Existence of waves for a reaction-diffusion-dispersion system

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Abstract. Existence of travelling waves is studied for a bistable reaction-diffusion system of equations with linear integral terms (dispersion) and with some conditions on the nonlinearity. The proof is based on the Leray-Schauder method using the topological degree theory for Fredholm and proper operators with the zero index and a priori estimates of solutions in properly chosen weighted spaces.

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1 Introduction

Conventional diffusion terms in reaction-diffusion equations describe random motion of atoms and molecules in multi-component continuous media or random motion of individuals in population dynamics. There exist also other types of motion in various applications, including a long range dispersion in ecology [12, 13], in neural models [9, 11] or in phase field models [4]. In this work we present a method to study the existence of travelling waves of reaction-diffusion-dispersion equations. Consider the system of equations

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + SJ(u) + F(u), \tag{1.1}$$

where $u = (u_1, ..., u_n)$, $F = (F_1, ..., F_n)$, $J = (J_1, ..., J_n)$,

$$J_k(u) = \int_{-\infty}^{\infty} \phi_k(x-y)u_k(y,t)dy\ , \quad k = 1, ..., n,$$

$\phi_k(x)$ are non-negative even functions such that $\phi_k(x)\exp(r|x|)$ are integrable for some $r > 0$, $\int_{-\infty}^{\infty} \phi_k(x)dx = 1$; D is a diagonal matrix with positive diagonal elements d_i, S is a matrix with constant elements $s_{ij} \geq 0$, $i, j = 1, ..., n$.

We will consider in this work system (1.1) on the whole axis, $x \in \mathbb{R}$, and will study the existence of travelling wave solutions. It is a solution $u(x,t) = w(x - ct)$, which satisfies the second-order equation

$$Dw'' + SJ(w) + cw' + F(w) = 0, \tag{1.2}$$

where c is an unknown constant, the wave speed. We will look for solutions with some limits at infinity,

$$w(\pm\infty) = w_\pm, \tag{1.3}$$

where $w_\pm$ are solutions of the equation

$$\Phi(w) \equiv Sw + F(w) = 0. \tag{1.4}$$

We assume that $0 < w_+ < w_-$ (the inequalities are understood component-wise), and we will consider monotonically decreasing solutions of problem (1.2), (1.3). Next, suppose that the vector-function $F(w)$ is continuous together with its second derivatives and satisfies the following condition:

$$F_i(w) \leq 0 \quad \Rightarrow \quad \frac{\partial F_i(w)}{\partial u_j} > 0 \ , \quad j \neq i, \ j = 1, ..., n. \tag{1.5}$$

This condition is automatically satisfied for the scalar equation ($n = 1$). Let us recall the definitions of some related classes of systems.

Monotone systems. The system is called monotone if the right-hand side inequality in (1.5) is satisfied for all $w \in R^n$. This is the class of systems for which the maximum principle is applicable. Existence, uniqueness and stability of waves for the monotone reaction-diffusion systems without dispersion ($S = 0$) was studied in [16, 18]. The scalar equation with dispersion was considered in [4, 7]. There are numerous works with a nonlinear dependence on the integral terms (see [15] and the references therein). There are various applications of such systems in chemical kinetics and combustion, in population dynamics and biomedicine [5, 18].

Locally monotone systems. The reaction-diffusion system (without dispersion) is called locally monotone if the right-hand side inequality in (1.5) is satisfied for such w that $F_i(w) = 0$ [16, 18]. It is a more general class of systems also encountered in various applications. Let us note that if a function $F(w)$ satisfies the monotonicity condition, then the function $\tilde{F} = (\tilde{F}_1, ..., \tilde{F}_n)$, $\tilde{F}_i = g_i(w)F_i(w)$ is locally monotone for any $g_i(w) > 0$. This remark opens a wide range of applications of such systems. Locally monotone systems do not satisfy the maximum principle. The existence of waves for such systems is proved in [16, 18, 21], the stability of waves, in general, may not hold. Locally monotone systems with dispersion are not studied since the same methods are not applicable for them.

In this work we study locally monotone systems with dispersion. Local monotonicity is understood here in the sense of condition (1.5). The presence of the integral terms makes this condition more restrictive in comparison with the reaction-diffusion systems without dispersion. The proof of the wave existence is based on the Leray-Schauder method using the topological degree for elliptic operators in unbounded domains and a priori estimates of solutions in some weighted spaces. The degree is constructed for Fredholm and proper operators with the zero index [14, 16] (Section 2.2). The main result of this work is given in the following theorem.

Theorem 1.1. *Suppose that the vector-function* $\Phi(w) = Sw + F(w)$ *has two zeros* w_+ *and* w_- *such that* $w_+ < w_-$ *(the inequalities between the vectors are understood component-wise), and a finite number of zeros* $w_0^k, k = 1, ..., m$ *such that* $w_+ \leq w_0^k \leq w_-$*. Assume, next, that all eigenvalues of the matrices* $\Phi'(w_\pm)$ *lie in the left-half plane, and the matrices* $\Phi'(w_0^k), k = 1, ..., m$ *have some eigenvalues in the right-half plane. If condition (1.5) is satisfied, then problem (1.2), (1.3) has a monotonically decreasing solution for some value of* c*.*

Thus, we consider the bistable case where the limiting values at infinity w_+ and w_- are stable as solutions of the ODE system $du/dt = \Phi(u)$. The intermediate zeros are unstable.

In the next section we introduce the functional setting: operators, spaces, topological degree for Fredholm and proper operators. Section 3 begins with the modification of the Leray-Schauder method applied for some subclasses of solutions [16, 18, 22]. It is followed by the separation of monotone and non-monotone solutions which represent these subclasses and by a priori estimates of monotone solutions. These results allow us to prove the main existence theorem.

2 Operators and spaces

For the functional setting let us introduce the Hölder space $C^{k+\alpha}(\mathbb{R})$ consisting of vector-functions of class C^k, which are bounded and continuous on the real axis $\mathbb{R}$ together with their derivatives of order k, and such that the derivatives of order k satisfy the Hölder condition with the exponent $\alpha \in (0, 1)$. The norm in this space is the usual Hölder norm. Set $E^1 = C^{2+\alpha}(\mathbb{R})$, $E^2 = C^\alpha(\mathbb{R})$. Next, we introduce the weighted spaces E^1_μ and E^2_μ with $\mu(x) = \sqrt{1 + x^2}$. These spaces are equipped with the norms:

$$\|w\|_{E^i_\mu} = \|w\mu\|_{E^i}, \quad i = 1, 2.$$

Following [16, 18] we introduce the operators which will allow us to study solutions of problem (1.2), (1.3). Consider an infinitely differentiable vector-function $\eta(x)$ such that

$$\eta(x) = \begin{cases} w_- & , \quad x \leq -1 \\ w_+ & , \quad x \geq 1 \end{cases} .$$

Set $w = u + \eta$ and consider the operator

$$A(u) = D(u+\eta)'' + SJ(u+\eta) + c(u+\eta)' + F(u+\eta), \tag{2.1}$$

acting from E^1_μ into E^2_μ.

2.1 Fredholm property of linear operators

Consider the linearized operator

$$Lu = Du'' + SJ(u) + cu' + F'(w(x))u$$

acting from E^1 into E^2, and the corresponding limiting operators

$$L_\pm u = Du'' + SJ(u) + cu' + F'(w_\pm)u,$$

where the function $w(x)$ is replaced by its limiting values at infinity. By definition, the operator L satisfies the Fredholm property if it is normally solvable, the dimension of its kernel and the codimension of its image (the number of solvability conditions) are finite. The set of all complex numbers λ for which the operator $L - \lambda$ does not satisfy the Fredholm property is the essential spectrum of the operator L. In order to determine the essential spectrum of the operator L, we consider the equations

$$L_\pm u = \lambda u. \tag{2.2}$$

The operator L is normally solvable with a finite dimensional kernel if and only if the only bounded solution of equations (2.2) is zero. This assertion is proved in [3] in the case of the scalar equation and in [1, 2] for systems of equations. Applying the Fourier transform we obtain that this condition is equivalent to the condition

$$\det\Big(-D\xi^2 + S\tilde{\phi}(\xi) + ci\xi + F'(w_\pm) - \lambda E\Big) = 0 \tag{2.3}$$

for some $\xi \in \mathbb{R}$. Here $\tilde{\phi}(\xi) = (\tilde{\phi}_1(\xi), ..., \tilde{\phi}_k(\xi))$, and $\tilde{\phi}_k(\xi)$ is the Fourier transform of the function $\phi_k(x)$, E is the identity matrix.

Condition 2.1. Equations (2.2) do not have nonzero bounded solutions for any real $\lambda \geq 0$ or equality (2.3) does not hold for such λ and any real ξ.

If Condition 2.1 is satisfied, then the operator L is Fredholm with the zero index. It is used in the construction of the topological degree [3, 14].

Lemma 2.2. *If all eigenvalues of the matrices $\Phi'(w_\pm)$ lie in the left-half plane, then Condition 2.1 is satisfied.*

Proof. Since the kernels $\phi_k(x)$ are even functions, then their Fourier transforms $\tilde{\phi}_k(\xi)$ are real-valued functions. Furthermore,

$$\tilde{\phi}_k(0) = \int_{-\infty}^{\infty} \phi_k(y)dy = 1,$$

and

$$\tilde{\phi}_k(\xi) = \int_{-\infty}^{\infty} \phi_k(y) \cos(\xi y) dy < 1, \;\; \forall \xi \neq 0.$$

Since the matrices

$$M_\pm(\xi) = -D\xi^2 + S\tilde{\phi}(\xi) + F'(w_\pm)$$

have positive off-diagonal elements, then their principal eigenvalues $\lambda_\pm(\xi)$ adopt the maximal values for $\xi = 0$. Indeed, the principal eigenvalues of such matrices are real and they decrease with the decrease of the elements of the matrix. On the other hand, $M_\pm(0) = \Phi'(w_\pm)$. Hence

$$\lambda_\pm(\xi) \le \lambda_\pm^0 \;, \quad \xi \in \mathbb{R},$$

where $\lambda_\pm^0$ are the principal eigenvalues of the matrices $\Phi'(w_\pm)$. Therefore, all eigenvalues of the matrices $M_\pm(\xi) + ci\xi$ lie in the half-plane Re $\lambda \le \lambda_\pm^0$.

□

2.2 Properness and topological degree

Homotopy. According to the Leray-Schauder (LS) method, we consider the equation

$$Dw'' + SJ_\tau(w) + cw' + F(w) = 0 \tag{2.4}$$

depending on parameter $\tau \in [0, 1]$. Here

$$J_\tau(w) = (1 - \tau)J(w) + \tau w.$$

The corresponding operator $A_\tau(u)$

$$A_\tau(u) = D(u + \eta)'' + SJ_\tau(u + \eta) + c(u + \eta)' + F(u + \eta), \tag{2.5}$$

acts from E_μ^1 into E_μ^2. For $\tau = 0$ we have the original operator (2.1) and for $\tau = 1$ the model operator for which the degree is different from 0. We need to obtain a priori estimates of solutions of the equation $A_\tau(u) = 0$ independent of τ. We will use a modification of the LS method for some subclasses of solutions.

Properness. Let us recall that an operator is called proper on closed bounded sets if the intersection of an inverse image of any compact set with a closed bounded set is compact. Properness of general elliptic problems in unbounded domains holds in properly chosen weighted spaces if the essential spectrum lies in the left-half plane [14]. They may not be proper in the spaces without weight. Properness of the integro-differential operators is proved in [3] in the case of the scalar operators. The proof remains similar in the vector case.

Topological degree. If Condition 2.1 is satisfied, then the operator linearized about any function in E^1_μ satisfies the Fredholm property and has the zero index. The nonlinear operator is proper on closed bounded sets. This means that the inverse image of a compact set is compact in any closed bounded set in E^1_μ. The topological degree can be defined for this operator. All these properties can be found in [18, 19, 20, 21, 14]. Let us note that most of the methods of nonlinear analysis use the Fredholm property of the corresponding operators. Few results are available for non-Fredholm operators [23, 24].

Functionalization of the parameter. Solution $w(x)$ of equation (1.2) is invariant with respect to translation in space. Along with any solution $w(x)$, the functions $w(x+h)$ also satisfy this equation for any real h. This property of solutions of autonomous problems on the whole axis implies the existence of a zero eigenvalue of the linearized operator A'. Consequently, we cannot find the index of the solution (the index is understood here as the value of the degree with respect to a small ball containing the solution). Moreover, this family of solutions is not bounded in the weighted norm. Therefore, we cannot apply the Leray-Schauder method to study the existence of solutions.

In order to overcome these difficulties we introduce functionalization of the parameter c [18] (Chapter 2). This means that instead of the unknown constant c we introduce some given functional $c(w)$ such that $c(w(\cdot+h))$ is a monotone function of h with the values from $-\infty$ to ∞. Hence, equation $c(w(\cdot+h)) = c$ has a unique solution h for any wave speed c. Therefore, we obtain an equivalent problem without invariance of solutions with respect to translation in space. The linearized operator A' does not have zero eigenvalue.

3 Leray-Schauder method

3.1 Subclasses of solutions

We consider the operator equation

$$A_\tau(u) = 0, \tag{3.1}$$

where the operator $A_\tau(u) : E^1_\mu \to E^2_\mu$ is defined in Section 2. The homotopy is constructed in such a way that $A_0(u)$ corresponds to the original problem (1.2), (1.3) and $A_1(u)$ to the model problem. In order to apply the Leray-Schauder method, we need to verify two conditions: a priori estimates of solutions of equation (3.1) hold in the space E^1_μ and the value of the topological degree for the model operator is different from 0.

Suppose that the set of solutions $\mathcal{K}$ of equation (3.1) in the space E^1_μ can be represented as a union of two subsets $\mathcal{K}_1$ and $\mathcal{K}_2$ such that the following two conditions are satisfied:

(i) for any $u \in \mathcal{K}_1$ and $v \in \mathcal{K}_2$, the estimate

$$\|u-v\|_{E^1_\mu} \geq r \tag{3.2}$$

holds with some positive constant r independent of the choice of u and v. We call this property separation of solutions.

(ii) for any $u \in \mathcal{K}_1$

$$\|u\|_{E^1_\mu} \le R \tag{3.3}$$

with some positive constant R independent of u. This is a priori estimate of solutions from the first subset.

Thus, we have a priori estimates of solutions which belong to the class $\mathcal{K}_1$ but not of all possible solutions. Therefore, we need to modify the Leray-Schauder method in the following way. Denote by B a ball in the space E^1_μ which contains all solutions from the class $\mathcal{K}_1$. Since the operator $A_\tau(u)$ is proper [14], that is the inverse image of the compact set is compact in any bounded closed set, then the set of solutions in B is compact. For each solution $u \in \mathcal{K}_1$, consider a ball $b_r(u)$ of radius r and center u. Set

$$\Omega_r = \cup_{u \in \mathcal{K}} b_r(u).$$

Let us choose r small enough such that Ω_r contains all solutions from $\mathcal{K}_1$ and does not contain other solutions. Consider the topological degree $\gamma(A_\tau, \Omega_r)$. It is well defined since $A_\tau(u) \neq 0$ for $u \in \partial\Omega_r$. We suppose that the degree is different from 0 for the model problem, $\gamma(A_1, \Omega_r) \neq 0$. Therefore, $\gamma(A_0, \Omega_r) \neq 0$, and equation $A_0(u) = 0$ has a solution in Ω_r.

In the following sections we will implement this approach.

3.2 Separation of solutions

The two subclasses of solutions separated in the function space are monotone and non-monotone solutions. By monotone solutions we understand vector-functions $w(x)$ all components of which are monotonically decreasing. Non-monotone solutions do not satisfy this property. Let us stress that monotonicity is considered for the function $w(x) = u(x) + \eta(x)$, and not for the function $u(x)$, but we will still call $u(x)$ monotone solutions.

We will show that the properties (i) and (ii) hold for the problem under consideration. Suppose that (i) is not valid. Then there are two sequences, $u^i \in \mathcal{K}_1$ (monotone solutions) and $v^i \in \mathcal{K}_2$ (non-monotone solutions) such that $\|u^i - v^i\|_{E^1_\mu} \to 0$ as $i \to \infty$. We will show that this assumption leads to a contradiction.

If condition (ii) is satisfied, then the sequence u^i is bounded. From the properness of the operator A_τ on closed bounded sets [14, 21] it follows that it has a convergent subsequence. Without loss of generality we can assume that $\|u^i + \eta - w\|_{E^1_\mu} \to 0$ for some function $w \in E^1_\mu$. Therefore, $w'(x) \le 0$ for all $x \in \mathbb{R}$ (component-wise). We show that this inequality is strict.

Lemma 3.1. *Let $w(x)$ be a solution of problem (1.2), (1.3). If $w'(x) \le 0$ for all $x \in \mathbb{R}$ (component-wise) and $w(x) \not\equiv const$, then $w'(x) < 0$.*

Proof. Suppose that $w_i'(x_0) = 0$ for some $i = 1, ..., n$ and x_0. Since $w'(x) \le 0$ for all x, then $w_i''(x_0) = 0$. Hence, by virtue of the ith equation of system (1.2),

$$\sum_{j=1}^{n} s_{ij} J_j(w)(x_0) + F_i(w(x_0)) = 0.$$

Since $s_{ij} \geq 0$ and $w(x) > 0$, we have $J_i(w)(x) \geq 0$ and $F_i(w(x_0)) \leq 0$. We set $u_i(x) = -w_i'(x)$ and differentiate the ith equation of system (1.2). By virtue of the equalities

$$\frac{d}{dx} J_j(w) = \frac{d}{dx} \int_{-\infty}^{\infty} \phi_j(z) w_j(x-z) dz = \int_{-\infty}^{\infty} \phi_j(z) w_j'(x-z) dz =$$
$$\int_{-\infty}^{\infty} \phi_j(x-y) w_j'(y) dy = J_j(w')$$

we get

$$d_i u_i'' + c u_i' + \frac{\partial F_i}{\partial w_i} u_i - \sum_{j \neq i} \frac{\partial F_i}{\partial w_j} w_j' + s_{ii} J_i(u_i) - \sum_{j \neq i} s_{ij} J_j(w_j') = 0. \tag{3.4}$$

Since $\frac{\partial F_i}{\partial w_j} > 0$ (see (1.5)) and $w_j'(x) \leq 0$, we obtain

$$T(x_0) \equiv -\sum_{j \neq i} \frac{\partial F_i}{\partial w_j} w_j'(x_0) + s_{ii} J_i(u_i) - \sum_{j \neq i} s_{ij} J_j(w_j') \geq 0.$$

Assume, first, $w_i'(x) \not\equiv 0$ in any small interval $I(x_0)$ around x_0. If we take it sufficiently small, then the inequalities $\frac{\partial F_i}{\partial w_j} > 0$ hold in this interval and, consequently, $T(x) \geq 0$ in $I(x_0)$. Hence we obtain a contradiction with the maximum principle for equation (3.4) in $I(x_0)$ since $u_i(x) \geq 0$ in $I(x_0)$, $u_i(x_0) = 0$ and $u_i(x) \not\equiv 0$.

If $w_i'(x) \equiv 0$ in some interval I_0, then we repeat the previous construction in a slightly larger interval I and obtain a similar contradiction.

□

Next, we consider the sequence of non-monotone solutions v^i. For each such solution there is at least one point x_i where the derivative of one of the components of the function $v^i + \eta$ vanish. Suppose, first, that this sequence is bounded. From the convergence $\|v^i + \eta - w\|_{E^1_\mu} \to 0$ as $i \to \infty$ it follows that the derivative of the limiting function $w'(x)$ also vanish (for one of the components). We obtain a contradiction with Lemma 3.1. Therefore, the sequence x_i is not bounded. Without loss of generality we can assume that $x_i \to \infty$ as $i \to \infty$. This gives a contradiction with the following lemma.

Lemma 3.2. *Let $v(x)$ be a solution of system (1.2) such that $w(x) \to w_+$ as $x \to \infty$. Moreover, the matrix $\Phi'(w_+)$ has positive off-diagonal elements and negative principal eigenvalue (i.e. with the maximal real part). If $w'(x_0) < 0$ (component-wise) for some x_0 sufficiently large, then $w'(x) < 0$ for all $x \geq x_0$.*

Proof. Set $u(x) = -w'(x)$ and differentiate equation (1.2):

$$Du'' + cu' + SJ(u) + B(x)u = 0, \tag{3.5}$$

where $B(x) = F'(w(x))$, $u(x_0) > 0$, $u(x) \to 0$ as $x \to \infty$. Since the matrix $\Phi'(w_+) = S + F'(w_+)$ has positive off-diagonal elements and negative principal eigenvalue, then $\Phi'(w_+)p < 0$, where p is the principal eigenvector. Therefore, we can choose x_0 sufficiently large such that $(S + B(x))p < 0$ for all $x \geq x_0$.

We need to prove that $u(x) > 0$ for $x \geq x_0$. Suppose that this is not true. If $u(x) \geq 0$ for all $x \geq x_0$ and $u_j(x_1) = 0$ for some j and x_1, then we obtain a contradiction with the maximum principle. Therefore, we consider the case where one of the components of the function $u(x)$ becomes negative. Hence there exists a positive number t such that the function $\hat{u}(x) = u(x) + tp$ satisfies the following conditions: $\hat{u} \geq 0$ for all $x \geq x_0$, $\hat{u}(x_0) > 0$, $\hat{u}_j(x_2) = 0$ for some j and $x_2 > x_0$. It satisfies the following equation:

$$D\hat{u}'' + c\hat{u}' + SJ(\hat{u}) + B(x)\hat{u} + f(x) = 0, \tag{3.6}$$

where $f(x) = -t(S + B(x))p > 0$. Therefore, we obtain again a contradiction with the maximum principle. This contradiction proves the lemma.

□

Thus, we have proved the following theorem.

Theorem 3.3. *Let the system*

$$Dw'' + SJ_\tau(w) + cw' + F(w) = 0 \tag{3.7}$$

satisfy condition (1.5), $\Phi_\tau(w_\pm) = 0$ for some $w_\pm$, $w_+ < w_-$ (component-wise), and the matrices $\Phi'_\tau(w_\pm)$ have all eigenvalues in the left-half plane. Suppose that for any monotonically decreasing solution w_m of system (3.7) with the limits

$$w(\pm\infty) = w_\pm \tag{3.8}$$

the estimate

$$\|w_m - \eta\|_{E^1_\mu} \leq R \tag{3.9}$$

holds with some positive constant R independent of the solution and of the value of $\tau \in [0, 1]$. Then there exists a positive constant r such that

$$\|w_m - w_n\|_{E^1_\mu} \geq r, \tag{3.10}$$

where w_n is any non-monotone solution of problem (3.7), (3.8) possibly for a different value of τ, r does not depend on solutions and on τ.

3.3 A priori estimates

3.3.1 The sign of the speed

We begin with the auxiliary results on the sign of the wave speed.

Lemma 3.4. *Suppose that $v(x)$ is a decreasing positive function, $\phi(x)$ is even and non-negative. Then for any N*

$$\int_N^\infty dx \int_{-\infty}^\infty \phi(x-y)v(y)dy \geqslant \int_N^\infty dx \int_{-\infty}^\infty \phi(x-y)v(x)dy = \int_N^\infty v(x)dx. \tag{3.11}$$

If v is a positive and increasing function, then

$$\int_{-\infty}^N dx \int_{-\infty}^\infty \phi(x-y)v(y)dy \geqslant \int_{-\infty}^N dx \int_{-\infty}^\infty \phi(x-y)v(x)dy = \int_{-\infty}^N v(x)dx. \tag{3.12}$$

It is assumed that all these integrals exist.

Proof We have

$$\int_N^\infty dx \int_N^\infty \phi(x-y)v(y)dy = \int_N^\infty dx \int_N^\infty \phi(x-y)v(x)dy.$$

If v is decreasing, then

$$\int_N^\infty dx \int_{-\infty}^N \phi(x-y)v(y)dy \geqslant \int_N^\infty dx \int_{-\infty}^N \phi(x-y)v(x)dy$$

since $y \leqslant N \leqslant x$ in the domain of integration and, consequently, $v(y) \geqslant v(N) \geqslant v(x)$. Taking a sum of the last two relations, we obtain (3.11).

Consider now the second case. From the equality

$$\int_{-\infty}^N dx \int_{-\infty}^N \phi(x-y)v(y)dy = \int_{-\infty}^N dx \int_{-\infty}^N \phi(x-y)v(x)dy$$

and inequality

$$\int_{-\infty}^N dx \int_N^\infty \phi(x-y)v(y)dy \geqslant \int_{-\infty}^N dx \int_N^\infty \phi(x-y)v(x)dy,$$

which takes place since in the domain of integration $x \leqslant N \leqslant y$ and $v(x) \leqslant v(N) \leqslant v(y)$, we obtain (3.12).

□

Lemma 3.5. *Suppose that there exists a monotonically decreasing solution $w(x)$ of equation (1.2) such that $w(x) \to w_0$ as $x \to \infty$. Then $c > 0$.*

Proof. Set $v(x) = \int_x^\infty (w(y) - w_0)dy$. Taking into account that $Sw_0 + F(w_0) = 0$, we write equation (1.2) as

$$Dw'' + cw' + (S + F'(w_0))(w - w_0) + S(J(w) - w) + o(|w - w_0|) = 0.$$

We integrate this equation from y to infinity:

$$(S+F'(w_0))\frac{v(x)}{|v(x)|}+\frac{o(v(x))}{|v(x)|}=\frac{1}{|v(x)|}\left(Dw'(x)+c(w(x)-w_0)-\int_x^\infty S(J(w)-w(y))dy\right). \tag{3.13}$$

If $c \le 0$, then by virtue of (3.11) the right-hand side of (3.13) is a negative vector for any x. On the other hand, by virtue of the minimax representation of the principal eigenvalue λ_0 of the matrix with positive off-diagonal elements [10] (Chapter XIII), we have

$$\min_{|q|=1,q>0}\max_i \frac{((S+F'(w_0))q)_i}{q_i}=\lambda_0>0.$$

Therefore, equality (3.13) gives a contradiction which proves that $c>0$.

□

Lemma 3.6. *Suppose that there exists a monotonically decreasing solution $w(x)$ of equation (1.2) such that $w(x)\to w_0$ as $x\to-\infty$. Then $c<0$.*

3.3.2 Estimates of solutions

In this section we will obtain a priori estimates of monotone solutions in weighted Hölder spaces. Since the essential spectrum of the operator L lies in the left-half plane and it satisfies the Fredholm property, then the solutions converge to their limiting values at infinity exponentially [14]. In the other words, the following estimates hold:

$$|w_m(x)-\eta(x)|\le K_1e^{-\mu_0x},\;\; x\ge N_+,\quad |w_m(x)-\eta(x)|\le K_1e^{\mu_0x},\;\; x\le N_- \tag{3.14}$$

with some positive constants K_1 and μ_0 independent of a monotone solution w_m and the value of τ. On the contrary, the values N_+ and N_- can depend on the solution. They are chosen in such a way that

$$|w_m(x)-\eta(x)|\le\epsilon,\;\; x\ge N_+,\quad |w_m(x)-\eta(x)|\le\epsilon,\;\; x\le N_-$$

for some small positive ϵ. This means that estimates (3.14) hold in some neighborhoods of the points $w_\pm$ in $\mathbb{R}^n$ (w-space).

Since the weight function $\mu(x)$ has polynomial growth at infinity, then we obtain the estimate

$$|(w_m(x)-\eta(x))\mu(x)|\le K_2 \tag{3.15}$$

for $x\ge N_+$ and $x\le N_-$. If N_+ and N_- are uniformly bounded for all solutions, the last estimate obviously holds for all $x\in\mathbb{R}$.

Introducing functionalization of parameter, we consider the given functional $c(w)$ instead of the unknown wave speed c_τ which depends on τ. The value of the functional $c(w(\cdot+h))$ depends on the shift h. Therefore, we choose a single value of h such that

$$c(w(\cdot + h)) = c_\tau \tag{3.16}$$

and, consequently, remove the invariance of solution with respect to translation. By virtue of a priori estimates of the wave speed, solution h of equation (3.16) is uniformly bounded for all τ. Indeed, $c(w(\cdot + h)) \to \pm\infty$ as $h \to \pm\infty$. Hence, solutions $w(x+h)$ and $w(x+N_\tau^+)$ differ by a final value of shift. Consequently, estimate (3.15) for the latter implies a similar estimate for the former.

Let us consider the case where these values are not uniformly bounded. Suppose that $N_+^i \to \infty$ for some sequence of solutions w^i, and N_-^i remains bounded. Consider the shifted functions $v^i(x) = w^i(x - N_+^i)$. We have the equality $|v^i(0) - \nu(0)| = \epsilon$. We can choose a subsequence of the sequence $v^i(x)$ locally converging to some limiting function $v^0(x)$. It is a solution of system (3.7) for some τ, it is monotonically decreasing, and $|v^0(0) - \nu(0)| = \epsilon$. Hence $v^0(x) \to w_+$ as $x \to \infty$, and there exists a limit $v^* = v^0(-\infty)$. Clearly, $\Phi(v^*) = 0$. Since $N_+^i - N_i^i \to \infty$, then $|v^* - w_-| \geq \epsilon$. Thus, we have constructed a solutions with the limits

$$v^0(-\infty) = v^*, \quad v^0(\infty) = w_+, \quad v^* \neq w_\pm. \tag{3.17}$$

Similarly, for the shifted functions $u^i(x) = w^i(x - N_-^i)$ we obtain a limiting solution $u^0(x)$ with the limits

$$u^0(-\infty) = w_-, \quad u^0(\infty) = v_*, \quad v_* \neq w_\pm. \tag{3.18}$$

We can now prove the following theorem.

Theorem 3.7. *Let the system (3.7) be locally monotone, $\Phi_\tau(w_\pm) = 0$ for some $w_\pm$, $w_+ < w_-$, and the matrices $\Phi'_\tau(w_\pm)$ have all eigenvalues in the left-half plane. Suppose that for any other zero w^0 of the function $\Phi(w)$ such that $w_+ \leq w^0 \leq w_-$, the principal eigenvalue of the matrix $\Phi'(w^0)$ is positive. Then the estimate*

$$\sup_x |(w_m(x) - \eta(x))\mu(x)| \leq K \tag{3.19}$$

holds for any monotonically decreasing solution $w_m(x)$ of problem (3.7), (3.8) with a constant K independent of the solution.

Proof. Suppose that the assertion of the theorem does not hold. Then, as it is shown above, the values $N_\pm$ in (3.14) are not uniformly bounded. Suppose that there is a sequence of solutions w^i for which $N_+^i \to \infty$ as $i \to \infty$, and N_-^i remain bounded. Then there are solutions $v^0(x)$ with limits (3.17) and $u^0(x)$ with limits (3.18). The existence of the former implies that $c < 0$ and of the latter that $c > 0$ (Lemmas 3.5, 3.6). This contradiction proves that the assumption on $N_\pm$ cannot hold.

Similarly, we can consider the case where N_-^i tends to $-\infty$ and N_+^i remains bounded, or both of them are unbounded. Since the solutions are invariant with respect to translation in space, all these cases can be reduced to the case where the values N_-^i are bounded. The shift remains bounded due to a priori estimates of the wave speed [16, 18].

□

Corollary 3.8. *Let $u = w_m - \eta$, where w_m is a monotone solution of problem (3.7), (3.8). Then $\|u\|_{E^1_\mu} \le K$, where a positive constant K does no depend on the solution.*

Thus, we obtain a priori estimates of monotone solutions.

3.3.3 Wave speed estimate

The estimate of solution in the previous section uses the estimate of the wave speed. We will show that it is uniformly bounded for all values of parameter τ. We will obtain the estimate from above. The estimate from below can be obtained similarly. We begin with some auxiliary results.

Lemma 3.9. *Let $v(x,t)$ be a solution of the Cauchy problem for the equation*

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} + SJ(v) + G(v), \tag{3.20}$$

with the initial condition $v(x,0)$, where $G(v) \ge F(v)$ for all $v \in \mathbb{R}^n$ (component-wise) and

$$\frac{\partial G_i}{\partial v_j} > 0\ , \quad i,j = 1,...,n, \quad j \ne i. \tag{3.21}$$

If $v(x,0) \ge u(x,0)$ for all $x \in \mathbb{R}$, then $v(x,t) \ge u(x,t)$ for all $x \in \mathbb{R}$ and $t \ge 0$, where $u(x,t)$ is the solution of equation (1.1) with the initial condition $u(x,0)$.

Proof. Set $z = v - u$. Then

$$\frac{\partial z}{\partial t} = D\frac{\partial^2 z}{\partial x^2} + SJ(z) + H(z) + f(x,t), \tag{3.22}$$

where $H = (H_1, ..., H_n)$,

$$H_i(z) = G_i(v) - G_i(u) = (G_i(v_1, v_2, ..., v_n) - G_i(u_1, v_2, ..., v_n))+$$
$$(G_i(u_1, v_2, ..., v_n) - G_i(u_1, u_2, ..., v_n)) + ... + (G_i(u_1, u_2, ..., v_n) - G_i(u_1, u_2, ..., u_n)) =$$

$$a_i(x,t)z_i + \sum_{j\ne i} b_{ij}(x,t)z_j\ .$$

If, for certainty, $i = 1$, then

$$a_1(x,t) = \frac{G_1(v_1, v_2, ..., v_n) - G_1(u_1, v_2, ..., v_n)}{v_1 - u_1}\ ,$$

$$b_{12}(x,t) = \frac{G_1(u_1, v_2, ..., v_n) - G_1(u_1, u_2, ..., v_n)}{v_2 - u_2}\ , ...$$

Since $b_{ij}(x,t) > 0$, $f(x,t) = G(u) - F(u) \geq 0$, and $v(x,0) \geq u(x,0)$, then by virtue of the positiveness theorem [8, 18], $z(x,t) \geq 0$.

□

Lemma 3.10. *Equation*

$$Dw'' + cw' + SJ(w) + Bw = 0, \tag{3.23}$$

where B is a constant matrix with positive off-diagonal elements such that the $S + B$ has a positive principle eigenvalue, has a solution $w(x) = p\exp(-\lambda x)$, where $p > 0$ is a constant vector, $\lambda > 0$ is a real number, $\lambda \to 0$ as $c \to \infty$.

Proof. Substituting solution $w(x) = p\exp(-\lambda x)$ into equation (3.23), we get

$$\big(D\lambda^2 - c\lambda + SJ(\lambda) + B\big)\, p = 0, \tag{3.24}$$

where $J(\lambda) = (J_1(\lambda), ..., J_n(\lambda))$, $J_k(\lambda) = \int_\infty^\infty \phi(y)e^{\lambda y}dy$. Set

$$M(\lambda) = D\lambda^2 + SJ(\lambda) + B.$$

This matrix has a real principle eigenvalue $\mu(\lambda)$ and the corresponding positive eigenvector p. According to the condition of the lemma, $\mu(0) > 0$. Moreover $\mu(\lambda) \to \mu(0)$ as $\lambda \to 0$. Put $c = \mu(\lambda)/\lambda$. Then equality (3.24) holds and $\lambda = c\mu(\lambda) \to \infty$ as $\lambda \to 0$.

□

Theorem 3.11. *The estimate $|c| \leq K$ holds with a constant K independent of τ.*

Proof. Let us recall that we consider the equation

$$Dw'' + SJ_\tau(w) + cw' + F(w) = 0, \tag{3.25}$$

where $J_\tau(w) = (1-\tau)J(w) + \tau w$. This homotopy does not change the stationary points and their stability. Let us now study how the essential spectrum of the corresponding operator

$$L_\tau u = Du'' + SJ_\tau(u) + cu' + F'(w(x))u$$

depends on τ. Set

$$M_\tau^\pm(\xi) = -D\xi^2 + S((1-\tau)\tilde{\phi}(\xi) + \tau) + F'(w_\pm).$$

The essential spectrum is given by the eigenvalues $\lambda_\pm(\xi)$ of the matrix $M_\tau^\pm(\xi) + ci\xi$ for all real ξ. Since the matrix $F'(w_\pm)$ has positive off-diagonal elements, the matrix S has non-negative elements, $\tilde{\phi}(\xi)$ is a real-valued vector since the functions $\phi_k(x)$ are even, and D is a diagonal matrix, then the principal eigenvalue $\mu_\tau^\pm(\xi)$ of the matrix $M_\tau^\pm(\xi)$ is real. Moreover, for each τ fixed its maximal value is reached for $\xi = 0$. Indeed,

$$(1-\tau)\tilde{\phi}_k(0) + \tau = 1\ , \quad (1-\tau)\tilde{\phi}_k(\xi) + \tau < 1\ , \quad \forall \xi \neq 0.$$

Therefore, all eigenvalues of the matrix $M_\tau^\pm(\xi)$ lie in the half-plane Re $\lambda \le \mu_\tau^\pm(0)$ for all real ξ and $\tau \in [0,1]$. However, $M_\tau^\pm(0) = S + F'(w_\pm)$. Thus, all eigenvalues of the matrix $M_\tau^\pm(\xi) + ci\xi$ lie in the half-plane Re $\lambda \le \mu_0^\pm$, where $\mu_0^\pm$ is the principal eigenvalues of the matrix $S + F'(w_\pm)$, and

$$\sigma_{ess}(L_\tau) \in \{\lambda \in \mathbb{C}, \ \ \text{Re}\, \lambda \le \max(\mu_0^+, \mu_0^-) < 0\}. \tag{3.26}$$

Choose a positive number κ such that $\max(\mu_0^+, \mu_0^-) < -\kappa$. Then it follows from estimate (3.26) that for any solution $w_1(x)$ of equation (3.25) such that $w_1(x) \to w_+$ as $x \to \infty$, the estimate

$$|w_1(x) - w_+| \le K e^{-\kappa x} \tag{3.27}$$

holds with a positive constant K that can depend on solution [14].

Next, we can choose a constant matrix B with positive elements such that $F(w) \le B(w - w_+)$ for all $w \ge w_+$. According to Lemma 3.10 we can choose $c = c_0$ such that $w_0(x) = w_+ + p\exp(-\kappa x)$ is a solution of (3.23). It follows from (3.27) that the estimate

$$w_1(x) \le w_0(x-h) \ , \quad x \in \mathbb{R} \tag{3.28}$$

holds for h sufficiently large.

We can now compare the speed c_1 of the solution $w_1(x)$ and the speed c_0 of the solution $w_0(x)$. We consider the solution $u(x,t) = w_1(x - c_1 t)$ of the equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + SJ(u) + F(u) \tag{3.29}$$

and the solution $v(x,t) = w_0(x - c_0 t)$ of the equation

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} + SJ(v) + B(v - w_+). \tag{3.30}$$

From Lemma 3.9 and inequality (3.28) it follows that $u(x,t) \le v(x,t)$ for all $x \in \mathbb{R}$ and $t \ge 0$. Hence $c_1 \le c_0$. It remains to note that c_0 does not depend on τ. Similarly, the estimate of the speed from below can be obtained.

□

3.4 Existence of solutions

Proof of Theorem 1.1. We consider the operator equation

$$A_\tau(u) = 0, \tag{3.31}$$

where the operator $A_\tau(u) : E_\mu^1 \to E_\mu^2$ is defined in Section 2. The homotopy is constructed in such a way that $A_0(u)$ corresponds to the original problem (1.2), (1.3) and $A_1(u)$ to the model problem. In order to apply the Leray-Schauder method, we need to verify two conditions: a priori estimates of solutions of equation (3.31) in the space E^1_μ and that the value of the topological degree for the model operator is different from 0.

In the previous section we obtained a priori estimates of solutions from the class $\mathcal{K}_1$ which consists of monotonically decreasing solutions (component-wise). We now apply the modification of the Leray-Schauder method presented in Section 3.1. Denote by B a ball in the space E^1_μ which contains all solutions from the class $\mathcal{K}$. Since the operator $A_\tau(u)$ is proper [14], that is the inverse image of the compact set is compact in any bounded closed set, then the set of solutions in B is compact. For each solution $u \in \mathcal{K}_1$, consider a ball $b_r(u)$ of radius r and center u. Set

$$\Omega_r = \cup_{u \in \mathcal{K}_1} b_r(u).$$

If r is sufficiently small, then the set Ω_r does not contain solutions $u \notin \mathcal{K}$. Indeed, suppose that this is not true and there exists a sequence $r_n \to 0$ such that the corresponding sequence of solutions u_n belongs to the sets Ω_{r_n}. By virtue of compactness of the set of solution we conclude that there is a subsequence of this sequence which converges in E^1_μ to a solution from $\mathcal{K}_1$. This assertion contradicts Theorem 3.3.

Let us choose r small enough such that Ω_r contains all solutions from $\mathcal{K}_1$ and does not contain other solutions. Consider the topological degree $\gamma(A_\tau, \Omega_r)$. It is well defined since $A_\tau(u) \neq 0$ for $u \in \partial\Omega_r$. It remains to note that the existence of solutions for the problem without the integral terms ($\tau = 1$) is proved in [15, 16, 18], and $\gamma(A_1, \Omega_r) \neq 0$. Therefore, $\gamma(A_0, \Omega_r) \neq 0$, and equation $A_0(u) = 0$ has a solution in Ω_r. Thus, Theorem 1.1 is proved. $\square$

4 Example of applications

We will apply the results of the existence of waves to the system of equations

$$d_1 u'' + k_1 J_1(u) + cu' + F_1(u, v) = 0, \tag{4.1}$$

$$d_2 v'' + k_2 J_2(v) + cv' + F_2(u, v) = 0, \tag{4.2}$$

where

$$F_1(u, v) = g_1(u, v) f_1(u, v), \quad F_2(u, v) = g_2(u, v) f_2(u, v),$$

$$\frac{\partial f_1(u, v)}{\partial v} > 0, \quad \frac{\partial f_2(u, v)}{\partial u} > 0. \tag{4.3}$$

If $g_i(u, v) \equiv \text{const}$, $i = 1, 2$, then

$$\frac{\partial F_1(u, v)}{\partial v} > 0, \quad \frac{\partial F_2(u, v)}{\partial u} > 0,$$

and the system satisfies the monotonicity condition. If $g_i(u,v) > 0$ for all $u, v \in \mathbb{R}$, then

$$F_1(u,v) = 0 \ \Rightarrow \frac{\partial F_1(u,v)}{\partial v} > 0, \quad F_2(u,v) = 0 \ \Rightarrow \frac{\partial F_2(u,v)}{\partial u} > 0,$$

and the system satisfies the local monotonicity condition.

Suppose that

$$g_i(u,v) > 0, \ \ i = 1,2 \ \ \forall u, v \in \mathbb{R} \tag{4.4}$$

and

$$\frac{\partial g_1(u,v)}{\partial v} < 0, \quad \frac{\partial g_2(u,v)}{\partial u} < 0. \tag{4.5}$$

It follows from (4.3)-(4.5) that Condition 2.1 is satisfied, and Theorem 1.1 is applicable.

Let us consider the following example:

$$f_1(u,v) = u(1 - u - \alpha v) \ , \quad f_2(u,v) = v(1 - v - \beta u).$$

If $g_i(u,v) \equiv$ const and $k_1 = k_2 = 0$, then (4.1), (4.2) is the classical system of competition of species. If we set $\tilde{u} = 1 - u$, then we reduce it to the monotone system. We get a locally monotone system under condition (4.4). If, in addition, (4.5) is satisfied, then Condition 2.1 holds, wave existence in the case with dispersion follows from Theorem 1.1.

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References

[1] N. Apreutesei, A. Ducrot, V. Volpert. Travelling waves for integro-differential equations in population dynamics. Discrete Contin. Dyn. Syst. Ser. B, 11 (2009), No. 3, 541-561.

[2] A. Apreutesei, A. Ducrot, V. Volpert. Competition of species with intra-specific competition. Math. Model. Nat. Phenom., 3 (2008), No. 4, 1-27.

[3] N. Apreutesei, V. Volpert. Properness and topological degree for nonlocal reaction-diffusion operators. Abstr. Appl. Anal., Art. ID 629692, 21 pp.

[4] P. Bates, F. Chen. Periodic traveling waves for a nonlocal integro-differential model. Electron. J. Differential Equations, 1999, no. 26, 19 pp.

[5] N. Bessonov, N. Reinberg, V. Volpert. Mathematics of Darwins Diagram. Math. Model. Nat. Phenom., 9 (2014), no. 3, 5-25.

[6] G. Bocharov, A. Meyerhans, N. Bessonov, S. Trofimchuk, V. Volpert. Modelling the dynamics of virus infection and immune response in space and time. The International Journal of Parallel, Emergent and Distributed Systems, 2017, https://doi.org/10.1080/17445760.2017.1363203

[7] X. Chen. Existence, uniqueness, and asymptotic stability of travelling waves in nonlocal evolution equations. Adv. Differential Equations, (1997), 1-27.

[8] A. Ducrot, M. Marion, V. Volpert. Spectrum of some integro-differential operators and stability of travelling waves. Nonlinear Anal. Series A: Theory, Methods and Applications, 74 (2011), no. 13, 4455-4473.

[9] B. Ermentrout, J.B. McLeod. Existence and uniqueness of travelling waves for a neural network. Proc. Roy. Soc. Edinburgh, 123 (1993), 461-478.

[10] F. Gantmakher. The theory of matrices. Nauka. Moscow, 1967.

[11] J. Modolo, B. Bhattacharya, R. Edwards, J. Campagnaud, A. Legros, A. Beuter. Using a virtual cortical module implementing a neural field model to modulate brain rhythms in Parkinsons disease Frontiers in Neuroscience, 4 (2010), article 45.

[12] S. Petrovskii, A. Morozov. Dispersal in a statistically structured population: fat tails revisited. The American Naturalist, 173 (2009), no. 2, 278-289.

[13] A.R.E. Sinclair, J.M. Fryxell, G. Caughley. Wildlife Ecology, Conservation, and Management, 2nd ed. Blackwell Publishing, Oxford, 2006.

[14] V. Volpert. Elliptic partial differential equations. Volume 1. Fredholm theory of elliptic problems in unbounded domains. Birkhäuser, Basel, 2011.

[15] V. Volpert. Elliptic partial differential equations. Volume 2. Reaction-diffusion equations. Birkhäuser, Basel, 2014.

[16] A.I. Volpert, V.A. Volpert. Application of the theory of the rotation of vector fields to the investigation of wave solutions of parabolic equations. Trans. Moscow Math. Soc., 52 (1990), 59-108.

[17] A. Volpert, V. Volpert. Existence of multidimensional travelling waves and systems of waves. Comm. Partial Differential Equations, 26 (2001), no. 3-4, 421 - 459.

[18] A. Volpert, Vit. Volpert, Vl. Volpert. Traveling wave solutions of parabolic systems. Translation of Mathematical Monographs, Vol. 140, Amer. Math. Society, Providence, 1994.

[19] A.I. Volpert, V.A. Volpert. Construction of the Leray-Schauder degree for elliptic operators in unbounded domains. Ann. Inst. H. Poincare Anal. Non Lineaire, 11 (1994), no. 3, 245-273.

[20] V. Volpert, A. Volpert, J.F. Collet. Topological degree for elliptic operators in unbounded cylinders. Adv. Differential Equations, 4 (1999), no. 6, 777-812.

[21] V. Volpert, A. Volpert. Properness and topological degree for general elliptic operators. Abstr. Appl. Anal., (2003), no. 3, 129-181.

[22] V. Volpert, V. Vougalter. Method of monotone solutions for reaction-diffusion equations. Contemporary Mathematics. Fundamental Directions, 2018, in press.

[23] V.Vougalter, V.Volpert. On the existence of stationary solutions for some non-Fredholm integro-differential equations, Doc. Math., 16 (2011), 561-580.

[24] V.Vougalter, V.Volpert. Solvability conditions for some linear and nonlinear non-Fredholm elliptic problems, Anal. Math. Phys., 2 (2012), No. 4, 473-496.