PERIODIC SOLUTIONS OF INVERSE QUANTUM ORTHOGONAL EQUATIONS

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Abstract. In the year 1939, the Mathematician G.H. Hardy proved that the only functions \( f \) which satisfy the classical orthogonality relation

\[
\int_0^1 f(\lambda_m t) f(\lambda_n t) \, dt = 0, \quad m \neq n,
\]

are the Bessel functions \( J_\nu(t) \) under certain constraints, where \( \nu > -1 \) is the order of the Bessel function, and \( \lambda_m, \lambda_n \) are the zeros of the Bessel function. More recently, the Mathematician L.D. Abreu proved that if a function \( f \in L^2_q(0,1) \) is \( q \)-orthogonal with respect to its own zeros in the interval \((0,1)\), then it satisfies the \( q \)-orthogonality relation

\[
\int_0^1 f(\lambda_m t) f(\lambda_n t) \, dq(t) = 0, \quad m \neq n,
\]

where the \( q \)-integral is a Riemann-Stieltjes integral with respect to a step function having infinitely many points of increase at the points \( q^\ell \), with the step size at the point \( q^\ell \) being \( q \), \( \forall \ell \in \mathbb{N}_0 \), where \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), and \( 0 < q < 1 \). Following these developments, herein we present an equivalence class of entire \( q^{-1} \)-periodic functions satisfying the \( q^{-1} \)-orthogonality relation

\[
\int_0^1 f(\lambda_m t) f(\lambda_n t) \, dq^{-1} t = 0, \quad m \neq n.
\]

1. Introduction

The quantum calculus, otherwise known as the \( q \)-calculus \[1\], has been found to have a wide variety of interesting applications in number theory \[2\], and the theory of orthogonal polynomials \[3, 4, 5\], for example. As such, herein we investigate a class of entire functions that are \( q^{-1} \)-orthogonal with respect to their own zeros,
and find that in this equivalence class, the only $q^{-1}$-periodic functions are nonzero constant-valued functions. It is well understood by the Fundamental Theorem of Algebra [6], that a nonzero constant function has no roots. Accordingly, this study aims to develop a novel approach to the field of $q^{-1}$-orthogonal polynomials [7], and the distribution of their zeros [8].

The paper is organized as follows: In Sec. 2 we introduce a class of entire functions, $q^{-1}$-orthogonal with respect to their own zeros, and demonstrate that the class is comprised of $q^{-1}$-periodic (i.e. constant) functions on the complex plane. Sec. 3 details the $q^{-1}$-Fourier series, and the completeness relations of the class. In Sec. 4, a first-order linear $q^{-1}$-difference equation is obtained for arriving at the value of the $q^{-1}$-periodic constant constituted by the class. Finally, concluding remarks are made in Sec. 5.

1.1. Preliminaries. If $q^{-1} \in \mathbb{R}$ is fixed, then a subset of $\mathbb{C}$ is named $\mathcal{A}$, and is also $q^{-1}$-geometric if $q^{-1}x \in \mathcal{A}$ whenever $x \in \mathcal{A}$. If $\mathcal{A} \subset \mathbb{C}$ is $q^{-1}$-geometric then it contains all geometric sequences $\{xq^{-\ell}\}_{\ell=0}^{\infty}$, where $x \in \mathcal{A}$ such that as $q \to 1$ then $\mathcal{A} \to \mathbb{C}$. Unless otherwise noted, herein $0 < q < 1$ [9].

**Definition 1.1.** A function $f$ defined on the $q$-geometric set $\mathcal{A}$, where $0 \in \mathcal{A}$, is said to be $q$-regular at zero if

$$
\lim_{\ell \to \infty} f(xq^\ell) = f(0), \quad \forall x \in \mathcal{A}.
$$

**Definition 1.2.** A function $f$ defined on the $q^{-1}$-geometric set $\mathcal{A}$, where $0 \in \mathcal{A}$, is said to be $q$-regular at infinity if there exists a constant $C$ such that

$$
\lim_{\ell \to \infty} f(xq^{-\ell}) = C, \quad \forall x \in \mathcal{A}.
$$

**Definition 1.3.** The Euler-Heine $q^{-1}$-difference operator [10, 11], is defined by

$$
\hat{D}_{q^{-1}} f(x) := \frac{f(x) - f(q^{-1}x)}{x - q^{-1}x}, \quad \forall x \in \mathcal{A} \setminus \{0\}.
$$
If $0 \in \mathcal{A}$, the $q$-derivative at zero is defined for $|q| < 1$ by

$$\hat{D}_{q^{-1}}f(0) := \lim_{\ell \to \infty} \frac{f(sq^{-\ell}) - f(0)}{sq^{-\ell}}, \quad \forall x \in \mathcal{A} / \{0\}.$$ \hspace{1cm} (1.4)

The $q^{-1}$-derivative at zero is denoted as $f'(0)$, assuming the limit exists and is independent of $x$.

The $q^{-1}$-product rule is \cite{12}

$$\hat{D}_{q^{-1}}[f(x)g(x)] = f(q^{-1}x)\hat{D}_{q^{-1}}g(x) + g(x)\hat{D}_{q^{-1}}f(x),$$ \hspace{1cm} (1.5)

and the $q^{-1}$-integral in the interval $(0, x)$ is

$$\int_0^x f(t)d_{q^{-1}}t = (1 - q) \sum_{\ell=0}^{\infty} f(xq^{-\ell})xq^{-\ell}.$$ \hspace{1cm} (1.6)

Now let $1 \leq p < \infty$, $x > 0$, and $\eta \in \mathbb{R}$. Also let $\mathcal{L}^p_{q^{-1},\eta}(0, x)$ be the space of all equivalence classes of functions satisfying

$$\int_0^x t^{|\eta|}f(t)^pd_{q^{-1}}t < \infty,$$ \hspace{1cm} (1.7)

where two functions are defined as equivalent if they are equivalent on the sequence $\{xq^{-\ell} : \ell \in \mathbb{N}_0\}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Hence, $f$ is a function in the Banach space $\mathcal{L}^p_{q^{-1},\eta}(0, x)$ with norm

$$||f||_{p,\eta,x} := \left( \int_0^x t^{|\eta|}f(t)^pd_{q^{-1}}t \right)^{\frac{1}{p}}.$$ \hspace{1cm} (1.8)

For the case when $p = 2$, it can be seen that the inner product

$$\langle f, g \rangle := \int_0^x t^{|\eta|}f(t)\overline{g(t)}d_{q^{-1}}t,$$ \hspace{1cm} (1.9)

is a separable Hilbert space, where $f, g \in \mathcal{L}^2_{q^{-1},\eta}(0, x)$. If $x = 1$, the resulting Hilbert space is $\mathcal{L}^2_{q^{-1},\eta}(0, 1)$, and the function $f \in \mathcal{L}^2_{q^{-1},\eta}(0, 1)$ is $q^{-1}$-orthogonal.
with respect to its own zeros in the interval \((0, 1)\) if

\[
\int_{0}^{1} f(\lambda_m t) f(\lambda_n t) d_{q^{-1}} t = \sum_{\ell=0}^{\infty} f(\lambda_m q^{-\ell}) f(\lambda_n q^{-\ell}) q^{-\ell} = 0, \quad m \neq n.
\]

Here, it should be pointed out that an orthonormal basis of \(L^2_{q^{-1}, \eta}(0, x)\) is \([13]\)

\[
\varphi_n(t) = \begin{cases} 
\frac{1}{\sqrt{\eta + (1-q)}}, & t = xq^{-\ell}, \quad \ell \in \mathbb{N}_0; \\
0, & \text{otherwise}.
\end{cases}
\]

2. \(q^{-1}\)-Periodicity

**Theorem 2.1.** If the class constituted by all entire functions \(f\) of order less than 1, or of order 1 and minimal type of the form

\[
f(x) = x^{\rho(x)} F(x),
\]

where \(f(0) = -1/2\), and \(\rho(x)\) is given by the natural logarithmic relation \([14]\)

\[
\rho(x) = \log \left( \frac{\log \left( \frac{1}{2(1-x)} \right) + 1}{\log(x)} \right) > -\frac{1}{2},
\]

where \(\Gamma\) is the gamma function, and the entire function \(F(x)\), with real but not necessarily positive zeros is

\[
F(x) = \exp(cx) \prod_{n=1}^{\infty} \left\{ \left( 1 - \frac{x}{\lambda_n} \right) \exp \left( \frac{x}{\lambda_n} \right) \right\},
\]

where \(c = \log(2\pi) - 1 - \gamma/2, \gamma\) is the Euler-Mascheroni constant; if \(F(x) \neq 0\) and \(f(x)\) is \(q^{-1}\)-orthogonal with respect to its zeros; \(\sum_{n} \lambda_n^{-1}\) is convergent, but not absolutely \([16]\); then \(f\) has the \(q^{-1}\)-periodic representation

\[
f_{q^{-1}}(x) = \prod_{\ell=0}^{\infty} \frac{1}{q^{2\ell+1} + q^2},
\]

defined on the \(q^{-1}\)-geometric set \(A\), i.e., \(f_{q^{-1}}(x)\) is constant in \(x\).
Proof. The proof depends on two lemmas. If

\[(2.5) \quad \int_0^1 \{f(\lambda_n t)\}^2 d\eta^{-1} t = (q^{-\ell})^{\eta+1}(1-q),\]

then the system

\[(2.6) \quad \varphi_n(t) = \frac{1}{\sqrt{(q^{-\ell})^{\eta+1}(1-q)}} f(\lambda_n t)\]

is orthonormal in \((0, 1)\). The following Theorem 2.2 demonstrates the system \(\varphi_n(t)\) is complete, independent of \(q^{-1}\)-orthogonality.

**Theorem 2.2.** If \(f\) satisfies the conditions of the previous Theorem 2.1, other than \(q^{-1}\)-orthogonality, \(g\) is \(q^{-1}\)-integrable, and

\[(2.7) \quad \int_0^1 g(t) f(\lambda_n t) d\eta^{-1} t = 0, \quad \forall \, n,\]

then \(g(t) \equiv 0\).

Proof. Let \(t = rq^{-\ell} \exp(i\theta)\), where \(\theta\) is the complex argument, \(i = \sqrt{-1}\), and

\[(2.8) \quad h(x) = \int_0^1 g(t) f(xt) d\eta^{-1} t.\]

It is clear that

\[(2.9) \quad h(x) = x^{\eta(x)} H(x),\]

where \(H(x)\) is an entire function. Here, we suppose that \(F(x)\) is of order less than 1, when \(H(x)\) is also of order less than 1. Since \(h(\lambda_n) = 0 \forall \, n\), it then follows that the ratio \([17]\)

\[(2.10) \quad \chi(x) = \frac{h(x)}{f(x)} = \frac{H(x)}{F(x)}\]

is also an entire function of order less than 1. Along the imaginary axis \(t = rq^{-\ell} \sin(\theta)\) it can be seen that \(|\exp(cx)| = |\exp(x\lambda_n^{-1})| = 1 \forall \, n\), where again
\[ c = \log(2\pi) - 1 - \gamma/2, \text{ and} \]

\[ \nu(x, t) = \left| \frac{F(xt)}{F(x)} \right| = \prod_{n=1}^{\infty} \left| \frac{\lambda_n - rt \sin(\theta)}{\lambda_n - r \sin(\theta)} \right|. \]

Here it should be pointed out that no factor exceeds 1, and the limit of each factor as \( r \to \infty \) is simply \( t \). Therefore \( |\nu| \leq 1 \forall r, t \). Moreover, for every fixed value of \( t < 1 \), as \( r \to \infty \) it can be seen that \( \nu \to \infty \). As such,

\[ |\chi(x)| = \left| \int_{0}^{1} g(t) \frac{F(xt)}{F(x)} d_q \right| \leq \int_{0}^{1} |g(t)| |\nu(x, t)| d_q \]

is bounded, and tends to zero along the imaginary axis \( t = rq^{-\ell} \sin(\theta) \). Furthermore, suppose that \( \chi(x) \) makes an angle of \( \pi/\alpha \) at the origin, and also along the imaginary axis. By denoting the bound on \( \chi(x) \) as \( B \), such that along the imaginary axis

\[ |\chi(x)| \leq B, \]

then as \( r \to \infty \), it can be seen that

\[ \chi(x) = O\left( \exp(\delta r^\alpha) \right) \]

for every positive \( \delta \), uniformly in the angle. It then follows that the boundedness holds in the region where \( f \) is entire and regular for \( t = rq^{-\ell} \exp(i\theta) \). Without loss of generality, suppose that \( \theta = \pm \pi/(2\alpha) \) for the two angles \( (-\pi/(2\alpha), 0) \) and \( (0, \pi/(2\alpha)) \). Also, by letting

\[ F(x) = \exp(-\varepsilon x^\alpha) f(x) \]

it can be seen that \( F(x) \) tends to zero on the real axis \( t = rq^{-\ell} \cos(\theta) \), and therefore has an upper bound, denoted \( B' \). Then, by denoting

\[ B'' = \max(B, B'), \]
it can be seen that

\[ |F(x)| = \left| \exp \left[ -\varepsilon \left( r \exp(i\theta) \right)^\alpha \right] f(x) \right|, \]

where again \( \theta = \pm \pi/(2\alpha) \). It then follows that throughout the angle, and along the imaginary axis \( t = rq^{-\ell} \sin(\theta) \), that

\[ |F(x)| \leq B''. \]  

Here, it should be pointed out that if \( B' \leq B \), then \( |F(x)| \) assumes the value \( B' \) at any point of the real axis \( t = rq^{-\ell} \cos(\theta) \). Consequently \( B' = B'' \), \( F(x) \) reduces to a constant, and \( B = B'' \). Otherwise \( B' < B'' \), such that \( B = B'' \) regardless. Thus,

\[ |F(x)| \leq B. \]

Accordingly,

\[ |f(x)| \leq B|\exp(-\varepsilon x^\alpha)|. \]

Taking \( \varepsilon \to 0 \) implies that \( B = 0 \), since \( \nu \to 0 \) for every fixed \( t < 1 \) as \( r \to \infty \). Therefore,

\[ \int_0^1 g(t) f(xt) dt q^{-1} t = 0. \]

However, we are interested in the class of functions of the form of Eq. (2.1), i.e.,

\[ f(x) = x^{\rho(x)} \sum_{\ell=0}^{\infty} a_\ell x^\ell, \]

where \( a_\ell \neq 0 \) for any \( \ell \). As such, we assume the following [15]:

1. There exists a class of series, larger than that of series known classically as convergent, such that a sum corresponds to each series of that class;
(2) Let $m$ and $n$, where $n < m$, be two positive integers. We then have the relation

\[
\frac{1 - x^n}{1 - x^m} = 1 - x^n + x^m - x^{n+m} + x^{2m} + \cdots.
\]  

(2.23)

At $t = q^{-l}$, we obtain the Euler series

\[
\frac{n}{m} = 1 - 1 + 1 - 1 + 1 - \cdots
\]  

(2.24)

which belongs to the class from assumption (1).

(3) Let $S$ be the sum of the series $x^\rho(x) \sum_n a_n$ of the class, where $x^\rho(x)$ is given by Eq. (2.2). Then the series itself belongs to the class, and has the sum $x^\rho(x) S$.

(4) If the series $a_0 + a_1 + \cdots + a_n + \cdots$ has the sum $S$, then the series $a_1 + \cdots + a_n + \cdots$ itself has the sum $S - a_0$. As such, it can be seen that

\[
S = 1 - 1 + 1 - 1 + 1 - \cdots
\]

\[
= 1 - (1 - 1 + 1 - \cdots)
\]

(2.25)

\[
= 1 - S,
\]

from which we obtain $S = 1/2$.

Hence,

\[
\int_0^1 g(t)t^\rho(xt)^n\,dt = 0, \quad \forall \ n,
\]

(2.26)

and therefore $g(t) \equiv 0$. \qed
3. $q^{-1}$-Fourier Series

The $q^{-1}$-Fourier series of $f(xt)$ with respect to the system Eq. (1.11) is

$$f(xt) \sim \sum_n a_n(x) \varphi_n(t)$$

$$= \sum_n a_n(x) \frac{1}{\sqrt{(q^{-t})^{n+1}(1-q)}},$$

where the Fourier coefficient

$$a_n(x) = \int_0^1 f(xt) \varphi_n(t) d_{q^{-1}}t$$

$$= \frac{1}{\sqrt{(q^{-t})^{n+1}(1-q)}} \int_0^1 f(xt)f(\lambda_n t) d_{q^{-1}}t;$$

and by the Parseval completeness theorem [19], we obtain

$$P(x, x') = \int_0^1 f(xt)f(x't)d_{q^{-1}}t$$

$$= \sum_{n=1}^{\infty} a_n(x) a_n(x').$$

The following theorem gives the value of $a_n(x)$.

**Theorem 3.1.** If the conditions of Theorem 2.1 are satisfied, and $x \neq \lambda_n$, then

$$\int_0^1 f(xt)f(\lambda_n t)d_{q^{-1}}t = \frac{(q^{-t})^{n+1}(1-q)}{f'(\lambda_n)} \cdot \frac{f(x)}{x - \lambda_n}. $$

**Proof.** First, supposing that $F(x)$ is of order less than 1, we write

$$h(x) = \int_0^1 f(xt)f(\lambda_n t)d_{q^{-1}}t,$$

$$f_n(x) = \frac{f(x)}{x - \lambda_n},$$

$$g(x) = \frac{h(x)}{f_n(x)},$$

$$G(x) = \frac{g(x)}{x+1}.$$
It then follows that \( g \) is an entire function of order less than \( 1 \); \( G \) is regular and of order less than \( 1 \) in the half-plane \( rq^{-t} \cos(\theta) > 0 \); and

\[
G(x) = \frac{x - \lambda_n}{x + 1} \int_0^1 \frac{f(xt)}{f(x)} f(\lambda_n t) d_{q^{-1}} t
\]

is bounded, and goes to zero along the angle \( \theta = \pm \pi/4 \). It then follows in the quadrant between \( \theta = \pm \pi/4 \) that

\[
g(x) = \mathcal{O}(|x|).
\]

In a similar fashion, the same result follows for the remaining three quadrants in the complex plane \( \mathbb{C} \). Obviously, \( g \) is linear and

\[
h(x) = g(x) f_n(x) = \frac{ax + b}{x - \lambda_n} f(x).
\]

However, \( G \) goes to zero along the angle \( \theta = \pi/4 \) such that \( a = 0 \), and

\[
h(x) = \frac{b}{x - \lambda_n} f(x).
\]

The constant \( b \) can be obtained by making \( x \to \lambda_n \), to obtain Eq. (3.4). \( \square \)

4. First-Order Linear \( q^{-1} \)-Difference Equation

From Eqs. (3.1), and (3.3)-(3.4) it follows that

\[
\mathcal{P}(x, x') = \int_0^1 f(xt)f(x't)d_{q^{-1}} t = -f(x)f(x') \frac{\tau(x) - \tau(x')}{x - x'},
\]

where

\[
\tau(x) = \sum_{\ell=1}^\infty \frac{(q^{-t})^{\ell+1}(1 - q)}{\{f'(\lambda_\ell)\}^2} \left( \frac{1}{x - \lambda_\ell} + \frac{1}{\lambda_\ell} \right),
\]

such that \( \tau(0) = 0 \). Eq. (4.1) will enable us to determine \( f \). By making \( x' \to 0 \), it follows that

\[
\int_0^1 t^n f(xt)d_{q^{-1}} t = -f(x) \frac{\tau(x)}{x},
\]
i.e.,
\begin{equation}
\int_0^x u^n f(u) d_u u = -x^n f(x) \tau(x).
\end{equation}

Hence,
\begin{equation}
\tau'(0) = (q-1)q^{-\ell}[1 + \eta(q^{-\ell} - 1)].
\end{equation}

Next, we write Eq. (4.1) in the form
\begin{equation}
\int_0^x u^\rho(u) F(u)(x')^\rho(x') F(x') d_u u = -x^\rho(x) F(x)(x')^\rho(x') F(x') \frac{\tau(x) - \tau(x')}{x - x'}.
\end{equation}

Differentiating with respect to $x'$, and evaluating at $x' = 0$, it can be seen that
\begin{equation}
\frac{\partial}{\partial x'} (x'^t)^\rho(x') F(x') \bigg|_{x'=0} = -\frac{t}{4}(2 + 2c + \gamma),
\end{equation}

\begin{equation}
-xf(x) \frac{\partial}{\partial x'} (x'^t)^\rho(x') F(x') \frac{\tau(x) - \tau(x')}{x - x'} \bigg|_{x'=0} = \frac{(2 + 2c + \gamma + 2x^{-1})\tau(x) f(x)}{4} \quad - \frac{\tau'(0)}{2} f(x).
\end{equation}

Using Eqs. (4.4)-(4.5), and by choosing $\eta = 1$ for brevity, we finally obtain the $q^{-1}$-integral equation for $f$, namely
\begin{equation}
\int_0^x u f(u) d_{q^{-1}} u = (1 - q)q^{-2\ell} x^2 f(x).
\end{equation}

By taking the $q^{-1}$-difference $\hat{D}_{q^{-1}}$, and using the $q^{-1}$-integration by parts, i.e.,
\begin{equation}
\int_0^x g(t) \left( \hat{D}_{q^{-1}} f(t) \right) d_{q^{-1}} t + \int_0^x \left( \hat{D}_{q^{-1}} g(t) \right) f(q^{-1}t) d_{q^{-1}} t = [fg](x)
\end{equation}

\begin{equation}
\hat{D}_{q^{-1}} \int_0^x u f(u) d_{q^{-1}} u = xf(x) - \lim_{\ell \to \infty} xq^\ell f(xq^\ell),
\end{equation}

it can be seen that since $f$ and $g$ are also $q$-regular at zero,
and

\begin{equation}
\hat{D}_{q^{-1}}[x^2 f(x)] = (\hat{D}_{q^{-1}} x^2) f(x) + (q^{-1} x)^2 \hat{D}_{q^{-1}} f(x).
\end{equation}

Hence, we arrive at the first-order linear $q^{-1}$-difference equation \cite{18}

\begin{equation}
\hat{D}_{q^{-1}} f(x) = \tilde{a}(x) f(x).
\end{equation}

Carrying out the $q^{-1}$-difference $\hat{D}_{q^{-1}}$ and upon making further simplifications,

\begin{equation}
f(x) = \left[ \frac{q}{q + x \tilde{a}(x)(1 - q)} \right] f(q^{-1} x),
\end{equation}

where

\begin{equation}
\tilde{a}(x) = \frac{q - q^2 (q^{2x} + q)}{(q - 1) x}.
\end{equation}

Repeating the above recurrence relation $N$ times,

\begin{equation}
f(x) = f(x_0) \prod_{t=qx_0}^x \frac{q}{q + t \tilde{a}(t)(1 - q)}.
\end{equation}

As $N \to \infty$ with $0 < q < 1$, then $q^{-N} \to \infty$, and

\begin{equation}
f(x) = f(q^{-N} x) \prod_{t=0}^{N-1} \frac{q}{q + x q^{-t} \tilde{a}(x q^{-t})(1 - q)}
\end{equation}

\begin{equation}
= f(\infty) \prod_{t=0}^{\infty} \frac{1}{q^{2t+1} + q^2}.
\end{equation}

Since by Eq. (2.1) we have $f(\infty) = 1$, it can be seen in the classical limit where $q \to 1$ and $\mathcal{A} \to \mathbb{C}$ that $f(x) = 1/2 \forall x \in \mathbb{C}$. \hfill \Box

5. Conclusion

By examining a class of entire first order $q^{-1}$-orthogonal functions $f \in \mathcal{L}_{q^{-1}}^2(0,1)$, it has been demonstrated that the class is indeed comprised of $q^{-1}$-periodic functions on the separable Hilbert space interval $(0,1)$. This was accomplished with
the $q^{-1}$-Fourier series, and a $q^{-1}$-integral equation for obtaining the value of the $q^{-1}$-periodic constant constituted by the class.

References
