunction u_D^ε using the formula

$$v_D^\varepsilon(x) = \begin{cases} 0, & x \in F^\varepsilon, \\ |B^\varepsilon|^{-1/2}, & x \in B^\varepsilon. \end{cases} \quad (3.22)$$

It is clear that $v_D^\varepsilon \in \text{dom}(\eta_D^\varepsilon)$. Taking into account that $\nabla v_D^\varepsilon = 0$ holds in $B^\varepsilon \cup F^\varepsilon$ we get

$$\eta_D^\varepsilon[v_D^\varepsilon, v_D^\varepsilon] = a^\varepsilon \int_{S^\varepsilon} ((v_D^\varepsilon)_+ - (v_D^\varepsilon)_-)^2 dx = \frac{a^\varepsilon |S^\varepsilon|}{|B^\varepsilon|} = \frac{a^\varepsilon |S|}{\varepsilon |B|} \sim \alpha \text{ as } \varepsilon \rightarrow 0, \quad (3.23)$$

$$\|v_D^\varepsilon\|_{L_2(Y^\varepsilon)} = 1, \quad (3.24)$$

and therefore $v_D^\varepsilon \in \mathcal{H}_D^\varepsilon$. Consequently, in view of (3.21) we can infer that

$$\eta_D^\varepsilon[u_D^\varepsilon, u_D^\varepsilon] \leq \eta_D^\varepsilon[v_D^\varepsilon, v_D^\varepsilon]. \quad (3.25)$$

It follows from (3.23) and (3.25) that

$$\eta_D^\varepsilon[u_D^\varepsilon, u_D^\varepsilon] \leq C, \quad (3.26)$$

and using (3.26) one arrives at the following Friedrichs- and Poincaré-type inequalities:

$$\|u_D^\varepsilon\|_{L_2(F^\varepsilon)}^2 \leq C\varepsilon^2 \|\nabla u_D^\varepsilon\|_{L_2(F^\varepsilon)}^2 \leq C\varepsilon^2 \eta_D^\varepsilon[u_D^\varepsilon, u_D^\varepsilon] \leq C_1\varepsilon^2, \quad (3.27)$$

$$\|u_D^\varepsilon - \langle u_D^\varepsilon \rangle_{B^\varepsilon}\|_{L_2(B^\varepsilon)}^2 \leq C\varepsilon^2 \|\nabla u_D^\varepsilon\|_{L_2(B^\varepsilon)}^2 \leq C\varepsilon^2 \eta_D^\varepsilon[u_D^\varepsilon, u_D^\varepsilon] \leq C_1\varepsilon^2, \quad (3.28)$$

where $\langle \cdot \rangle$ as usual denotes the mean value. Moreover, using (3.19), (3.27), and (3.28) we get

$$|\langle u_D^\varepsilon \rangle_{B^\varepsilon}|^2 |B^\varepsilon| = 1 - \|u_D^\varepsilon\|_{L_2(F^\varepsilon)}^2 - \|u_D^\varepsilon - \langle u_D^\varepsilon \rangle_{B^\varepsilon}\|_{L_2(B^\varepsilon)}^2 = 1 + \mathcal{O}(\varepsilon^2),$$

which implies, taking into account (3.20), that

$$\left(\langle u_D^\varepsilon \rangle_{B^\varepsilon} |B^\varepsilon|^{1/2} - 1\right) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (3.29)$$

Finally, using (3.28) and (3.29) we conclude that

$$\begin{aligned} \|u_D^\varepsilon - |B^\varepsilon|^{-1/2}\|_{L_2(B^\varepsilon)}^2 &\leq 2\|u_D^\varepsilon - \langle u_D^\varepsilon \rangle_{B^\varepsilon}\|_{L_2(B^\varepsilon)}^2 \\ &\quad + 2|B^\varepsilon| \left| \langle u_D^\varepsilon \rangle_{B^\varepsilon} - |B^\varepsilon|^{-1/2} \right|^2 \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{aligned} \quad (3.30)$$

In the next step we represent the eigenfunction u_D^ε in the form of a sum,

$$u_D^\varepsilon = v_D^\varepsilon + w_D^\varepsilon, \quad (3.31)$$

and estimate the remainder w_D^ε . Plugging (3.31) into (3.25) we obtain

$$\begin{aligned} \eta_D^\varepsilon[w_D^\varepsilon, w_D^\varepsilon] &\leq -2\eta_D^\varepsilon[v_D^\varepsilon, w_D^\varepsilon] \\ &= -2a^\varepsilon \int_{S^\varepsilon} \left((v_D^\varepsilon)_+ - (v_D^\varepsilon)_- \right) \left((w_D^\varepsilon)_+ - (w_D^\varepsilon)_- \right) ds. \end{aligned} \quad (3.32)$$

Let us recall the following standard trace inequalities:

$$\forall u \in H^1(F), v \in H^1(B) : \quad \|u\|_{L_2(S)}^2 \leq C\|u\|_{H^1(F)}^2, \quad \|v\|_{L_2(S)}^2 \leq C\|v\|_{H^1(B)}^2.$$

Using them together with a change of variables, $x \mapsto x\varepsilon$, one can derive the estimates

$$\int_{S^\varepsilon} |(u_D^\varepsilon)_+|^2 ds \leq C\varepsilon^{-1} \left(\|u_D^\varepsilon\|_{L_2(F^\varepsilon)}^2 + \varepsilon^2 \|\nabla u_D^\varepsilon\|_{L_2(F^\varepsilon)}^2 \right), \quad (3.33)$$

$$\int_{S^\varepsilon} |(u_D^\varepsilon)_- - |B^\varepsilon|^{-1/2}|^2 ds \leq C\varepsilon^{-1} \left(\|u_D^\varepsilon - |B^\varepsilon|^{-1/2}\|_{L_2(B^\varepsilon)}^2 + \varepsilon^2 \|\nabla u_D^\varepsilon\|_{L_2(B^\varepsilon)}^2 \right). \quad (3.34)$$

Then taking into account (2.4) and (3.33)–(3.34) we obtain from (3.32) the inequalities

$$\begin{aligned} |\eta_D^\varepsilon[w_D^\varepsilon, w_D^\varepsilon]|^2 &\leq 4(a^\varepsilon)^2 \int_{S^\varepsilon} |(v_D^\varepsilon)_+ - (v_D^\varepsilon)_-|^2 ds \cdot \int_S |(w_D^\varepsilon)_+ - (w_D^\varepsilon)_-|^2 ds \\ &\leq 8 \frac{(a^\varepsilon)^2 |S^\varepsilon|}{|B^\varepsilon|} \int_{S^\varepsilon} \left(|(u_D^\varepsilon)_+|^2 + |(u_D^\varepsilon)_- - |B^\varepsilon|^{-1/2}|^2 \right) ds \\ &\leq C \left(\|u_D^\varepsilon\|_{L_2(F^\varepsilon)}^2 + \|u_D^\varepsilon - |B^\varepsilon|^{-1/2}\|_{L_2(B^\varepsilon)}^2 + \varepsilon^2 \|\nabla u_D^\varepsilon\|_{L_2(B^\varepsilon \cup F^\varepsilon)}^2 \right), \end{aligned}$$

thus in view of (3.26), (3.27), and (3.30) we find

$$\eta_D^\varepsilon[w_D^\varepsilon, w_D^\varepsilon] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (3.35)$$

It follows from (3.23), (3.31), and (3.35) that

$$\lambda_1^D(\varepsilon) = \eta_D^\varepsilon[u_D^\varepsilon, u_D^\varepsilon] \sim \eta_D^\varepsilon[v_D^\varepsilon, v_D^\varepsilon] \sim \alpha \text{ as } \varepsilon \rightarrow 0, \quad (3.36)$$

which is the claim we have set up to prove. \square

Next we need an analogous claim for the second Neumann eigenvalue which we use to estimate the right gap edge from below.

Lemma 3.3. *One has*

$$\lambda_2^N(\varepsilon) \rightarrow \beta \text{ as } \varepsilon \rightarrow 0. \quad (3.37)$$

Proof. Let u_N^ε be the eigenfunction of $\mathcal{A}_N^\varepsilon$, which corresponds to $\lambda_2^N(\varepsilon)$ and satisfying the following requirement:

$$\|u_N^\varepsilon\|_{L_2(Y^\varepsilon)} = 1, \quad (3.38)$$

$$u_N^\varepsilon \text{ is real-valued, } \int_{B^\varepsilon} u_N^\varepsilon dx \geq 0. \quad (3.39)$$

Since $\lambda_1^N(\varepsilon) = 0$ and the corresponding eigenspace consists of constant functions, we have

$$\int_{Y^\varepsilon} u_N^\varepsilon dx = 0, \quad (3.40)$$

and in view of the min-max principle it holds

$$\lambda_2^N(\varepsilon) = \eta_N^\varepsilon[u_N^\varepsilon, u_N^\varepsilon] = \min_{u \in \mathcal{H}_N^\varepsilon} \eta_N^\varepsilon[u, u], \quad (3.41)$$

where $\mathcal{H}_N^\varepsilon = \left\{ u \in \text{dom}(\eta_N^\varepsilon) : \|u\|_{L_2(Y^\varepsilon)} = 1, \int_{Y^\varepsilon} u dx = 0 \right\}$. As in the previous proof, we construct an approximation v_N^ε to the eigenfunction u_N^ε , this time defined

by the formula

$$v_N^\varepsilon(x) = \begin{cases} \kappa_F^\varepsilon := -\sqrt{\frac{|B^\varepsilon|}{|F^\varepsilon| \cdot |Y^\varepsilon|}}, & x \in F^\varepsilon, \\ \kappa_B^\varepsilon := \sqrt{\frac{|F^\varepsilon|}{|B^\varepsilon| \cdot |Y^\varepsilon|}}, & x \in B^\varepsilon. \end{cases} \quad (3.42)$$

Obviously $v_N^\varepsilon \in \text{dom}(\eta_N^\varepsilon)$ and it is easy to see that

$$\eta_N^\varepsilon[v_N^\varepsilon, v_N^\varepsilon] = a^\varepsilon |S^\varepsilon| (\kappa_F^\varepsilon - \kappa_B^\varepsilon)^2 = \frac{a^\varepsilon |S| \cdot |Y|}{\varepsilon |B| \cdot |F|} \sim \beta \text{ as } \varepsilon \rightarrow 0, \quad (3.43)$$

$$\|v_N^\varepsilon\|_{L_2(Y^\varepsilon)} = 1, \quad (3.44)$$

$$\int_{Y^\varepsilon} v_N^\varepsilon dx = 0. \quad (3.45)$$

In view of (3.44) and (3.45), v_N^ε belongs to $\mathcal{H}_N^\varepsilon$, and therefore using (3.41) and taking into account (3.43) we get

$$\eta_N^\varepsilon[u_N^\varepsilon, u_N^\varepsilon] \leq \eta_N^\varepsilon[v_N^\varepsilon, v_N^\varepsilon] \leq C. \quad (3.46)$$

Using (3.46) one can derive the following Poincaré-type inequalities:

$$\|u_N^\varepsilon - \langle u_N^\varepsilon \rangle_{F^\varepsilon}\|_{L_2(F^\varepsilon)}^2 \leq C\varepsilon^2 \|\nabla u_N^\varepsilon\|_{L_2(F^\varepsilon)}^2 \leq C\varepsilon^2 \eta_N^\varepsilon[u_N^\varepsilon, u_N^\varepsilon] \leq C_1\varepsilon^2, \quad (3.47)$$

$$\|u_N^\varepsilon - \langle u_N^\varepsilon \rangle_{B^\varepsilon}\|_{L_2(B^\varepsilon)}^2 \leq C\varepsilon^2 \|\nabla u_N^\varepsilon\|_{L_2(B^\varepsilon)}^2 \leq C\varepsilon^2 \eta_N^\varepsilon[u_N^\varepsilon, u_N^\varepsilon] \leq C_1\varepsilon^2. \quad (3.48)$$

Moreover, using (3.38), (3.40), (3.47), and (3.48) we infer that

$$\begin{aligned} & |\langle u_N^\varepsilon \rangle_{F^\varepsilon}|^2 |F^\varepsilon| + |\langle u_N^\varepsilon \rangle_{B^\varepsilon}|^2 |B^\varepsilon| \\ &= 1 - \|u_N^\varepsilon - \langle u_N^\varepsilon \rangle_{F^\varepsilon}\|_{L_2(F^\varepsilon)}^2 + \|u_N^\varepsilon - \langle u_N^\varepsilon \rangle_{B^\varepsilon}\|_{L_2(B^\varepsilon)}^2 = 1 + \mathcal{O}(\varepsilon^2), \\ & \langle u_N^\varepsilon \rangle_{F^\varepsilon} |F^\varepsilon| + \langle u_N^\varepsilon \rangle_{B^\varepsilon} |B^\varepsilon| = \int_{Y^\varepsilon} u_N^\varepsilon dx = 0, \end{aligned}$$

from where, taking into account (3.39), we obtain the following asymptotics:

$$\langle u_N^\varepsilon \rangle_{F^\varepsilon} = \kappa_F^\varepsilon (1 + \mathcal{O}(\varepsilon^2))^{1/2}, \quad \langle u_N^\varepsilon \rangle_{B^\varepsilon} \sim \kappa_B^\varepsilon (1 + \mathcal{O}(\varepsilon^2))^{1/2} \text{ as } \varepsilon \rightarrow 0. \quad (3.49)$$

Finally, using (3.47)–(3.49) one can easily arrive at the relation

$$\|u_N^\varepsilon - \kappa_F^\varepsilon\|_{L_2(F^\varepsilon)}^2 + \|u_N^\varepsilon - \kappa_B^\varepsilon\|_{L_2(B^\varepsilon)}^2 \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (3.50)$$

In the next step, we again represent the eigenfunction u_N^ε in the form of a sum,

$$u_N^\varepsilon = v_N^\varepsilon + w_N^\varepsilon \quad (3.51)$$

and estimate the remainder w_N^ε . We plug (3.51) into (3.46) and obtain

$$\begin{aligned} \eta_N^\varepsilon[w_N^\varepsilon, w_N^\varepsilon] &\leq -2\eta_N^\varepsilon[v_N^\varepsilon, w_N^\varepsilon] \\ &= -2a^\varepsilon \int_{S^\varepsilon} \left((v_N^\varepsilon)_+ - (v_N^\varepsilon)_- \right) \left((w_N^\varepsilon)_+ - (w_N^\varepsilon)_- \right) ds. \end{aligned} \quad (3.52)$$

In analogy with relation (3.34) in the proof of Lemma 3.2 we employ the following trace inequalities:

$$\begin{aligned} \int_{S^\varepsilon} |(u_N^\varepsilon)_+ - \kappa_F^\varepsilon|^2 ds &\leq C\varepsilon^{-1} \left(\|u_N^\varepsilon - \kappa_F^\varepsilon\|_{L_2(F^\varepsilon)}^2 + \varepsilon^2 \|\nabla u_N^\varepsilon\|_{L_2(F^\varepsilon)}^2 \right), \\ \int_{S^\varepsilon} |(u_N^\varepsilon)_- - \kappa_B^\varepsilon|^2 ds &\leq C\varepsilon^{-1} \left(\|u_N^\varepsilon - \kappa_B^\varepsilon\|_{L_2(B^\varepsilon)}^2 + \varepsilon^2 \|\nabla u_N^\varepsilon\|_{L_2(B^\varepsilon)}^2 \right). \end{aligned}$$

Using them we infer from (3.52) that

$$\begin{aligned} |\eta_N^\varepsilon[w_N^\varepsilon, w_N^\varepsilon]|^2 &\leq 4(a^\varepsilon)^2 \int_{S^\varepsilon} |(v_N^\varepsilon)_+ - (v_N^\varepsilon)_-|^2 ds \cdot \int_S |(w_N^\varepsilon)_+ - (w_N^\varepsilon)_-|^2 ds \\ &\leq 8(a^\varepsilon)^2 |S^\varepsilon| (\kappa_F^\varepsilon - \kappa_B^\varepsilon)^2 \int_{S^\varepsilon} \left(|(u_N^\varepsilon)_+ - \kappa_F^\varepsilon|^2 + |(u_N^\varepsilon)_- - \kappa_B^\varepsilon|^2 \right) ds \\ &\leq C \left(\|u_N^\varepsilon - \kappa_F^\varepsilon\|_{L_2(F^\varepsilon)}^2 + \|u_N^\varepsilon - \kappa_B^\varepsilon\|_{L_2(B^\varepsilon)}^2 + \varepsilon^2 \|\nabla u_N^\varepsilon\|_{L_2(B^\varepsilon \cup F^\varepsilon)}^2 \right), \end{aligned}$$

and thus by virtue of (3.46) and (3.50) we get

$$\eta_N^\varepsilon[w_N^\varepsilon, w_N^\varepsilon] \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (3.53)$$

Finally, it follows (3.43), (3.51), and (3.53) that

$$\lambda_2^N(\varepsilon) = \eta_N^\varepsilon[u_N^\varepsilon, u_N^\varepsilon] \sim \eta_N^\varepsilon[v_N^\varepsilon, v_N^\varepsilon] \sim \beta \quad \text{as } \varepsilon \rightarrow 0, \quad (3.54)$$

which is nothing else than the claim of the lemma. \square

Finally, one has to inspect the behaviour of the other eigenvalues involved in estimates of the gap edges in the limit $\varepsilon \rightarrow 0$.

Lemma 3.4. *One has*

$$\text{if } \varphi = \pi, \text{ then } \lambda_1^\varphi(\varepsilon) \rightarrow \alpha \quad \text{as } \varepsilon \rightarrow 0, \quad (3.55)$$

$$\text{if } \varphi = 0, \text{ then } \lambda_2^\varphi(\varepsilon) \rightarrow \beta \quad \text{as } \varepsilon \rightarrow 0. \quad (3.56)$$

Proof. The proof of (3.56) is similar to the argument used to demonstrate (3.37). Specifically, we approximate the eigenfunction u_φ^ε of $\mathcal{A}_\varphi^\varepsilon$ that corresponds to $\lambda_2^\varphi(\varepsilon)$ and satisfies the conditions (3.38)–(3.40), with the index N replaced by φ , by the function v_N^ε given by (3.42). It is clear that $v_N^\varepsilon \in \text{dom}(\eta_\varphi^\varepsilon)$ if $\varphi = 0$. The check of the asymptotic equality

$$\lambda_2^\varphi(\varepsilon) \sim \eta_\varphi^\varepsilon[v_N^\varepsilon, v_N^\varepsilon], \quad \varphi = 0$$

repeats word-by-word the proof of (3.54).

Consider next $\varphi = \pi$. By u_φ^ε we denote the eigenfunction of $\mathcal{A}_\varphi^\varepsilon$ that corresponds to $\lambda_1^\varphi(\varepsilon)$ and satisfies the conditions (3.19)–(3.20), with the index D replaced by φ . Obviously, for $\varphi = \pi$ such an eigenfunction exists.

Using the Cauchy inequality, a standard trace inequality and the Poincaré inequality one can employ the following estimate: for any $v \in H^1(F)$ we have

$$\begin{aligned} |\langle v \rangle_{S_{\pm}} - \langle v \rangle_F|^2 &\leq \frac{1}{|S_{\pm}|} \|v - \langle v \rangle_F\|_{L_2(S_{\pm})}^2 \\ &\leq C \left(\|v - \langle v \rangle_F\|_{L_2(F)}^2 + \|\nabla v\|_{L_2(F)}^2 \right) \leq C_1 \|\nabla v\|_{L_2(F)}^2. \end{aligned} \quad (3.57)$$

Via a change of variables, $x \mapsto x\varepsilon$, one can easily derive from (3.57) the estimate

$$|\langle u_{\varphi}^{\varepsilon} \rangle_{S_{\pm}^{\varepsilon}} - \langle u_{\varphi}^{\varepsilon} \rangle_{F^{\varepsilon}}|^2 \leq C\varepsilon^{2-n} \|\nabla u_{\varphi}^{\varepsilon}\|_{L_2(F^{\varepsilon})}^2. \quad (3.58)$$

Furthermore, from (3.58) and the fact that $u_{\varphi}^{\varepsilon}$ satisfies condition (3.1) we can infer that

$$\begin{aligned} |\langle u_{\varphi}^{\varepsilon} \rangle_{F^{\varepsilon}}|^2 &= |1 - \exp(i\pi)|^{-2} \left| \langle u_{\varphi}^{\varepsilon} \rangle_{F^{\varepsilon}} - \langle u_{\varphi}^{\varepsilon} \rangle_{S_{+}^{\varepsilon}} + \exp(i\pi) \langle u_{\varphi}^{\varepsilon} \rangle_{S_{-}^{\varepsilon}} - \exp(i\pi) \langle u_{\varphi}^{\varepsilon} \rangle_{F^{\varepsilon}} \right|^2 \\ &\leq 2 |1 - \exp(i\pi)|^{-2} \left(|\langle u_{\varphi}^{\varepsilon} \rangle_{F^{\varepsilon}} - \langle u_{\varphi}^{\varepsilon} \rangle_{S_{+}^{\varepsilon}}|^2 + |\langle u_{\varphi}^{\varepsilon} \rangle_{S_{-}^{\varepsilon}} - \langle u_{\varphi}^{\varepsilon} \rangle_{F^{\varepsilon}}|^2 \right) \\ &\leq C\varepsilon^{2-n} \|\nabla u_{\varphi}^{\varepsilon}\|_{L_2(F^{\varepsilon})}^2. \end{aligned} \quad (3.59)$$

It follows from (3.59) and the Poincaré inequality that

$$\|u_{\varphi}^{\varepsilon}\|_{L_2(F^{\varepsilon})}^2 = \|u_{\varphi}^{\varepsilon} - \langle u_{\varphi}^{\varepsilon} \rangle_{F^{\varepsilon}}\|_{L_2(F^{\varepsilon})}^2 + |\langle u_{\varphi}^{\varepsilon} \rangle_{F^{\varepsilon}}|^2 \cdot |F^{\varepsilon}| \leq C\varepsilon^2 \|\nabla u_{\varphi}^{\varepsilon}\|_{L_2(F^{\varepsilon})}^2. \quad (3.60)$$

This means that similarly to the Dirichlet eigenfunction case the function $u_{\varphi}^{\varepsilon}$ satisfies the Friedrichs inequality in F^{ε} , despite the fact that $u_{\varphi}^{\varepsilon}$ does not vanish on ∂Y . The remaining part of the argument leading to (3.55) repeats literally the proof of (3.18): we approximate the eigenfunction $u_{\varphi}^{\varepsilon}$ by the function v_D^{ε} (3.22), noting that since v_D^{ε} vanishes in the vicinity of ∂Y^{ε} it belongs to $\text{dom}(\eta_{\varphi}^{\varepsilon})$ for an arbitrary φ , and then check the asymptotic equality

$$\lambda_1^{\varphi}(\varepsilon) \sim \eta_{\varphi}^{\varepsilon}[v_D^{\varepsilon}, v_D^{\varepsilon}], \quad \varphi = \pi. \quad (3.61)$$

Finally, the proof of (3.61) repeats word-by-word that of (3.36) taking into account the inequality (3.60). \square

Remark 3.2. With some slight modifications of the proof one is able to show that (3.55) holds in fact for all values of the parameter $\varphi \neq 0$. For our purposes, however, it is sufficient to consider antiperiodic (i.e. $\varphi = \pi$) eigenvalues only.

Now, with the preliminaries represented by Lemmata 3.1–3.4 it is not difficult to prove Theorem 2.1. Indeed, it follows from (3.3) that

$$\sigma(\mathcal{A}^{\varepsilon}) = \bigcup_{k=1}^{\infty} [\lambda_k^{-}(\varepsilon), \lambda_k^{+}(\varepsilon)], \quad (3.62)$$

where the spectral bands, i.e. the compact intervals $[\lambda_k^{-}(\varepsilon), \lambda_k^{+}(\varepsilon)]$, are defined by

$$[\lambda_k^{-}(\varepsilon), \lambda_k^{+}(\varepsilon)] = \bigcup_{\varphi \in [0, 2\pi)} \{\lambda_k^{\varphi}(\varepsilon)\}. \quad (3.63)$$

We set $\varphi_- = 0$, $\varphi_+ = \pi$. It follows from (3.4) and (3.63) that

$$\lambda_k^N(\varepsilon) \leq \lambda_k^-(\varepsilon) \leq \lambda_k^{\varphi_-}(\varepsilon), \quad (3.64)$$

$$\lambda_k^{\varphi_+}(\varepsilon) \leq \lambda_k^+(\varepsilon) \leq \lambda_k^D(\varepsilon). \quad (3.65)$$

Obviously, the left- and right-hand-sides of (3.64) are equal to zero if $k = 1$. It follows from (3.37), (3.56) that in the case $k = 2$ they both converge to β as $\varepsilon \rightarrow 0$, and consequently

$$\lambda_1^-(\varepsilon) = 0, \quad \lim_{\varepsilon \rightarrow 0} \lambda_2^-(\varepsilon) = \beta. \quad (3.66)$$

Similarly, in view of (3.5), (3.18), and (3.55) we obtain

$$\lim_{\varepsilon \rightarrow 0} \lambda_1^+(\varepsilon) = \alpha, \quad \lim_{\varepsilon \rightarrow 0} \lambda_2^+(\varepsilon) = \infty. \quad (3.67)$$

Then the relations (2.5)–(2.6) follow directly from (3.62) and (3.66)–(3.67) in combination with the monotonicity of the sequences $\{\lambda_k^\pm(\varepsilon)\}_{k \in \mathbb{N}}$. In this way, Theorem 2.1 is proved.

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