EIGENVALUE BOUNDS FOR SCHRODINGER OPERATORS WITH RANDOM COMPLEX POTENTIALS

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ABSTRACT. We consider the Schrödinger operator perturbed by a random complex-valued potential. We obtain an estimate on the rate of accumulation of the eigenvalues of this operator to the positive half-line.

1. MAIN RESULT

In this paper, we study the behavior of eigenvalues of the operator

\[ H = -\Delta + V. \]

The potential \( V \) is assumed to be a complex-valued function of the form

\[ V = \sum_{n \in \mathbb{Z}^d} \omega_n v_n \chi(x - n), \quad v_n \in \mathbb{C}, \quad x \in \mathbb{R}^d, \]

where \( \omega_n \) are independent random variables taking values in the interval \([0, 1]\) and \( \chi \) is the characteristic function of the unit cube \([0, 1)^d\). We impose the condition \( \mathbb{E}[\omega_n] = 0 \) on \( \omega_n \) guaranteeing oscillations of \( V \). The coefficients \( v_n \) do not have to be real.

To formulate the main result, we set

\[ \tilde{V} = \sum_{n \in \mathbb{Z}^d} |v_n| \chi(x - n). \]

Note that \( \tilde{V} \) is a non-negative function such that \( |V| \leq \tilde{V} \).

Theorem 1.1. Let \( d \geq 3 \), let \( 0 < R_0 \leq 1 \), and let \( 1 < \nu < q < 2 \). Then the eigenvalues \( \lambda_j \) of the operator \(-\Delta + V\) satisfy

\[
\mathbb{E} \left[ \sum_{|\lambda_j| < R_0^d} \text{Im} \sqrt{\lambda_j}|\lambda_j|^{(q-1)/2} \right] \leq C |R_0|^{q-\nu} \left( \int_{\mathbb{R}^d} |\tilde{V}(x)|^p dx \right)^2, \tag{1.1}
\]

with

\[
p = \frac{d}{2} + \frac{d - q}{2(d - 2)}.
\]

It is assumed that \( \text{Im} \sqrt{\lambda_j} \geq 0 \). The constant \( C \) in (1.1) depends only on \( d, \nu \) and \( q \).

It is known that, if \( v_n \in \mathbb{R} \), then the eigenvalues \( \lambda_j \) obey the Lieb-Thirring estimate

\[
\sum_j |\lambda_j|^{\gamma} \leq C \int_{\mathbb{R}^d} |V(x)|^{d/2 + \gamma} dx, \quad V = \tilde{V}, \quad d \geq 3, \quad \gamma \geq 0. \tag{1.2}
\]

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Theorem 1.1 allows one to consider real potentials $V$ for which the right hand side of (1.2) is infinite for

$$1 < 2\gamma < d/(d - 1),$$

while the left hand side is finite almost surely. Similar results for real random potentials $V = \tilde{V}$ were obtained earlier in [5]. However, there is a big difference between Theorem 1.1 and the results of [5], since the only point of accumulation of eigenvalues of the operator $H$ in the case considered by the authors of [5] is the point $\lambda = 0$. When one studies complex-valued potentials, the fact that the eigenvalues $\lambda_j$ might accumulate to points other than $\lambda = 0$ should not be excluded. Examples of decaying complex potentials $V$ such that eigenvalues of $H = -\Delta + V$ accumulate to points of the positive real line $\mathbb{R}_+$ are constructed in [1].

Because of the difference between the cases of real and complex potentials, it would be more appropriate to ask what new information does Theorem 1.1 provide compared to [4] and [3], rather than realize that this theorem does not follow from the Lieb-Thirring estimate.

The next statement is an improvement of Theorem 1.1 for $3 \leq d \leq 5$.

**Theorem 1.2.** Let $3 \leq d \leq 5$ and let $0 < R_0 \leq 1$. Assume that $\tau_1$ satisfies

$$0 \leq \left( \left( \frac{d}{2} + \frac{(\eta - 1)(d + 1)}{7d} + \frac{d - \eta}{2(d - 2)} \right) - 2 \right) \tau_1 \leq \frac{(\nu - 1)(d + 1)}{7d}$$

with $\eta$ and $\nu$ such that $1 < \nu < \eta < 2$. If $d = 3$, then we assume additionally that $8\nu + 9\eta < 26$.

Let $p$, $q$ and $r$ be the numbers defined by

$$p = \frac{d}{7\tau_1}, \quad \frac{1}{q} = 1 - \frac{\theta}{p} + \frac{\theta}{2}, \quad \text{and} \quad \frac{1}{r} = \frac{1 - \theta}{2p} + \frac{\theta}{2},$$

where $\theta$ is the solution of the equation

$$\tau_1(1 - \theta) + \frac{\theta}{2} \left( \frac{d}{2} + \frac{d - \eta}{2(d - 2)} \right) = 1.$$

Then the eigenvalues $\lambda_j$ of the operator $-\Delta + V$ satisfy

$$\mathbb{E} \left[ \sum_{|\lambda_j| \leq R_0^2} \Im \sqrt{\lambda_j} |\lambda_j|^{(\sigma - 1)/2} \right] \leq C_{\tau_1, \sigma} |R_0|^{\sigma - \theta q \nu / 2} \left( \int_{\mathbb{R}^d} |\tilde{V}(x)|^r dx \right)^{2q/r}, \quad \sigma > \theta q \nu / 2.$$

Besides its dependence on $d$, the constant $C_{\tau_1, \sigma}$ in this inequality depends on a choice of the parameters $\tau_1$ and $\sigma$.

Theorem 1.2 gives new information about eigenvalues of $H$. Even in the case $V = \tilde{V}$, this theorem does not follow from the Lieb-Thirring estimates. It turns into Theorem 1.1 for dimensions $3 \leq d \leq 5$ once we set $\tau_1 = 0$. However, Theorem 1.2 allows one to consider ratios $\sigma/r$ which are smaller than the ratios $q/p$ allowed in Theorem 1.1. Thus, Theorem 1.2 is an improvement of Theorem 1.1 for dimensions $3 \leq d \leq 5$.

One of the difficulties we encountered in this paper is that Theorem 1.1 and Theorem 1.2 can not be obtained by taking expectations in the inequalities obtained by Borichev, Golinski, and Kupin [2]. This difficulty was overcome through an application of the Joukowsky transform to a half-plane with a removed semi-disk and consecutive integration with respect to the radius.
2. Preliminaries

Everywhere below, $\mathcal{S}_p$ denotes the class of compact operators $K$ obeying

$$\|K\|_p^p = \text{Tr}(K^*K)^{p/2} < \infty, \quad p > 1.$$  

Note that if $K \in \mathcal{S}_p$ for some $p > 1$, then $K \in \mathcal{S}_q$ for $q > p$ and $\|K\|_q \leq \|K\|_p$.

Let $z_j$ be the eigenvalues of a compact operator $K \in \mathcal{S}_n$ where $n \in \mathbb{N} \setminus \{0\}$. We define the $n$-th determinant of $I + K$ by

$$\det_n(I + K) = \prod_j (1 + z_j) \exp\left(\sum_{m=1}^{n-1} (-1)^m \frac{z_j^m}{m}\right), \quad n \geq 2;$$

$$\det(I + K) = \prod_j (1 + z_j), \quad n = 1.$$  

There exists a constant $C_n > 0$ depending only on $n$ such that

$$|\det_n(I + X)| \leq e^{C_n\|X\|_n^n}, \quad \forall X \in \mathcal{S}_n.$$  

Moreover, the following statement holds:

**Proposition 2.1.** Let $n \geq 2$. Then for any $n - 1 \leq p \leq n$, there exists a constant $C_{p,n} > 0$ depending only on $p$ and $n$ such that

$$|\det_n(I + X)| \leq e^{C_{p,n}\|X\|_p^p}, \quad \forall X \in \mathcal{S}_p, \quad n \geq 2.$$  

Let

$$X(k) = |V|^{1/2}(-\Delta - z)^{-1}V(-\Delta - z)^{-1}V|V|^{-1/2}, \quad z = k^2, \quad k \in \mathbb{C}_+.$$  

Let us also set

$$D_n(k) = \det_n(I - X(k)), \quad n = [p] + 1,$$

for a compactly supported $V$.

**Proposition 2.2.** Let $V$ be compactly supported. If a point $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ is an eigenvalue of $H = -\Delta + V$, then $D_n(k) = 0$ for $k = \sqrt{\lambda}$. The algebraic multiplicity of the eigenvalue $\lambda$ does not exceed the multiplicity of the root of the function $D_n(\cdot)$.

**Proof.** The statement about the multiplicity follows from the fact that eigenvalues of $H$ can be turned into simple eigenvalues by a small perturbation (which does not have to be a function). The rest is implied by the Birman-Schwinger principle, saying that a point $\lambda$ is an eigenvalue of $H$ if and only if $-1$ is an eigenvalue of $|V|^{1/2}(-\Delta - \lambda)^{-1}V|V|^{-1/2}$. Therefore, $1$ is an eigenvalue of $X(k)$.  

3. Operators of the Birman-Schwinger type

Let $a$, $b$ and $V$ be functions on $\mathbb{R}^d$. Define

$$A_\zeta = |a|^{\zeta} F V^{\zeta} F^* |b|^{\zeta},$$

where $F$ is the unitary Fourier transform operator. For any complex number $z$, we understand $V^z$ as the sum

$$V^z(x) := \sum_n \omega_n |v_n|^z e^{i \arg v_n} \chi(x - n).$$
Also, we will use the following notation
\[ \tilde{V} = \sum_n |v_n| \chi(x-n). \]

**Proposition 3.1.** Let \( a \in L^2 \), \( b \in L^2 \) and \( \tilde{V} \in L^2 \). Then
\[ \left( \mathbb{E} \left[ \| A_\zeta \|_{\mathcal{S}_p}^p \right] \right)^{1/p} \leq (2\pi)^{-2d/p} \| a \|_2^{2/p} \| b \|_2^{2/p} \| \tilde{V} \|_2^{2/p}, \quad \Re \zeta = 2/p. \tag{3.3} \]

*Proof.* We are going to prove (3.3) for one point \( \zeta_0 \) such that \( \Re \zeta_0 = 2/p \). For that purpose, we define the operator \( K(\omega) = |A_{\zeta_0}|^{p/2} \). Then, obviously,
\[ \beta := \mathbb{E} \left( \| K \|_{\mathcal{S}_2}^2 \right) = \mathbb{E} \left[ \| A_{\zeta_0} \|_{\mathcal{S}_p}^p \right]. \]

Let \( \Omega = \Omega(\omega) \) be the partially isometric operator appearing in the polar decomposition of \( A_{\zeta_0} = \Omega(\omega) |A_{\zeta_0}| \).

We introduce the analytic function
\[ f(\zeta) = \mathbb{E} [ \text{Tr} A_\zeta |K|^{2-\zeta} |K|^{i\Im \zeta} \Omega^*], \]
which will be treated by the three lines lemma. Since \( \| A_\zeta \| \leq 1 \) for \( \Re \zeta = 0 \), we obtain that
\[ |f(\zeta)| \leq \beta, \quad \text{for} \quad \Re \zeta = 0. \tag{3.4} \]

On the other hand,
\[ |f(\zeta)| \leq (2\pi)^{-d} \beta^{1/2} \| \tilde{V} \|_2 \| a \|_2 \| b \|_2, \quad \text{for} \quad \Re \zeta = 1, \tag{3.5} \]
by Hölder’s inequality. Using the three lines lemma, we obtain from (3.4) and (3.5) that
\[ |f(\zeta)| \leq (2\pi)^{-d \Re \zeta} \beta^{1-\Re \zeta/2} \| \tilde{V} \|_2^{\Re \zeta} \| a \|_2^{\Re \zeta} \| b \|_2^{\Re \zeta}. \]

Note now that \( f(\zeta_0) = \beta \). Consequently,
\[ \beta^{1/p} \leq (2\pi)^{-2d/p} \| \tilde{V} \|_2^{2/p} \| a \|_2^{2/p} \| b \|_2^{2/p}. \]

\[ \square \]

**Corollary 3.2.** Let \( T \) be a random operator of the form
\[ T = |a| F V F^* |b|, \]
with
\[ V(x) := \sum_n \omega_n v_n \chi_n(x). \]

Let \( a \in L^p \), \( b \in L^p \), \( v_n \in \ell^p \) and \( p \geq 2 \). Then
\[ \left( \mathbb{E} \left[ \| T \|_{\mathcal{S}_p}^p \right] \right)^{1/p} \leq (2\pi)^{-2d/p} \| a \|_p \| b \|_p \| \tilde{V} \|_p. \]

*Proof.* Observe that the functions \( |a|^{p/2}, \ |b|^{p/2} \) and \( V^{p/2} \) belong to \( L^2 \). Therefore, according to the proposition, the \( \mathcal{S}_p \)-norm of the operator
\[ \tilde{K} = |a|^{p/2} F V^{p/2} F^* |b|^{p/2} \]
obeys the inequality
\[ \left( \mathbb{E} \left[ \| \tilde{K} \|_{\mathcal{S}_p}^p \right] \right)^{1/p} \leq (2\pi)^{-2d/p} \| a \|_2^{p/2} \| b \|_2^{p/2} \| V^{p/2} \|_2^{p/2}, \quad \Re \zeta = 2/p. \]
The following result is a very well known bound obtained by E. Seiler and B. Simon [6]. Moreover, the reader can easily prove it using standard interpolation.

**Proposition 3.3.** Let $a$ and $W$ be two functions on $\mathbb{R}^d$. Let $T$ be the operator

$T = aFW,$

where $F$ is the operator of Fourier transform. Then

$$\|T\|_{L^p} \leq (2\pi)^{-d/p} \|a\|_p \|W\|_p, \quad p \geq 2.$$  

**Corollary 3.4.** Let $q \geq p \geq 2$. Let $T$ be a random operator of the form

$T = |a|FVF^*|b|$, 

with

$$V(x) := \sum_n \omega_n v_n \chi_n(x).$$

Let $a \in L^p$, $b \in L^q$ and $v_n \in \ell^p$. Then

$$\left( \mathbb{E}[\|T\|_{L^q}] \right)^{1/q} \leq (2\pi)^{-d/p - d/q} \|a\|_p \|b\|_q \|\tilde{V}\|_p.$$  

**Proof.** According to Proposition 3.3,

$$\|T\|_{L^p} \leq (2\pi)^{-d/p} \|a\|_p \|b\|_\infty \|\tilde{V}\|_p, \quad p \geq 2.$$  

On the other hand, according to Corollary 3.2,

$$\left( \mathbb{E}[\|T\|_{L^p}] \right)^{1/p} \leq (2\pi)^{-2d/p} \|a\|_p \|b\|_p \|\tilde{V}\|_p.$$  

It remains to interpolate between the two cases.

For that purpose, we introduce the function

$$f(\zeta) = \mathbb{E}\left[ \left( \text{Tr} K^{(1+q-p)(1-\zeta)/p+\zeta(p-1)(q-p)/p^2} \right) \text{Tr}|a|FVF^*|b|^{q/p}K^{p-1}\Omega^* \right],$$

where $K = |a|FVF^*|b|$ and $\Omega$ is the partially isometric operator appearing in the polar decomposition

$$|a|FVF^*|b| = \Omega K.$$  

For our convenience, we denote

$$\beta := \mathbb{E}\left[ \left( \text{Tr} K^{q/p} \right) \right]$$

If $\text{Re} \zeta = 0$, then by Hölder’s inequality,

$$|f(\zeta)| \leq (2\pi)^{-d/p} \beta \|a\|_p \|\tilde{V}\|_p.$$

If $\text{Re} \zeta = 1$, then

$$|f(\zeta)| \leq \mathbb{E}\left[ \left( \text{Tr} K^{(p-1)(q-p)/p^2} \right)^{(p-1)(q-p)/p^2} \|a|FVF^*|b|^{q/p}\|_{p} \left( \text{Tr} K^{(p-1)/p} \right)^{(p-1)/p} \right],$$

which leads to

$$|f(\zeta)| \leq \beta^{1-1/p} (2\pi)^{-2d/p} \|a\|_p \|b\|_q^{q/p} \|\tilde{V}\|_p.$$  

Observe also that

$$f(p/q) = \beta.$$
Thus, by the three lines lemma, we obtain that
\[
\beta \leq \beta^{1-1/q}(2\pi)^{-d/p-\delta/q}\|a\|_p\|b\|_q\|\hat{V}\|_p.
\]
The proof is completed.\qed

4. Small values of Re\(\zeta\)

The arguments of this paper will often rely on the properties of the operator \((-\Delta - z)^{-\zeta}\) for different values of \(\zeta\). In this section, we discuss the case \(0 \leq \text{Re}\ \zeta < 1\).

Let \(R > 0\). Let \(\chi_{0,k}\) be the characteristic function of the ball
\[
\mathfrak{B} = \left\{ \xi \in \mathbb{R}^d : |\xi| \leq \frac{2|k|}{R} \right\},
\]
and let \(\chi_{1,k} = 1 - \chi_{0,k}\) be the characteristic function of its complement
\[
\mathbb{R}^d \setminus \mathfrak{B} = \left\{ \xi \in \mathbb{R}^d : |\xi| > \frac{2|k|}{R} \right\}.
\]
We introduce the operators
\[
P_{n,k} = F\chi_{n,k}F^*,
\]
which are the spectral projections of \(-\Delta\) corresponding to the intervals \([0, 4|k|^2]\) and \((4|k|^2, \infty)\).

For any complex number \(z\), we understand \(V^z\) as the sum
\[
V^z(x) := \sum_n \omega_n|v_n|^2e^{i\arg v_n}\chi_n(x).
\]

In the next two propositions, we discuss the properties of the operators
\[
X_{n,m}(\zeta) = e^{\alpha_0\zeta^2}\left(WP_{n,k}(-\Delta - z)^{-\zeta}V(-\Delta - z)^{-\zeta}P_{m,k}W\right)
\]
for \(\text{Re}\ \zeta = \gamma/2\) and \(0 < \gamma < 3/2\). We will study the spectral properties of the operator
\[
Y_\gamma(\zeta) = X_{0,0}(\zeta) + X_{0,1}(\zeta) + X_{1,0}(\zeta).
\]
However, the terms in this representation will be studied separately. A this point, we do not discuss \(X_{1,1}(\zeta)\) at all. For the sake of convenience, we set
\[
\sigma = \frac{3}{\gamma}.
\]

**Proposition 4.1.** Let \(d \geq 2\). Let \(z \in \mathbb{C} \setminus \mathbb{R}_+\) and let \(2 \leq 2p < 3/\gamma\). Assume that \(0 < R \leq 1\). If \(\text{Re}\ \zeta = \gamma/2\) and \(\hat{V} \in L^{2p}\), then \(X_{0,0}(\zeta) \in \mathfrak{g}_p\) almost surely. Moreover,
\[
\mathbb{E}\left(||X_{0,0}(\zeta)||_{\mathfrak{g}_p}^p\right)^{1/p} \leq C_{p,\gamma}e^{-\alpha_{0}|\text{Im}\ \zeta|^2/2}\left(\frac{|k|}{R}\right)^{3d/2p-2\gamma}\|\hat{V}\|_p\|W\|_p^2.
\]

**Proof.** This statement follows from Corollary 3.2 and Proposition 3.3. If \(r = q/2 = 2p\), then \(1/r + 2/q = 1/p\). Moreover, since
\[
X_{0,0}(\zeta) = e^{\alpha_0\zeta^2}\left(W(-\Delta - z)^{-\zeta/3}P_{0,k}(-\Delta - z)^{-2\zeta/3}V(-\Delta - z)^{-2\zeta/3}P_{0,k}(-\Delta - z)^{-\zeta/3}W\right),
\]
we obtain the estimate
\[
\|\hat{X}_{0,0}(\zeta)\|_p \leq |e^{\alpha_0\zeta^2}| \cdot \|W(-\Delta - z)^{-\zeta/3}P_{0,k}\|_q.
\]
\[
\|P_{0,k}(-\Delta - z)^{-2\zeta/3}V(-\Delta - z)^{-2\zeta/3}P_{0,k}\|_r \|P_{0,k}(-\Delta - z)^{-\zeta/3}W\|_q.
\]
It remains to realize that
\[
\left( \int_{\mathbb{R}^d} \frac{\chi_{0,k}\,d\xi}{|\xi|^2 - z} \right)^{2/p} \leq \left( \int_{|\xi| > 2|k|} \frac{d\xi}{|\xi|^2 - z} \right)^{2/p} + C_{p,\gamma} e^{c} \text{Im} \xi \left( \int_{|\xi| < 2|k|/R} \frac{d\xi}{|\xi|^2} \right)^{2/r} \leq C_{p,\gamma} e^{c} \text{Im} \xi \left( \frac{|k|}{R} \right)^{2(d-2r/\gamma)/r}, \quad \gamma r < 3,
\]
while a similar argument shows that
\[
\left( \int_{\mathbb{R}^d} \frac{\chi_{0,k}\,d\xi}{|\xi|^2 - z} \right)^{2/q} \leq \tilde{C}_{p,\gamma} e^{c} \text{Im} \xi \left( \frac{|k|}{R} \right)^{2(d-q\gamma/\gamma)/q} = \tilde{C}_{p,\gamma} e^{c} \text{Im} \xi \left( \frac{|k|}{R} \right)^{d/2p-2\gamma/3}.
\]

\[\square\]

**Proposition 4.2.** Let \(2 \leq d \leq 5\). Let \(z \in \mathbb{C} \setminus \mathbb{R}_+\) and let \(2 \leq 2p < 3/\gamma\). Assume that \(4p\gamma > d\) and \(0 < R \leq 1\). If \(\text{Re} \xi = \gamma/2\) and \(\tilde{V} \in L^{2p}\), then \(X_{0,1}(\xi) \in \mathfrak{S}_p\) for all \(\omega\). Moreover,
\[
\|X_{0,1}(\xi)\|_{\mathfrak{S}_p} \leq C_{p,\gamma} e^{-\alpha|\text{Im} \xi|^{2/2}} \left( \frac{|k|}{R} \right)^{d/p-2\gamma} \|\tilde{V}\|_{2p} \|W\|_{4p}^2.
\]

**Proof.** Since
\[
X_{0,1}(\xi) = e^{\alpha\xi^2} \left( W(-\Delta - z)^{-\xi/3} P_{0,k}(-\Delta - z)^{-2\xi/3} V P_{1,k}(-\Delta - z)^{-\xi} W \right),
\]
we obtain the estimate
\[
\|X_{0,1}(\xi)\|_{\mathfrak{S}_p} \leq |e^{\alpha\xi^2}| \cdot \|W(-\Delta - z)^{-\xi/3} P_{0,k}\|_{4p} \|P_{1,k}(-\Delta - z)^{-2\xi/3} V\|_{2p} \|W\|_{4p}.
\]
It remains to realize that
\[
\left( \int_{\mathbb{R}^d} \frac{\chi_{0,k}\,d\xi}{|\xi|^2 - z} \right)^{1/(2p)} \leq \tilde{C}_{p,\gamma} e^{c} \text{Im} \xi \left( \frac{|k|}{R} \right)^{d/(2p)-2\gamma/3},
\]
while
\[
\left( \int_{\mathbb{R}^d} \frac{\chi_{0,k}\,d\xi}{|\xi|^2 - z} \right)^{1/(4p)} \leq \tilde{C}_{p,\gamma} e^{c} \text{Im} \xi \left( \frac{|k|}{R} \right)^{d/(4p)-\gamma/3}.
\]
Finally,
\[
\left( \int_{\mathbb{R}^d} \frac{\chi_{1,k}\,d\xi}{|\xi|^2 - z} \right)^{1/(4p)} \leq 2 e^{c} \text{Im} \xi \left( \int_{|\xi| > 2|k|/R} \frac{d\xi}{|\xi|^2} \right)^{1/(4p)} \leq \tilde{C}_{p,\gamma} e^{c} \text{Im} \xi \left( \frac{|k|}{R} \right)^{d/(4p)-\gamma}.
\]
\[\square\]

Let us now talk about the operator
\[
Y_{\gamma}(\xi) = X_{0,0}(\xi) + X_{0,1}(\xi) + X_{1,0}(\xi).
\]
The following estimate plays a very important role in our arguments.

**Corollary 4.3.** Let \(2 \leq d \leq 5\). Let \(|k| \geq R\) where \(0 < R \leq 1\). Assume that \(2 \leq 2p < 3/\gamma\) and \(4p\gamma > d\). If \(\text{Re} \xi = \gamma/2\) and \(\tilde{V} \in L^{2p}\), then
\[
\mathbb{E}(\|Y_{\gamma}(\xi)\|_{\mathfrak{S}_p})^{1/p} \leq C_{p,\gamma} e^{-\alpha|\text{Im} \xi|^{2/2}} \left( \frac{|k|}{R} \right)^{3d/2p-2\gamma} \|\tilde{V}\|_{2p}.
\]
In particular, we can set \(p = 1\) and prove the following statement.
Proposition 4.4. Let $2 \leq d \leq 5$. Let $|k| \geq R$ where $0 < R \leq 1$. Assume that

$$\frac{d}{8} < \frac{\gamma}{2} = \Re\zeta < \frac{3}{4}.$$  

Then

$$\mathbb{E}(||Y_\gamma(\zeta)||_{E_1}) \leq C_{\Re\zeta} e^{-\alpha_0|\Im\zeta|^2/2} \left(\frac{|k|}{R}\right)^{3d/2 - 4\Re\zeta} \|\tilde{V}\|_2.$$  

5. Large values of $\Re\zeta$

The arguments of this paper will often rely on the properties of the operator $(-\Delta - z)^{-\zeta}$ for different values of $\zeta$. In this section, we discuss the case $(d-1)/2 \leq \Re\zeta \leq (d+1)/2$.

Let $R > 0$. Let $\chi_{0,k}$ be the characteristic function of the ball

$$\mathfrak{B} = \left\{ \xi \in \mathbb{R}^d : |\xi| \leq \frac{2|k|}{R} \right\},$$

and let $\chi_{1,k} = 1 - \chi_{0,k}$ be the characteristic function of its complement

$$\mathbb{R}^d \setminus \mathfrak{B} = \left\{ \xi \in \mathbb{R}^d : |\xi| > \frac{2|k|}{R} \right\}.$$

We introduce the operators

$$P_{n,k} = F\chi_{n,k}F^*,$$

which are the spectral projections of $-\Delta$ corresponding to the intervals $[0, 4|k|^2]$ and $(4|k|^2, \infty)$. For any complex number $z$, we understand $V^z$ as the sum

$$V^z(x) := \sum_n \omega_n |v_n|^2 e^{i\arg v_n} \chi_n(x).$$

The following proposition plays an important role in our arguments.

Proposition 5.1. Let $d \geq 2$ and let $(d-1)/2 \leq \Re\zeta \leq (d+1)/2$. The integral kernel of the operator $(-\Delta - z)^{-\zeta}$ satisfies the estimate

$$|(-\Delta - z)^{-\zeta}(x, y)| \leq \beta e^{\alpha(|\Im\zeta|^2)|k|^{(d-1)/2} - \Re\zeta}|x - y|^{\Re\zeta - (d+1)/2}$$

for $z \notin \mathbb{R}_+$. The positive constants $\beta$ and $\alpha$ in this inequality depend only on $d$ and $\Re\zeta$.

The proof of this proposition, as well as related references, can be found in [4].

Proposition 5.2. Let $R \leq 1$. Let $d \geq 2$ and let $(d-1)/2 < \Re\zeta \leq (d+1)/2$. The integral kernel of the operator $P_{j,k}(-\Delta - z)^{-\zeta}$ satisfies the estimate

$$|P_{j,k}(-\Delta - z)^{-\zeta}(x, y)| \leq \beta e^{\alpha(|\Im\zeta|^2)|k|^{(d-1)/2} - \Re\zeta}|x - y|^{\Re\zeta - (d+1)/2}$$

for $z \notin \mathbb{R}_+$ and $j = 0, 1$. The positive constants $\beta$ and $\alpha$ in this inequality depend only on $d$ and $\Re\zeta$.

Proof. Due to Proposition 5.1, it is sufficient to prove only one of the inequalities (5.8). Let us first estimate the integrals

$$I_n = \int_{2^n|k| < R|\xi| < 2^{n+1}|k|} \frac{e^{i\zeta(x-y)}}{|\xi|^2 - k^2 \zeta} = -|x - y|^{-2} \int_{2^n|k| < R|\xi| < 2^{n+1}|k|} \frac{\Delta \zeta e^{i\zeta(x-y)}}{|\xi|^2 - k^2 \zeta} =$$
for \( n \geq 1 \). We will show that

\[
|I_n| \leq \beta e^{\alpha \left(\text{Re}\, \zeta\right)} \left(2^n |k|/R\right)^{(d-1)/2 - \text{Re}\, \zeta} |x - y|^{\text{Re}\, \zeta - (d+1)/2}
\]

for some \( \beta > 0 \) and \( \alpha > 0 \). A priori,

\[
|I_n| \leq C_d e^{2\pi|\text{Im}\, \zeta|} \left(2^n |k|/R\right)^{d - 2 \text{Re}\, \zeta},
\]

but the representation (5.9) leads to

\[
|I_n| \leq C_d e^{2\pi|\text{Im}\, \zeta|} \left(2^n |k|/R\right)^{d - 2 \text{Re}\, \zeta - 1} |x - y|^{-1}.
\]

The first estimate (5.11) implies (5.10) for \( 2^n |k||x - y| < R \), because in this case,

\[
|I_n| \leq C_d e^{2\pi|\text{Im}\, \zeta|} \left(2^n |k|/R\right)^{d - 2 \text{Re}\, \zeta} \left(2^n |k|/R\right)^{\text{Re}\, \zeta - (d+1)/2}.
\]

The second inequality (5.12) implies (5.10) for \( 2^n |k||x - y| \geq R \), because \((d+1)/2 - \text{Re}\, \zeta \leq 1\) and, therefore,

\[
\left(2^n |k|/R\right)^{d - 2 \text{Re}\, \zeta - 1} |x - y|^{-1} \leq \left(2^n |k|/R\right)^{d - 2 \text{Re}\, \zeta + \text{Re}\, \zeta - (d+1)/2} |x - y|^{\text{Re}\, \zeta - (d+1)/2}.
\]

The estimates (5.10) imply (5.8) for \( j = 1 \), because

\[
P_{1,k}(-\Delta - z)^{-\zeta}(x, y) = (2\pi)^{-d} \sum_{n=1}^{\infty} I_n.
\]

\[\square\]

**Corollary 5.3.** Let \((d - 1)/2 < \text{Re}\, \zeta < (d + 1)/2\), where \(d \geq 2\). Let \( 2 \leq r < 2d/2\text{Re}\, \zeta - 1 \). Suppose that \( W \) is a function of the form

\[
W(x) = \sum_{n \in \mathbb{Z}^d} w_n \chi_n(x), \quad w_n \in \mathbb{C}, \quad x \in \mathbb{R}^d.
\]

Then

\[
\|WP_{j,k}(-\Delta - z)^{-\zeta} \chi_l\|_{\mathfrak{A}_2} \leq \beta e^{\alpha (\text{Im}\, \zeta)^2} \left|k\right|^{(d-1)/2 - \text{Re}\, \zeta} \|W\|_r,
\]

for \( z \notin \mathbb{R}_+ \) and \( j = 0, 1 \). The positive constants \( \beta \) and \( \alpha \) in this inequality depend only on \( d \) and \( \text{Re}\, \zeta \). If \( \text{Re}\, \zeta = (d + 1)/2 \) and \( d \geq 2 \), then (5.13) holds with \( r = 2 \).

**Proof.** It follows from (5.8) that

\[
\|WP_{j,k}(-\Delta - z)^{-\zeta} \chi_l\|_{\mathfrak{A}_2}^2 \leq C e^{2\alpha (\text{Im}\, \zeta)^2} \left|k\right|^{(d-1) - 2 \text{Re}\, \zeta} \sum_{n \in \mathbb{Z}^d} \left|n - l + 1\right|^{2 \text{Re}\, \zeta - (d+1)} |w_n|^2.
\]

A simple application of Hölder’s inequality leads to (5.13). \(\square\)

On the other hand, we have the following inequality:

\[
\|WP_{j,k}(-\Delta - z)^{-\zeta} \chi_l\| \leq \beta e^{\alpha (\text{Im}\, \zeta)^2} \|W\|_{\infty},
\]

for \( \text{Re}\, \zeta = 0 \).

By interpolation we obtain from (5.13) and (5.14) that
Proposition 5.4. Let \((d - 1)/2 < \kappa < (d + 1)/2\), where \(d \geq 2\). Let \(2 \leq r < \frac{2d}{2\kappa - 1}\). Suppose that \(W\) is a function of the form
\[
W(x) = \sum_{n \in \mathbb{Z}^d} w_n \chi_n(x), \quad w_n \in \mathbb{C}, \quad x \in \mathbb{R}^d.
\]
Then, for any \(\Re \zeta = \tau \in (0, \kappa), \ z \notin \mathbb{R}_+\) and \(j = 0, 1\),
\[
\| WP_{j,k}(-\Delta - z)^{-\zeta} \chi_i \|_{\mathcal{S}_{2\kappa/\tau}} \leq \beta e^{\alpha(\Im \zeta)^2} |k|^{(d-1)/(2\kappa-1)\tau} \|W\|_{r\kappa/\tau}.
\] (5.15)
The positive constants \(\beta\) and \(\alpha\) in this inequality depend only on \(d\) and \(\tau\). If \(\kappa = (d + 1)/2\) and \(d \geq 2\), then (5.16) holds with \(r = 2\).

Proof. Indeed, let \(\Re \zeta_0 = \tau\) and let
\[
A = \Omega |A|
\]
be the polar decomposition of the operator
\[
A = |W|^{\zeta_0/\tau} P_{j,k}(-\Delta - z)^{-\zeta_0} \chi_i.
\]
Consider the function
\[
f(\zeta) = e^{\alpha \zeta^2} \Tr \left( |W|^{\zeta/\tau} P_{j,k}(-\Delta - z)^{-\zeta} \chi_i |A|^{(2\kappa - \zeta + \Im \zeta_0)/\tau} \Omega^* \right).
\]
If \(\Re \zeta = 0\), then
\[
|f(\zeta)| \leq C_1 \|A\|^{2\kappa/\tau}. \quad \text{(5.15)}
\]
If \(\Re \zeta = \kappa\), then
\[
|f(\zeta)| \leq C_2 |k|^{(d-1)/2 - \kappa} \|W\|^{2\kappa/\tau} \|A\| \|W\|^{2\kappa/\tau} \|\chi_i\|_{\mathcal{S}_{2\kappa/\tau}}. \quad \text{(5.15)}
\]
Consequently, by the three lines lemma,
\[
|f(\zeta)| \leq C |k|^\theta (d-1)/2 - \kappa \|W\|^{\theta \kappa/\tau} \|A\|^{(2 - \theta) \kappa/\tau}, \quad \theta = \tau / \kappa.
\]
Put differently,
\[
|e^{\alpha \zeta_0^2} \|A\|^{2\kappa/\tau} \|\chi_i\|_{\mathcal{S}_{2\kappa/\tau}} \leq C |k|^\theta (d-1)/2 - \kappa \|W\|^{\theta \kappa/\tau} \|A\|^{(2 - \theta) \kappa/\tau}, \quad \theta = \tau / \kappa.
\]
The latter inequality implies (5.16). The proof is completed. \(\square\)

In particular, once we set \(r\kappa/\tau = 4\), we obtain

Corollary 5.5. Let \((d - 1)/2 < \kappa < (d + 1)/2\), where \(d \geq 2\). Suppose that \(W\) is a function of the form
\[
W(x) = \sum_{n \in \mathbb{Z}^d} w_n \chi_n(x), \quad w_n \in \mathbb{C}, \quad x \in \mathbb{R}^d.
\]
Then
\[
\| WP_{j,k}(-\Delta - z)^{-\zeta} \chi_i \|_{\mathcal{S}_{4}} \leq \beta e^{\alpha(\Im \zeta)^2} |k|^{((d-1)/(2\kappa)-1)\Re \zeta} \|W\|_{4},
\] (5.16)
for any \(\kappa/2 \leq \Re \zeta < \min\{\kappa, d\kappa/(4\kappa - 2)\}\), \(z \notin \mathbb{R}_+\) and \(j = 0, 1\). The positive constants \(\beta\) and \(\alpha\) in this inequality depend only on \(d\) and \(\Re \zeta\). If \(\kappa = (d + 1)/2\) and \(d \geq 2\), then (5.16) holds with \(\Re \zeta = \kappa/2\).

We aslo need to introduce the operator
\[
X_{n,m}(\zeta) = e^{\alpha \zeta^2} WP_{n,k}(-\Delta - z)^{-\zeta} V(-\Delta - z)^{-\zeta} P_{m,k} W.
\]
Proposition 5.6. Let \((d - 1)/2 < \kappa < (d + 1)/2\), where \(d \geq 2\). Let \(\kappa/2 \leq Re \zeta < \min\{\kappa, d\kappa/(4\kappa - 2)\}\). Assume that \(V \in L^2(\mathbb{R}^d)\) and \(\alpha_0 > 2\alpha\). Then
\[
\left( E(\|X_{n,m}(\zeta)\|_{\mathcal{E}_2}^2) \right)^{1/2} \leq C Re e^{(2\alpha - \alpha_0)\|\Im \zeta\|^2/2} |k|^{((d-1)/\kappa - 2)Re \zeta} \|\hat{V}\|_2^2. \tag{5.17}
\]

If \(\kappa = (d + 1)/2\) and \(d \geq 2\), then (5.17) holds with \(Re \zeta = \kappa/2\).

Proof. Obviously,
\[
E\left(\|X_{n,m}(\zeta)\|_{\mathcal{E}_2}^2\right) = E\left(\text{Tr} X_{n,m}(\zeta)^* X_{n,m}(\zeta)\right) \leq e^{2\alpha_0 Re \zeta^2} \sum_{l \in \mathbb{Z}^d} |v_l|^2 \|WP_{n,k}(-\Delta - z)^{-\zeta} \|_{\mathcal{E}_4}^2 \|x_l(-\Delta - z)^{-\zeta} P_{m,k} W\|_{\mathcal{E}_4}^2.
\]

Together with Corollary 5.5, this implies (5.17). \(\Box\)

Corollary 5.7. Let \((d-1)/2 < \kappa < (d+1)/2\), where \(d \geq 2\). Let \(\kappa/2 \leq Re \zeta < \min\{\kappa, d\kappa/(4\kappa - 2)\}\). Assume that \(V \in L^2(\mathbb{R}^d)\) and \(\alpha_0 > 2\alpha\). Then
\[
\left( E(\|Y_{e}(\zeta)\|_{\mathcal{E}_2}^2) \right)^{1/2} \leq C Re e^{(2\alpha - \alpha_0)\|\Im \zeta\|^2/2} |k|^{((d-1)/\kappa - 2)Re \zeta} \|\hat{V}\|_2^2. \tag{5.18}
\]

If \(\kappa = (d + 1)/2\) and \(d \geq 2\), then (5.18) holds with \(Re \zeta = \kappa/2\).

6. An estimate for the square of the Birman-Schwinger operator

To obtain our first result about eigenvalues, we can interpolate between the case \(V \in L^\infty\) for \(Re \zeta = 0\) and the case \(V \in L^2\) for \(Re \zeta\) described in Corollary 5.5. Let
\[
\tilde{X}(k) = W(-\Delta - z)^{-1}V(-\Delta - z)^{-1}W, \quad z = k^2, \quad k \in \mathbb{C}_+.
\]

Proposition 6.1. Let \((d-1)/2 < \kappa < (d+1)/2\) if \(d \geq 3\); let \(1/2 < \kappa < 1\) if \(d = 2\). Let
\[
\max\{2, \kappa\} \leq p \leq \min\{2\kappa, d\kappa/(2\kappa - 1)\}. \tag{6.19}
\]

Let \(W = |V|^{1/2}\). Assume that \(\hat{V} \in L^p(\mathbb{R}^d)\). Then
\[
\left( E(\|\tilde{X}(k)\|_{\mathcal{E}_p}^p) \right)^{1/p} \leq C |k|^{((d-1)/\kappa - 2)} \|\hat{V}\|_p^2. \tag{6.20}
\]

If \(\kappa = (d + 1)/2\) and \(d \geq 3\), then (6.25) holds with \(p = \kappa\).

Let \(d \geq 3\). Fix the value of \(\nu\) so that \(1 < \nu < 2\) and choose \(q\) so that \(\nu < q < 2\). Let us now define \(p\) by
\[
p = \frac{d(d - 1) - q}{2(d - 2)} = \frac{d}{2} + \frac{d - q}{2(d - 2)}. \tag{6.21}
\]

Observe that the assumption \(\nu < q < 2\) leads to the inequalities
\[
\frac{d + 1}{2} < p < \frac{d(d - 1) - \nu}{2(d - 2)}. \tag{6.22}
\]

We also introduce the parameter \(\kappa\) setting
\[
\kappa = \frac{(d - 1)p}{2p - \nu}.
\]
The latter relation simply means that
\[ \nu = \left( 2 - \frac{(d-1)}{\kappa} \right) p. \] 
(6.23)

The second inequality in (6.22) implies
\[ \kappa > \frac{d(d-1) - \nu}{2(d-\nu)} > \frac{d-1}{2}, \] 
(6.24)
while the first inequality in (6.22) combined with the condition \( \nu < 2 \) implies that
\[ \kappa < \frac{d+1}{2}. \]

One can also see that the first inequality in (6.24) is equivalent to the estimate
\[ p = \frac{\kappa \nu}{2\kappa - (d-1)} < \frac{d\kappa}{2\kappa - 1}, \]
Finally, note that in \( d \geq 3 \), the condition \( p < 2 \) follows from the fact that \( \nu + q > 2 \).

Now, we can formulate the following

**Theorem 6.2.** Let \( d \geq 3 \) and let \( 1 < \nu < q < 2 \). Assume that \( W = |V|^{1/2} \). Then
\[ \mathbb{E}(\|X(k)\|_{\mathbb{E}_p}^p) \leq C|k|^{-\nu}\|	ilde{V}\|_p^2 \] 
(6.25)
for \( p \) defined by (6.21).

7. **Proof of Theorem 1.1**

We will work with the function
\[ d(z) = \det_n(I - X(k)), \quad n = [p] + 1, \]
where \( z \) is related to \( k \) via the Joukowski mapping
\[ z = \frac{R}{k} + k, \quad R > 0, \]
which maps the set \( \{ k \in \mathbb{C} : \text{Im } k > 0, |k| > R \} \) onto the upper half-plane \( \{ z \in \mathbb{C} : \text{Im } z > 0 \} \). Rather standard arguments lead to the estimate
\[ \sum_j \text{Im } z_j \leq C \int_{-\infty}^{\infty} \ln |d(z)|dz, \] 
(7.26)
where \( z_j \) are zeros of the function \( d(z) \) situated in the upper half-plane \( \mathbb{C}_+ \). In fact, (7.26) could be established in the same way as Jensen’s inequality for zeros of an analytic function on a unit disc. In (7.26) we assume that \( V \) is compactly supported. The relation (7.26) leads to the estimate
\[ \sum_j \left( \frac{|k_j|^2 - R^2}{|k_j|^2 R} \right) \text{Im } k_j \leq C \left( \int_{-\infty}^{\infty} \|X(k)\|_p^p \left( \frac{1}{R} - \frac{R}{K^2} \right) dk + \int_0^{\pi} \|X(R \cdot e^{i\theta})\|_p^p \sin \theta d\theta \right), \]
Taking the expectation we obtain
\[ \sum_j \mathbb{E} \left[ \frac{\text{Im } k_j(|k_j|^2 - R^2)}{|k_j|^2 R} \right] \leq C \left( \int_{-\infty}^{\infty} \mathbb{E}[\|X(k)\|_p^p] \left( \frac{1}{R} - \frac{R}{K^2} \right) dk + \int_0^{\pi} \mathbb{E}[\|X(R \cdot e^{i\theta})\|_p^p] \sin \theta d\theta \right). \]
Due to Theorem 6.2, the latter inequality leads to
\[
\sum_j \mathbb{E} \left[ \frac{\text{Im} k_j (|k_j|^2 - R^2)}{|k_j|^2 R} \right] \leq C R^{-\nu} \| \hat{V} \|_{2p}^2.
\] (7.27)

Now, suppose that we consider only the eigenvalues \( \lambda_j = k_j^2 \) that satisfy the inequality
\[ |k_j| \leq R_0. \]

Multiplying (9.42) by \( R^{q-1} \) and integrating with respect to \( R \) from 0 to \( R_0 \), we obtain
\[
\sum_{|k_j| \leq R_0} \mathbb{E} [\text{Im} k_j |k_j|^{q-1}] \leq C |R_0|^{q-\nu} \| \hat{V} \|_{2p}^2, \quad q > \nu.
\] (7.28)

This implies Theorem 1.1

8. Interpolation between small and large values of Re \( \zeta \)

Let us recall two theorems that hold for the operator
\[ Y_{\gamma}(\zeta) = X_{0,0}(\zeta) + X_{0,1}(\zeta) + X_{1,0}(\zeta). \]

By small values of Re \( \zeta \) we mean the values that are considered in Corollary 4.3, which states that, for any \( p \geq 1 \) and \( d/(8p) < \text{Re} \zeta < 3/(4p) \),
\[
\mathbb{E} (\| Y_{\gamma}(\zeta) \|_{\mathcal{H}_p}^p)^{1/p} \leq C_{\text{Re} \zeta, p} e^{-a_0 |\text{Im} \zeta|^{2}/2} \left( \frac{|k|}{R} \right)^{3d/2p-4 \text{Re} \zeta} \| \hat{V} \|_{2p}^2.
\] (8.29)

In this corollary we had to assume that \( 2 \leq d \leq 5 \) and \( |k| \geq R \) where \( 0 < R \leq 1 \). One should not forget also that our assumptions about \( \gamma = 2 \text{Re} z \) imply that \( \text{Re} z < 3/4 \).

The next result follows once we change \( 4 \text{Re} z \) to \( d/(2p) \) in the right hand side of (8.29).

**Theorem 8.1.** Let \( 2 \leq d \leq 5 \). Let
\[ 0 < \text{Re} \zeta < 3/4. \]

Assume that
\[ \frac{d}{8 \text{Re} \zeta} < p < \frac{3}{4 \text{Re} \zeta}, \quad p \geq 1, \]
and \( 0 < R \leq 1 \). Then
\[
\mathbb{E} (\| Y_{\gamma}(\zeta) \|_{\mathcal{H}_p}^p)^{1/p} \leq C_{\text{Re} \zeta, p} e^{-a_0 |\text{Im} \zeta|^{2}/2} \left( \frac{|k|}{R} \right)^{d/p} \| \hat{V} \|_{2p}^2
\]
for \( |k| \geq R \).

For the sake of simplicity, we choose
\[ p = \frac{d}{7 \text{Re} \zeta} \]
In this case, because of the assumption \( p \geq 1 \) that we made, we have to assume that
\[ 0 < \text{Re} \zeta \leq \frac{d}{7}. \]

Note that \( d/7 < 3/4 \). Thus, we can formulate the following assertion:
Corollary 8.2. Let $2 \leq d \leq 5$. Let $0 < \Re \zeta \leq d/7$ and let $p = \frac{d}{\Re \zeta}$. Assume that $0 < R \leq 1$. Then
\[
\mathbb{E}(||Y_{\gamma}(\zeta)||_{L^p}^p)^{1/p} \leq C_{\Re \zeta, p} e^{-\alpha_0} |\Im \zeta|^{d/2} \left(\frac{|k|}{R}\right)^{d/p} \|\tilde{V}\|_2^2
\]
for $|k| \geq R$.

By the large values of $\Re \zeta$ we mean the values appearing in Corollary 5.7. We will use only a simpler version of this result. Let $d \geq 3$. Fix the value of $\nu$ so that $1 < \nu < 2$ and choose $\eta$ so that $\nu < \eta < 2$. Let us now consider $\Re \zeta$ given by
\[
2 \Re \zeta = \frac{d(d - 1) - \eta}{2(d - 2)} = \frac{d}{2} + \frac{d - \eta}{2(d - 2)}.
\] (8.30)

Observe that the assumption $\nu < \eta < 2$ leads to the inequalities
\[
\frac{d + 1}{2} < 2 \Re \zeta < \frac{d(d - 1) - \nu}{2(d - 2)}.
\] (8.31)

We also introduce the parameter $\kappa$ setting
\[
\kappa = \frac{2(d - 1) \Re \zeta}{4 \Re \zeta - \nu}.
\]

The latter relation simply means that
\[
\nu = \left(2 - \frac{(d - 1)}{\kappa}\right)2 \Re \zeta.
\] (8.32)

The second inequality in (8.31) implies
\[
\kappa > \frac{d(d - 1) - \nu}{2(d - \nu)} > \frac{d - 1}{2},
\] (8.33)

while the first inequality in (8.31) combined with the condition $\nu < 2$ implies that
\[
\kappa < \frac{d + 1}{2}.
\]

One can also see that the first inequality in (8.33) is equivalent to the estimate
\[
2 \Re \zeta = \frac{\kappa \nu}{2\kappa - (d - 1)} < \frac{d\kappa}{2\kappa - 1}.
\]

Finally, note that in $d \geq 3$, the condition $\Re \zeta < \kappa$ follows from the fact that $\nu + \eta > 2$.

Consequently, Corollary 5.7 implies the following statement:

Theorem 8.3. Let $d \geq 3$. Let $1 < \nu < \eta < 2$. Let
\[
2 \Re \zeta = \frac{d}{2} + \frac{d - \eta}{2(d - 2)}.
\]

Assume that $V \in L^2(\mathbb{R}^d)$ and $\alpha_0 > 2\alpha$. Then
\[
\left(\mathbb{E}(||Y_{\gamma}(\zeta)||_{L^p}^2)^{1/2} \leq C_{\Re \zeta} e^{|(2\alpha - \alpha_0)(\Im \zeta)|^2} |k|^{-\nu/2} \|\tilde{V}\|_2^2.
\] (8.34)
We will interpolate between Corollary 8.2 and Theorem 8.3. Let us choose now \( \tau_1 \) so that
\[
0 < \tau_1 \leq d/7 \quad \text{and set} \quad p = \frac{d}{\tau_1}.
\]
Consider now \( Y_\gamma(\zeta) \) for
\[
\tau_1 \leq \text{Re} \zeta \leq \frac{d}{4} + \frac{d - \eta}{4(d - 2)}
\]
After that we find \( \theta \in (0, 1) \) satisfying the equation
\[
(1 - \theta)\tau_1 + \frac{\theta}{2} \left( \frac{d}{2} + \frac{d - \eta}{2(d - 2)} \right) = 1
\]
and define \( q \) and \( r \) so that
\[
\frac{1}{q} = \frac{1 - \theta}{p} + \frac{\theta}{2}, \quad \text{and} \quad \frac{1}{r} = \frac{1 - \theta}{2p} + \frac{\theta}{2}.
\]
Since \( p = \frac{d}{\tau_1} \), we obtain by interpolation that
\[
\left( E\left( \|Y_\gamma(1)\|_{L_q^p} \right) \right)^{1/q} \leq C_q \left( \frac{|k|}{R} \right)^{d(1 - \theta)/p} |k|^{-\theta \nu/2} \|\tilde{V}\|_2^p.
\]
To guarantee integrability of
\[
E\left( \|Y_\gamma(1)\|_{L_q^p} \right)
\]
at infinity, we need the parameters to satisfy the condition
\[
qd(1 - \theta)/p - q\theta\nu/2 < -1,
\]
which is equivalent to the inequality
\[
\tau_1(1 - \theta) < \frac{\theta\nu}{14} - \frac{1}{7q} = \frac{\theta(\nu - 1)}{14} - \frac{(1 - \theta)\tau_1}{d},
\]
implicating that
\[
\tau_1(1 - \theta) < \frac{\theta(\nu - 1)(d + 1)}{14d}.
\]
The latter can be written in the form
\[
1 - \frac{\theta}{2} \left( \frac{d}{2} + \frac{d - \eta}{2(d - 2)} \right) < \frac{\theta(\nu - 1)(d + 1)}{14d}.
\]
Put differently,
\[
2 < \theta \left( \frac{d}{2} + \frac{(\nu - 1)(d + 1)}{7d} + \frac{d - \eta}{2(d - 2)} \right) \quad (8.35)
\]
The condition that \( \theta \) is large can be converted into an inequality showing that \( \tau_1 \) is small. The relation (8.35) is satisfied, if
\[
\left( \left( \frac{d}{2} + \frac{(\nu - 1)(d + 1)}{7d} + \frac{d - \eta}{2(d - 2)} \right) - 2 \right) \tau_1 < \frac{(\nu - 1)(d + 1)}{7d}
\]
Since \( \eta > \nu \), that condition is obviously fulfilled, if
\[
0 \leq \left( \left( \frac{d}{2} + \frac{(\eta - 1)(d + 1)}{7d} + \frac{d - \eta}{2(d - 2)} \right) - 2 \right) \tau_1 \leq \frac{(\nu - 1)(d + 1)}{7d}. \quad (8.36)
\]
In this case, we also have

$$
\tau_1 < \begin{cases} 
\frac{2(\nu-1)(d+1)}{Ne^d - d} \leq \frac{d}{r}, & \text{if } d > 3, \\
\frac{8(\nu-1)}{21(2-\eta)} \leq \frac{d}{r}, & \text{if } 8\nu + 9\eta < 26, \text{ and } d = 3,
\end{cases}
$$

Let us try to formulate the result.

**Theorem 8.4.** Let $3 \leq d \leq 7$. Assume that $\tau_1$ satisfies (8.36) with $\eta$ and $\nu$ such that $1 < \nu < \eta < 2$. If $d = 3$, then we assume additionally that $8\nu + 9\eta < 26$. Let $p$, $q$ and $r$ be the numbers defined by

$$
p = \frac{d}{7}, \quad \frac{1}{q} = \frac{1 - \theta}{p} + \frac{\theta}{2}, \quad \text{and} \quad \frac{1}{r} = \frac{1 - \theta}{2p} + \frac{\theta}{2}, \tag{8.37}
$$

where $\theta$ is the solution of the equation

$$
\tau_1(1 - \theta) + \frac{\theta}{2} \left( \frac{d}{2} + \frac{d - \eta}{2(d - 2)} \right) = 1. \tag{8.38}
$$

Then

$$
\left( \mathbb{E}(\|Y_\gamma(1)\|_{\ell_q}) \right)^{1/q} \leq C_q \left( \frac{|k|}{R} \right)^{d(1-\theta)/p} |k|^{-\theta\nu/2} \|\tilde{V}\|_p^2,
$$

for $|k| \geq R$ and $0 < R \leq 1$.

In the next statement, we estimate the remainder $X_{1,1}(\zeta)$ for $\zeta = 1$.

**Theorem 8.5.** Let $p > 3d/4$ and let $\zeta = 1$. Then

$$
\mathbb{E}[\|X_{1,1}(\zeta)\|_{\ell_p}]^{1/p} \leq C \left( \frac{|k|}{R} \right)^{-d} \|\tilde{V}\|_p^2.
$$

*Proof.* In this theorem, we deal with the operator

$$
W(-\Delta - z)^{-1}P_{1,k}V(-\Delta - z)^{-1}P_{1,k}W
$$

On the one hand, we see that

$$
\mathbb{E}[\|(-\Delta - z)^{-2/3}P_{1,k}V(-\Delta - z)^{-2/3}P_{1,k}\|_{\ell_p}]^{1/p} \leq C \left( \int_{|\xi| > 2|k|/R} |\xi|^2 - z \right)^{-2p/3} d\xi \mathbb{E}[\|\tilde{V}\|_p^2],
$$

which implies the inequality

$$
\mathbb{E}[\|(-\Delta - z)^{-2/3}P_{1,k}V(-\Delta - z)^{-2/3}P_{1,k}\|_{\ell_p}]^{1/p} \leq C \left( \frac{|k|}{R} \right)^{-8/3} \|\tilde{V}\|_p, \quad p > 3d/4.
$$

On the other hand,

$$
\|W(-\Delta - z)^{-1/3}P_{1,k}\|_{\ell_{2p}}^2 \leq C \left( \frac{|k|}{R} \right)^{-4/3} \|\tilde{V}\|_p, \quad p > 3d/4.
$$

Consequently,

$$
\mathbb{E}[\|W(-\Delta - z)^{-1}P_{1,k}V(-\Delta - z)^{-1}P_{1,k}W\|_{\ell_p}]^{1/p} \leq C \left( \frac{|k|}{R} \right)^{-4} \|\tilde{V}\|_p^2, \quad p > 3d/4.
$$

The next statement follows by Hölder’s inequality.
Corollary 8.6. Let \( q > 3d/8 \geq 1 \) and let \( \zeta = 1 \). Then

\[
\mathbb{E}[||X_{1,1}(\zeta)||_{\mathfrak{E}_q}^q]^{1/q} \leq C \left( \frac{|k|}{R} \right)^{-4} \|\tilde{V}\|_{2q}^2.
\]

Surprisingly, \( q \) in (8.37) satisfies the inequality \( q > 3d/8 \geq 1 \). Thus, we obtain the following result.

Theorem 8.7. Let \( 3 \leq d \leq 5 \). Assume that \( \tau_1 \) satisfies (8.36) with \( \eta \) and \( \nu \) such that \( 1 < \nu < \eta < 2 \). If \( d = 3 \), then we assume additionally that \( 8\nu + 9\eta < 26 \). Let \( p, q \) and \( r \) be the numbers defined by

\[
p = \frac{d}{\tau_1}, \quad \frac{1}{q} = \frac{1 - \theta}{p} + \frac{\theta}{2}, \quad \text{and} \quad \frac{1}{r} = \frac{1 - \theta}{2p} + \frac{\theta}{2},
\]

where \( \theta \) is the solution of the equation

\[
\tau_1(1 - \theta) + \frac{\theta}{2} \left( \frac{d}{2} + \frac{d - \eta}{2(d - 2)} \right) = 1.
\]

Then

\[
\left( \mathbb{E}(||X(k)||_{\mathfrak{E}_q}^q) \right)^{1/q} \leq C_q \left[ \left( \frac{|k|}{R} \right)^{d(1 - \theta)/p} |k|^{-\theta \nu/2} + \left( \frac{|k|}{R} \right)^{-4} \right] \|\tilde{V}\|_r^r,
\]

for \( |k| \geq R \) and \( 0 < R \leq 1 \).

9. Proof of Theorem 1.2

We will work with the function

\[
d(z) = \det_n (I - X(k)), \quad n = [q] + 1,
\]

where \( z \) is related to \( k \) via the Joukowski mapping

\[
z = \frac{R}{k} + \frac{k}{R}, \quad R > 0,
\]

which maps the set \( \{k \in \mathbb{C} : \text{Im} \ k > 0, \ |k| > R \} \) onto the upper half-plane \( \{z \in \mathbb{C} : \text{Im} \ z > 0 \} \). Rather standard arguments lead to the estimate

\[
\sum_j \text{Im} \ z_j \leq C \int_{-\infty}^{\infty} \ln |d(z)| dz,
\]

where \( z_j \) are zeros of the function \( d(z) \) situated in the upper half-plane \( \mathbb{C}_+ \). In fact, (9.41) could be established in the same way as Jensen’s inequality for zeros of an analytic function on a unit disc. In (9.41) we assume that \( V \) is compactly supported. The relation (7.26) leads to the estimate

\[
\sum_j \left( \frac{|k_j|^2 - R^2}{|k_j|^2 R} \right) \text{Im} k_j \leq C \left( \int_{-\infty}^{\infty} \|X(k)\|^q \left( \frac{1}{R} - \frac{R}{k^2} \right) dk + \int_0^\pi \|X(R \cdot e^{i\theta})\|^q \sin \theta d\theta \right),
\]

Taking the expectation we obtain

\[
\sum_j \mathbb{E} \left[ \frac{\text{Im} k_j(|k_j|^2 - R^2)}{|k_j|^2 R} \right] \leq C \left( \int_{-\infty}^{\infty} \mathbb{E}[\|X(k)\|^q] \left( \frac{1}{R} - \frac{R}{k^2} \right) dk + \int_0^\pi \mathbb{E}[\|X(R \cdot e^{i\theta})\|^q] \sin \theta d\theta \right).
\]
Due to Theorem 6.2, the latter inequality leads to
\[
\sum_j \mathbb{E}\left[ \frac{\text{Im} k_j (|k_j|^2 - R^2)_+}{|k_j|^2 R} \right] \leq C|R|^{-\theta q/2} \parallel \tilde{V} \parallel^{2q}_r.
\] (9.42)

Now, suppose that we consider only the eigenvalues \( \lambda_j = k_j^2 \) that satisfy the inequality
\[|k_j| \leq R_0.\]

Multiplying (9.42) by \( R^{\sigma - 1} \) and integrating with respect to \( R \) from 0 to \( R_0 \), we obtain
\[
\sum_{|k_j| \leq R_0} \mathbb{E}[\text{Im} k_j |k_j|^{\sigma - 1}] \leq C|R_0|^{\sigma - \theta q/2} \parallel \tilde{V} \parallel^{2q}_r, \quad \sigma > \theta q/2.
\] (9.43)

REFERENCES


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