Let $[\lambda, \mu]$ be an interval contained in a spectral gap of a periodic Schrödinger operator $H$. Consider $H(\alpha) = H - \alpha V$ where $V$ is a fast decaying positive function. We study the asymptotic behavior of the number of eigenvalues of $H(\alpha)$ in $[\lambda, \mu]$ as $\alpha \to \infty$.

1. Statement of the main results

Let $H$ be the Schrödinger operator with a bounded real potential $q \in L^\infty(\mathbb{R}^d)$

$$H = -\Delta + q(x), \quad x \in \mathbb{R}^d.$$  

We will often assume that $q$ is periodic, but some of the results do not require that. Operators with periodic potentials $q$ play a very important role in the so-called one-electron approximation model used in the quantum theory of crystals. According to this theory, the "allowed" energies of an electron moving in a crystal lie in $\sigma(H)$, the spectrum of $H$, consisting of bands. For a typical insulator, there is at least one gap in $\sigma(H)$. The $\text{Al}_2\text{O}_3$-crystal is one of examples of periodic media with this property. The theory also says that a photon whose energy is smaller than the length of the gap can not be absorbed by the crystal, because all states in the first band of the spectrum are already filled (see also [8]). That is the reason why Corundum (i.e. $\text{Al}_2\text{O}_3$) is colorless. However, replacing some of the $\text{Al}^{3+}$- ions by either $\text{Cr}^{3+}$ or $\text{Ti}^{3+}$-ions, one obtains Ruby or Sapphire which absorb green and yellow colors. That is the reason why Ruby or Sapphire look red and blue correspondingly. Due to the fact that the crystal is not pure, there are additional isolated energy levels in spectral gaps of the pure crystal. Since the distances from these levels to the edges of the gap are smaller than the length of the gap, it is easier for the light to be absorbed by a crystal with impurities.

Let $V \geq 0$ be the operator of multiplication by a non-negative bounded function $V$ having the property

$$V \in L^p(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$$  

for some $p \leq 1$ such that $d/4 < p \leq d/2$. The function $V$ will play the role of the impurity potential. Define $H(\alpha)$ setting

$$H(\alpha) = H - \alpha V, \quad \alpha > 0.$$  

The condition (1.1) guarantees that, in gaps of $\sigma(H)$, the spectrum of $H(\alpha)$ is discrete, which means it consists of isolated eigenvalues possibly accumulating to the edges of the gaps. One can easily show that, with the growth of $\alpha$, eigenvalues of $H(\alpha)$ move from the right to the left. However, for some potentials $V$, the spectral flow through one point of a gap is not
exactly the same as the flow through another point, which results in the fact that the number
of all eigenvalues of $H(\alpha)$ situated between the two points tends to infinity as $\alpha \to \infty$.

Our first theorem gives an upper estimate for the weighted integral of the number of eigen-
values of $H(\alpha)$ located in a fixed interval.

**Theorem 1.1.** Let $d \leq 3$. Let $\lambda$ and $\mu$ be two points in a spectral gap of the operator $H$. Let $N(\alpha)$ be the number of eigenvalues of the operator $H(\alpha)$ in $[\lambda, \mu]$. Assume that $V$ satisfies (1.1) for some $p \leq 1$ such that $d/4 < p \leq d/2$. Then there is a positive constant $C$ independent of $V$ such that

$$\int_0^\infty N(\alpha) \alpha^{-p-1} d\alpha \leq C \sum_{n \in \mathbb{Z}^d} \left( \int_{[0,1)^d + n} V(x) dx \right)^p.$$  \hspace{1cm} (1.2)

The estimate (1.2) is sharp. If $V(x) = (1 + |x|)^{-\nu}$, with $\nu > 2$, then

$$\liminf_{\alpha \to \infty} \alpha^{-d/\nu} N(\alpha) > 0.$$

**Corollary 1.2.** Assume that conditions of Theorem 1.1 are fulfilled. Let $\tau > 1$. Then for almost every $\theta \in [1, \tau]$,

$$N(\tau^n \theta) = o(\tau^{np}), \quad \text{as} \quad n \to \infty.$$

In our next result, we assume that $q$ is a periodic function. In this case, one can talk about
the integrated density of states whose description is given below. Let $\Omega(L, x_0)$ be a domain
of the following form

$$\Omega(L, x_0) = \{ x \in \mathbb{R}^d : x \in L \cdot (\mathbb{Q} + x_0) \}, \quad L > 0, \quad x_0 \in \mathbb{R}^d, \quad \mathbb{Q} = [0, 1)^d.$$

Let now $H^\pm(L, x_0) = -\Delta + q(x)$ be the operators on $\Omega(L, x_0)$ with either Dirichlet or Neumann boundary conditions (depending on the choice of the $\pm$ sign). The integrated density of states is defined as the limit

$$\rho(\lambda) = \lim_{L \to \infty} \frac{\text{the number of eigenvalues of } H^\pm(L, x_0) \text{ below } \lambda}{L^d},$$  \hspace{1cm} (1.3)

which does not depend on the choice of the boundary conditions and the choice of $x_0$.

In the present paper we address the following problem posed not later than in 1991. Assume
that $V$ is a positive bounded function such that

$$V(x) = \Phi(\theta)|x|^{-\nu}(1 + o(1)), \quad \text{as} \quad |x| \to \infty, \quad \theta = x/|x|.$$  \hspace{1cm} (1.4)

What is the asymptotic behavior of $N(\alpha)$ as $\alpha \to \infty$ in the case $\nu > 2$?

For $d = 1$ and a differentiable $V$, the answer to this question is given in [19]. My paper [17]
deals with the case of a spherically symmetric $V$. Finally, the assumptions of my other paper
[18] require $V$ to be a smooth function. The claim of all three papers is that, if $\nu < 2+2/(d-1)$, then

$$\lim_{\alpha \to \infty} \alpha^{-d/\nu} N(\alpha) = \int_{\mathbb{R}^d} \left( \rho(\mu + \Phi|x|^{-\nu}) - \rho(\lambda + \Phi|x|^{-\nu}) \right) dx.$$  \hspace{1cm}

Our main theorem does not require $V$ to be either smooth or spherically symmetric. The price we pay for not having these conditions is that the limit on the left hand side is understood in some average sense.
Theorem 1.3. Let $d \leq 3$. Let $H$ be the Schrödinger operator with a periodic potential $q$. Let $N(\alpha)$ be the number of eigenvalues of $H(\alpha)$ in $[\lambda, \mu]$. Suppose that $p \in (0, 1]$ and $\nu$ satisfy the conditions

$$
\frac{d}{4} < p < \frac{d}{\nu}, \quad 2 < \nu < 2 + \frac{4p}{d(d - 2p)}.
$$

Assume also that $V$ obeys (1.4), where $\Phi$ is continuous on the unit sphere. Then

$$
\lim_{\alpha \to \infty} \alpha^{p - d/\nu} \int_0^\alpha N(t) \frac{dt}{t^{p+1}} = \frac{\nu}{d - \nu p} \int_{\mathbb{R}^d} \left( \rho(\mu + \Phi |x|^{-\nu}) - \rho(\lambda + \Phi |x|^{-\nu}) \right) dx.
$$

Note that the values of the parameter $\nu$ allowed by this theorem are

$$
2 < \nu < 2 + \frac{4}{d}.
$$

In $d = 3$, the range of these values is wider than the one allowed by previous results. However, in $d = 2$, the results of [17], [18] allow $2 < \nu < 4$ under very restrictive conditions on $V$. In $d = 1$, the results of [19] allow all $\nu > 2$ under some additional conditions involving differentiability of $V$.

Perturbations of the form (1.4) have been studied for a long time. The main result of [10] says that if $\nu > 2$ then the number $N(\lambda, \alpha)$ of eigenvalues of $H(t)$ passing a regular point $\lambda \notin \sigma(H)$ as $t$ increases from 0 to $\alpha$ satisfies

$$
N(\lambda, \alpha) \sim (2\pi)^{-d} c_d \alpha^{d/2} \int_{\mathbb{R}^d} V^{d/2} dx, \quad \text{as} \; \alpha \to \infty.
$$

Here $c_d$ is the volume of the unit ball in $\mathbb{R}^d$. An interesting short proof of (1.6) was given by M. Birman in [3]. The author shows in [3] that the asymptotics of $N(\lambda, \alpha)$ does not depend on the point $\lambda$ and the potential $q$. Consequently, one can take $\lambda = -1$ and set $q = 0$. After that, one can use previously known results.

Observe that $N(\alpha) = N(\mu, \alpha) - N(\lambda, \alpha)$. However, since the expression on the right hand side of (1.6) does not depend on $\lambda$, this formula only implies that

$$
N(\alpha) = o(\alpha^{d/2}), \quad \text{as} \; \alpha \to \infty.
$$

In order to obtain (1.5) one would need to know the second term in the asymptotics of $N(\lambda, \alpha)$. The second term in (1.6) has never been obtained. This explains why the problem is challenging.

To help the reader understand whether other mathematicians studied similar problems, we gave a list of articles treating close (but not the same) questions (see, for instance, [1], [4], [5], [9], [11], [12],[13], [14], [15], [20]).

2. Compact operators

For a compact operator $T$, the symbols $s_k(T)$ denote the singular values of $T$ enumerated in the non-increasing order ($k \in \mathbb{N}$) and counted in accordance with their multiplicity. Observe that $s_k^2(T)$ are eigenvalues of $T^*T$. We set

$$
n(s, T) = \# \{ k : s_k(T) > s \}, \quad s > 0.
$$

For a self-adjoint compact operator $T$ we also set

$$
n_{\pm}(s, T) = \# \{ k : \pm \lambda_k(T) > s \}, \quad s > 0.
$$
where $\lambda_k(T)$ are eigenvalues of $T$. Observe that

$$n_+(s_1 + s_2, T_1 + T_2) \leq n_+(s_1, T_1) + n_+(s_2, T_2), \quad s_1, s_2 > 0.$$ 

A similar inequality holds for the function $n$. Also,

$$n(s_1 s_2, T_1 T_2) \leq n(s_1, T_1) + n(s_2, T_2), \quad s_1, s_2 > 0.$$ 

**Theorem 2.1.** Let $A$ and $B$ be two compact operators in the same Hilbert space. Then for any $r \in \mathbb{N}$,

$$\sum_{1}^{r} s_k^p(A + B) \leq \sum_{1}^{r} s_k^p(A) + \sum_{1}^{r} s_k^p(B), \quad \forall p \in (0, 1], \quad (2.1)$$

and

$$\sum_{1}^{r} s_k^p(AB) \leq \sum_{1}^{r} s_k^p(A)s_k^p(B), \quad \forall p > 0. \quad (2.2)$$

The first inequality was discovered by S. Rotfeld [16]. The second estimate is called Horn’s inequality.

Below we use the following notation for the positive and negative part of a self-adjoint operator $T$:

$$T_\pm = \frac{1}{2}(|T| \pm T).$$

**Proposition 2.2.** Let $p > 0$. Let $A$ and $B$ be two compact operators. If $s_r(A) \geq s_r(B)$ for some $r$, then

$$p \int_{s_r(A)}^{\infty} \left( n(s, A) - n(s, B) \right) s^{p-1} ds \leq \sum_{k=1}^{r} \left( s_k^p(A) - s_k^p(B) \right).$$

Additionally, if $A$ and $B$ are self-adjoint and $s_r(A_+) \geq s_r(B_+)$ for some $r$, then

$$p \int_{s_r(A_+)}^{\infty} \left( n_+(s, A) - n_+(s, B) \right) s^{p-1} ds \leq \sum_{k=1}^{r} \left( s_k^p(A_+) - s_k^p(B_+) \right). \quad (2.3)$$

**Proof.** Let’s prove the second relation (2.3). Observe that for any pair of compact operators $A = A^*$ and $B = B^*$,

$$\sum_{k=1}^{r} s_k^p(A_+) = - \int_{s_r(A_+)}^{\infty} s^p d\eta_+(s, A) = r s_r^p(A_+) + p \int_{s_r(A_+)}^{\infty} s^{p-1} n_+(s, A) ds, \quad (2.4)$$

$$\sum_{k=1}^{r} s_k^p(B_+) = - \int_{s_r(B_+)}^{\infty} s^p d\eta_+(s, B) = r s_r^p(B_+) + p \int_{s_r(B_+)}^{\infty} s^{p-1} n_+(s, B) ds. \quad (2.5)$$

Note also that

$$p \int_{s_r(A_+)}^{s_r(B_+)} s^{p-1} n_+(s, B) ds \leq r \left( s_r^p(A_+) - s_r^p(B_+) \right), \quad (2.6)$$

because $n_+(s, B) \leq r$ for $s > s_r(B_+)$. To finish the proof, one needs to subtract (2.5) from (2.4) and take into account (2.6). □

The next statement follows from S. Rotfeld’s estimate (2.1).
Corollary 2.3. Let $0 < p \leq 1$. Let $A$ and $B$ be two compact operators. If $n(s, A) \geq n(s, B)$ for some $s > 0$, then

$$p \int_s^\infty \left( n(t, A) - n(t, B) \right) t^{p-1} dt \leq s^p + \sum_{k=1}^{n(s, A)+1} s_k^p (A - B).$$  \hfill (2.7)

Additionally, if $A$ and $B$ are self-adjoint and $n_+(s, A) \geq n_+(s, B)$ for some $s > 0$, then

$$p \int_s^\infty \left( n_+(t, A) - n_+(t, B) \right) t^{p-1} dt \leq s^p + \sum_{k=1}^{n_+(s, A)+1} s_k^p (A - B).$$  \hfill (2.8)

Moreover, if $B \leq A$, then

$$p \int_s^\infty \left( n_+(t, A) - n_+(t, B) \right) t^{p-1} dt \leq s^p + \sum_{k=1}^{n_+(A)+1} s_k^p (A - B), \quad \forall s > 0.$$

Proof. We will only prove (2.8). Let $r = n_+(s, A)$. If $s r + 1 (A +) = s$, then we do not change the operator $A$. If $s r + 1 (A +) < s$, then we set

$$\tilde{A} = A + (s - s r + 1 (A +)) P_{r+1},$$

where $P_{r+1}$ is the projection onto the eigenspace of $A$ corresponding to the eigenvalue $s r + 1 (A +)$. Note that the left hand side of (2.8) does not change when one replaces $A$ by $\tilde{A}$. According to (2.3), it is sufficient to show that

$$\sum_1^r s_k^p (\tilde{A} +) \leq \sum_1^r s_k^p (B +) + \sum_1^r s_k^p (\tilde{A} - B),$$

The latter inequality follows from the fact that the estimate

$$\tilde{A} \leq B_+ + (\tilde{A} - B)$$

implies that

$$\sum_1^r s_k^p (\tilde{A} +) \leq \sum_1^r s_k^p (B_+ + (\tilde{A} - B)).$$

It remains to apply (2.1) to the right hand side and take into account the fact that

$$\sum_1^r s_k^p (\tilde{A} - A) \leq s^p.$$

□

Now we can easily prove the following very important for our purposes statement.

Theorem 2.4. Let $0 < p \leq 1$. Let $A$ and $B$ be two compact selfadjoint operators. Then for any $s > 0$

$$p \int_s^\infty \left( n_+(t, A) - n_+(t, B) \right) t^{p-1} dt \leq \|B\|_p^p + \sum_{k=1}^{n_+(s, A)+1} s_k^p (A - B).$$  \hfill (2.9)
Proof. The argument is very simple. First one has to find the maximal number \( \tilde{s} \geq s \) such that

\[
n_+(t, A) - n_+(t, B) < 0, \quad \text{for all} \quad t \in [s, \tilde{s}].
\]  

(2.10)

Then it will be sufficient to show that

\[
p \int_{\tilde{s}}^{\infty} \left( n_+(t, A) - n_+(t, B) \right) t^{p-1} dt \leq \tilde{s}^p + \sum_{k=1}^{n_+(s, A)+1} s_k^p(A - B).
\]  

(2.11)

It remains to note that (2.11) follows from (2.8), because \( n_+(\tilde{s}, A) \geq n_+(\tilde{s}, B) \). \( \square \)

The next statement follows from Corollary 2.3.

**Proposition 2.5.** Let \( 0 < p \leq 1 \). Let \( X_\lambda \) be the family of operators defined by (2.7). Then for any two points \( \lambda < \mu \) in the same gap and any \( s > 0 \),

\[
p \int_{s}^{\infty} \left( n_+(s, X_\mu) - n_+(s, X_\lambda) \right) s^{p-1} ds \leq \sum_{k=1}^{n_+(s, X_\mu)+1} \left( s_k^p(X_\mu - X_\lambda) \right).
\]

Proof. Here one needs to use the fact that \( X_\lambda \leq X_\mu \). \( \square \)

The following proposition is called the Birman-Schwinger principle:

**Proposition 2.6.** Let \( H \) and \( V \geq 0 \) be self-adjoint operators in a Hilbert space. Let \( \mathcal{N}(\lambda, \alpha) \) be the number of eigenvalues of \( H(t) = H - tV \) passing through a regular point \( \lambda \notin \sigma(H) \) as \( t \) increases from 0 to \( \alpha \). Then

\[
\mathcal{N}(\lambda, \alpha) = n_+(s, X_\lambda), \quad \text{for} \quad s \alpha = 1, \text{and} \quad W = \sqrt{V}.
\]  

(2.12)

The idea of the proof of (2.12) is the following. First, one shows that \( \lambda \in \sigma(H(\alpha)) \), if and only if \( \alpha^{-1} \in \sigma(W(H - \lambda)^{-1}W) \). This relation holds with multiplicities taken into account. After that, one simply uses the definition of the distribution function \( n_+(s, X_\lambda) \).

**Corollary 2.7.** Let \( H \) and \( V \geq 0 \) be self-adjoint operators in a Hilbert space. Let \( N(\alpha) \) be the number of eigenvalues of the operator \( H(\alpha) \) in \( [\lambda, \mu] \) contained in a gap of the spectrum \( \sigma(H) \). Then

\[
N(\alpha) = n_+(s, W(H - \mu)^{-1}W) - n_+(s, W(H - \lambda)^{-1}W), \quad s \alpha = 1.
\]  

(2.13)

If \( H = -\Delta + q \) is the Schrödinger operator with a bounded potential, then the following operator is bounded on \( L^2(\mathbb{R}^d) \):

\[
G = (\mu - \lambda)(-\Delta + I)(H - \mu)^{-1}\left((-\Delta + I)(H - \lambda)^{-1}\right)^*.
\]

Now we are ready to prove the following result.

**Theorem 2.8.** Let \( 0 < p \leq 1 \). Let \( \lambda < \mu \) be two points in a spectral gap of the operator \( H \). Let \( N(\alpha) \) be the number of eigenvalues in \( [\lambda, \mu] \). Then there exists a positive constant \( C \) which does not depend on the potential \( V \) such that

\[
\int_{0}^{\alpha} N(t) \frac{dt}{t^{p+1}} \leq C \sum_{k=1}^{n_+(s, X_\mu)+1} s_k^{2p}(W(-\Delta + I)^{-1}), \quad s = \alpha^{-1}, \quad W = \sqrt{V}.
\]
Proof. Obviously, Proposition 2.5 and Corollary 2.7 imply the inequality
\[ p \int_0^{\infty} t^{p-1} N(t^{-1}) dt \leq \sum_{k=1}^{n+(s,X_p)+1} s_k^p (X_\mu - X_\lambda) \]
Consequently, it is enough to show that, for any \( \lambda \notin \sigma(H) \) and \( \mu \notin \sigma(H) \), there is a constant \( C > 0 \) such that
\[ s_k(X_\mu - X_\lambda) \leq C s_k^2 (W(-\Delta + I))^{-1}. \]
The latter follows from the fact that
\[ X_\mu - X_\lambda = (\mu - \lambda) W(H - \mu)^{-1}(H - \lambda)^{-1} W = W(-\Delta + I)^{-1} G(-\Delta + I)^{-1} W, \]
where \( G \) is a positive bounded operator. The proof is completed. \( \square \)

Let \( p > 0 \). The class of compact operators \( T \) whose singular values satisfy
\[ \| T \|_{p, S^p} := \sum_k s_k^p(T) < \infty \]
is called the Schatten class \( S_p \). One can also consider weak Schatten classes \( \Sigma_p \) which are sets of operators \( T \) such that
\[ \| T \|_{p, \Sigma_p} := \sup_k (k^{1/p} s_k(T)) < \infty. \]

For a function \( f: \mathbb{R}^d \to \mathbb{C} \), define \( \| f \|_p \) setting
\[ \| f \|_p^p = \begin{cases} \int_{\mathbb{R}^d} |f(x)|^p dx & \text{if } p > 2; \\ \sum_n \| f \|_{L^p(Q+n)}^p & \text{if } p < 2; \\ \sum_n \| f \|_{L^q(Q+n)}^q & q > 2, \quad \text{if } p = 2. \end{cases} \]
The next statement could be found in [3] and should be viewed as a generalization of the Cwikel inequality [7].

**Proposition 2.9.** There is a constant \( C > 0 \) independent of \( W \) and \( s > 0 \) such that
\[ n(s, W(-\Delta + I)^{-1/2}) \leq C s^{-d/\ell} \| W \|_{d/\ell} \]
In particular, if \( d \leq 3 \), then
\[ \| W(-\Delta + I)^{-1} \|_{\Sigma_{d/2}} \leq C \| W \|_{\ell^{d/2}(Z^d; L^2(\mathbb{Q}))} \]
where
\[ \| W \|_{\ell^{d/2}(Z^d; L^2(\mathbb{Q}))} = \sum_n \left( \int_{Q+n} W^2 dx \right)^{d/4} \]
This implies that if \( \chi_n \) is the characteristic function of the cube \( Q+n \), then for any \( p > d/4 \),
\[ \| \chi_n W(-\Delta + I)^{-1} \|_{\mathcal{E}_{2p}} \leq C \left( \int_{Q+n} W^2 dx \right)^{1/2} \]
which leads to
\[ \| W(-\Delta + I)^{-1} \|_{\mathcal{E}_{2p}}^{2p} \leq \sum_n \| \chi_n W(-\Delta + I)^{-1} \|_{\mathcal{E}_{2p}}^{2p} \leq C \sum_n \left( \int_{Q+n} W^2 dx \right)^{p} \]
if \( 2p \leq 1 \).
Proposition 2.10. Let $d = 1$ and let $1/4 < p \leq 1/2$. Then
\[ \| W(-\Delta + I)^{-2} W \|_{\mathcal{E}_p}^p \leq C \sum_n \left( \int_{Q+n} V(x) dx \right)^p. \] (2.16)

Let us now consider an operator of the form $WF a$, where $F$ is the unitary operator of Fourier transform, while $W$ and $a$ are operators of multiplication by real valued functions $W$ and $a$.

The following estimate is stated in Theorem 11 of Section 11.8 of the book by M. Birman and M. Solomyak [6]:
\[ \| WF a \|_{\mathcal{E}_1} \leq C \sum_n \left( \int_{Q+1} W^2 dx \right)^{1/2} \sum_n \left( \int_{Q+1} a^2 dx \right)^{1/2}. \] (2.17)

One the other hand, one can always state the trivial estimate
\[ \| WF a \|_{\mathcal{E}_2} \leq C \left( \int_{\mathbb{R}^d} W^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^d} a^2 dx \right)^{1/2}. \] (2.18)

Proposition 2.11. Let $W$ and $a$ be two functions from the class $\ell^p(\mathbb{Z}^d; L^2(\mathbb{Q}))$ where $p \in [1, 2]$. Then
\[ \| WF a \|_{\mathcal{E}_p}^p \leq C \sum_n \left( \int_{Q+n} W^2 dx \right)^{p/2} \sum_n \left( \int_{Q+n} a^2 dx \right)^{p/2}. \] (2.19)

Proof. Let $W$ and $a$ be in $\ell^p(\mathbb{Z}^d; L^2(\mathbb{Q}))$. Consider the functions
\[ W_z(x) = \sum_n \| W \|_{L^2(Q+n)}^{-1} \chi_n(x) W(x) \]
and
\[ a_z(x) = \sum_n \| a \|_{L^2(Q+n)}^{-1} \chi_n(x) a(x). \]

For a finite rank operator $K$ we set
\[ K_z = \| K \|_{L^q(\mathbb{Z}^d; L^2(\mathbb{Q}))}^{p-1} K \]
and consider the analytic function
\[ f(z) = \text{Tr} \left( W_z F a_z K_z \right) \]
on the vertical strip
\[ \{ z \in \mathbb{C} : \ p/2 \leq \text{Re} z \leq p \}. \]

By the three lines lemma, we derive from (2.17) and (2.18) that
\[ |f(z)| \leq C \| W \|^{\text{Re} z}_{\ell^p(\mathbb{Z}^d; L^2(\mathbb{Q}))} \| a \|^{\text{Re} z}_{\ell^p(\mathbb{Z}^d; L^2(\mathbb{Q}))} \| K \|_{L^q(\mathbb{Z}^d; L^2(\mathbb{Q}))}^{p-\text{Re} z/q}, \quad q = \frac{p}{p-1}. \] (2.20)

Since (2.20) holds for all $K$, the relation (6.4) follows from (2.20) with $z = 1$. \hfill \Box

Corollary 2.12. Let $p \in [1/2, 1]$ satisfy the condition $p > d/4$. Then
\[ \| W(-\Delta + I)^{-2} W \|_{\mathcal{E}_p}^p \leq C \sum_n \left( \int_{Q+n} V(x) dx \right)^p. \] (2.21)
While the previous statements are useful only for \(d \leq 3\), the next assertion might be applied in higher dimensions.

**Proposition 2.13.** Let \(p \geq 1\) satisfy the condition \(p > d/4\). Then

\[
\|W(-\Delta + I)^{1/2}W\|_{p} \leq C \int_{\mathbb{R}^d} V^p(x) \, dx
\]  

(2.22)

This statement immediately follows from the so-called Simon-Seiler estimate.

3. **Main ingredients of the proof of Theorem 1.3**

Define \(W_1\) and \(W_2\) setting

\[
W_1(x) = \begin{cases} 
W(x) & \text{if } |x| < \varepsilon \cdot s^{-1/\nu} \\
0 & \text{if } |x| > \varepsilon \cdot s^{-1/\nu}
\end{cases}
\]

and

\[
W_2 = W - W_1.
\]

The next result is essentially due to S. Alama, P. Deift, and R. Hempel [2].

**Theorem 3.1.** Let \(\rho(\cdot)\) be the integrated density of states for the operator \(H\). Then

\[
n_{\pm}(s, W_2(H - \lambda)^{-1}W_2) \sim \pm s^{-d/\nu} \int_{|x| > \varepsilon} (\rho(\lambda \pm \Phi |x|^{-\nu}) - \rho(\lambda)) \, dx
\]  

(3.1)

as \(s \to 0\). Moreover,

\[
n_{-}(s, W(H - \lambda)^{-1}W) \sim s^{-d/\nu} \int_{\mathbb{R}^d} (\rho(\lambda) - \rho(\lambda - \Phi |x|^{-\nu})) \, dx
\]  

(3.2)

as \(s \to 0\).

**Proof.** In order to describe the approach of [2], we need to introduce \(V_R\) setting

\[
V_R(x) = \begin{cases} 
V(x) & \text{if } x \in [-R, R]^d \text{ and } |x| > \varepsilon \alpha^{1/\nu}; \\
0 & \text{if } x \notin [-R, R]^d \text{ or } |x| \leq \varepsilon \alpha^{1/\nu},
\end{cases}
\]

for \(\alpha > 0\). After that, we need to consider the operator \(H_R = -\Delta + q\) on the cube \([-2R, 2R]^d\) with the Dirichlet conditions on the boundary. Let \(N_{\pm}(\alpha, \lambda, A, B)\) be the number of eigenvalues of \(A \mp tB\) that pass the point \(\lambda\) as \(t\) increases from 0 to \(\alpha\). Then according to Lemma 1.12 of [2], for any \(\delta > 0\), there exist \(C > 0\) and \(\lambda^+_R\) such that \(|\lambda^+_R - \lambda| < \delta\) and

\[
N_{\pm}((1 - \delta)\alpha, \lambda^+_R, H_R, V_R) \leq N_{\pm}(\alpha, \lambda, H, V_R) \leq N_{\pm}((1 + \delta)\alpha, \lambda^+_R, H_R, V_R), \quad \forall R > C\alpha^{1/\nu}.
\]

(3.3)

Lemma 1.10 of the same article [2] claims that there is a constant \(C_0\) such that

\[
N_{\pm}(\alpha, \lambda, H, V_R) \leq N_{\pm}(\alpha, \lambda, H, V) \leq N_{\pm}((1 + \delta)\alpha, \lambda, H, V_R), \quad \forall R > C_0\left(\frac{\alpha}{\delta}\right)^{1/\nu}.
\]

(3.4)

Combining (3.3) with (3.4), one obtains

\[
N_{\pm}((1 - \delta)\alpha, \lambda^+_R, H_R, V_R) \leq N_{\pm}(\alpha, \lambda, H, V) \leq N_{\pm}((1 + \delta)^2\alpha, \lambda^+_R, H_R, V_R)
\]

(3.5)

for \(R\) of the form

\[
R = R_\delta \alpha^{1/\nu},
\]

(3.6)

where \(R_\delta \to \infty\) as \(\delta \to 0\).
Define the set \( K_{\varepsilon,\delta} \) by
\[
K_{\varepsilon,\delta} := \{ x \in \mathbb{R}^d : |x| > \varepsilon, \quad x \in [-2R_{\delta}, 2R_{\delta}]^d \}
\]
where \( R_{\delta} \) is the same as in (3.6). We are going to show that
\[
\liminf_{\alpha \to \infty} \alpha^{-d/\nu} N_\pm((1 - \delta)\alpha, \lambda^\pm_{R}, H_{R}, V_{R}) \geq \pm (1 - \delta)^{d/\nu} \int_{K_{\varepsilon,\delta}} \left( \rho(\lambda \mp \delta \pm \Phi|x|^{-\nu}) - \rho(\lambda) \right) dx, \tag{(3.7)}
\]
and
\[
\limsup_{\alpha \to \infty} \alpha^{-d/\nu} N_\pm((1 + \delta)^2\alpha, \lambda^\pm_{R}, H_{R}, V_{R}) \leq \pm (1 + \delta)^{2d/\nu} \int_{K_{\varepsilon,\delta}} \left( \rho(\lambda \pm \delta \pm \Phi|x|^{-\nu}) - \rho(\lambda) \right) dx \tag{(3.8)}
\]
Relations (3.5), (3.7) and (3.8) would imply then that
\[
\pm (1 - \delta)^{d/\nu} \int_{K_{\varepsilon,\delta}} \left( \rho(\lambda \mp \delta \pm \Phi|x|^{-\nu}) - \rho(\lambda) \right) dx \leq \liminf_{\alpha \to \infty} \alpha^{-d/\nu} N_\pm(\alpha, \lambda, H, V), \tag{(3.9)}
\]
and
\[
\limsup_{\alpha \to \infty} \alpha^{-d/\nu} N_\pm(\alpha, \lambda, H, V) \leq \pm (1 + \delta)^{2d/\nu} \int_{K_{\varepsilon,\delta}} \left( \rho(\lambda \pm \delta \pm \Phi|x|^{-\nu}) - \rho(\lambda) \right) dx. \tag{(3.10)}
\]
Taking the limits as \( \delta \to 0 \) in (3.9), (3.10) and using the Birman-Schwinger principle, we obtain (3.1).

To establish the relations (3.7) and (3.8) one needs to approximate \( \Phi|x|^{-\nu} \) by a piecewise constant function \( \Psi \geq 0 \) on a set containing \( K_{\varepsilon,\delta} \). Let \( \Psi \) be such an approximation. Assume that \( \Psi(x) > \Phi|x|^{-\nu} \) for all \( x \in K_{\varepsilon,\delta} \) and that the sets where \( \Psi \) is constant are cubes. Since some pieces of the boundary of \( K_{\varepsilon,\delta} \) are round, while the sets where the function is constant are cubes, we can not assume that \( \Psi \) is zero outside of \( K_{\varepsilon,\delta} \). However, we can always assume that the measure of \( (\text{supp} \Psi) \setminus K_{\varepsilon,\delta} \) is small. The following property plays a crucial role in our arguments:
\[
\alpha V_{R}(x) \leq \Psi(\alpha^{-1/\nu} x), \quad \text{for sufficiently large } \alpha. \tag{(3.11)}
\]
Let \( Q_j \subset (\text{supp} \Psi) \) be the cubes on which \( \Psi \) is constant. Denote the value of \( \Psi \) on \( Q_j \) by \( \Psi_j \). Let \( H_{Q_j}^\pm = -\Delta + q(x) \) be the operator on \( \alpha^{1/\nu} Q_j \) with either Neumann or Dirchlet conditions on the boundary depending on the choice of the sign \( \pm \). Finally, let \( E_A(\cdot) \) denote the operator valued spectral measure of an operator \( A = A^* \). Then, using the Dirichlet-Neumann bracketing philosophy and (3.11), we obtain that
\[
N_\pm((1 + \delta)^2\alpha, \lambda^\pm_{R}, H_{R}, V_{R}) \leq \sum_j \text{rank} E_{H_{Q_j}^\pm - (1 + \delta)^2 \Psi_j}(-\infty, \lambda + \delta) - \text{rank} E_{H_{Q_j}^\pm - (1 - \delta)^2 \Psi_j}(-\infty, \lambda).
\]
According to the definition of the integrated density of states, The asymptotics of the right hand side is
\[
\alpha^{d/\nu} \sum_j \left( \rho(\lambda + \delta + (1 + \delta)^2 \Psi_j) - \rho(\lambda) \right)|Q_j|.
\]
Since the approximation \( \Psi \) of \( \Phi|x|^{-\nu} \) is arbitrary, we obtain that
\[
\limsup_{\alpha \to \infty} \frac{N_\pm((1 + \delta)^2\alpha, \lambda^\pm_{R}, H_{R}, V_{R})}{\alpha^{d/\nu}} \leq \int_{K_{\varepsilon,\delta}} \left( \rho(\lambda + \delta + (1 + \delta)^2 \Phi|x|^{-\nu}) - \rho(\lambda) \right) dx. \tag{(3.12)}
\]
The remaining relations are established in the same way. \( \Box \)
The next proposition is obtained from (3.1) by an application of the Lebesgue dominated convergence theorem requiring an integrable bound for the quantity in the left hand side of (3.1).

**Proposition 3.2.** Let \( p < d/\nu \). Then

\[
\int_0^\infty n_\pm(t, W_2(H - \lambda)^{-1}W_2) t^{p-1} dt \sim \pm s^{p-d/\nu} \int_1^{\infty} t^{p-1} \int_{|x| > \varepsilon} \left( \rho(\lambda \pm t^{-1} \Phi|x|^{-\nu}) - \rho(\lambda) \right) dx dt
\]

(3.13) as \( s \to 0 \).

**Proof.** According to (3.1),

\[
s^{d/\nu} n_\pm(st, W_2(H - \lambda)^{-1}W_2) \to \pm \int_{|x| > \varepsilon} \left( \rho(\lambda \pm t^{-1} \Phi|x|^{-\nu}) - \rho(\lambda) \right) dx,
\]

(3.14) as \( s \to 0 \), for all \( t > 0 \). On the other hand, according to the Cwikel-Lieb-Rozenblum estimate,

\[
n_\pm(st, W_2(H - \lambda)^{-1}W_2) \leq (st)^{-d/2} C_0 \int_{\mathbb{R}^d} W_2^d dx \leq Cs^{-d/\nu} t^{-d/2} \varepsilon^{d(1-\nu)/2}.
\]

(3.15) It follows by the Lebesgue dominated convergence theorem from (3.14) and (3.15) that

\[
s^{d/\nu} \int_1^{\infty} n_\pm(st, W_2(H - \lambda)^{-1}W_2) t^{p-1} dt \to \pm \int_1^{\infty} t^{p-1} \int_{|x| > \varepsilon} \left( \rho(\lambda \pm t^{-1} \Phi|x|^{-\nu}) - \rho(\lambda) \right) dx dt,
\]

as \( s \to 0 \). Changing the variables we obtain

\[
s^{-p+d/\nu} \int_s^{\infty} n_\pm(t, W_2(H - \lambda)^{-1}W_2) t^{p-1} dt \to \pm \int_1^{\infty} t^{p-1} \int_{|x| > \varepsilon} \left( \rho(\lambda \pm t^{-1} \Phi|x|^{-\nu}) - \rho(\lambda) \right) dx dt,
\]

as \( s \to 0 \). \(\square\)

**Proposition 3.3.** Let \( p \leq 1 \) satisfy the condition \( d/4 < p < d/2 \). Assume that

\[
2 < \nu < 2 + \frac{4p}{d(d - 2p)}
\]

(3.16) and \( r(s) = O(s^{-d/2}) \) as \( s \to 0 \). Then

\[
\sum_{k=1}^{r(s)} s_k^p W_1(H - \lambda)^{-1}W_2 = o(s^{p-d/\nu}),
\]

(3.17) as \( s \to 0 \).

**Proof.** Let \( \zeta \) be a smooth \( C^\infty(\mathbb{R}) \)-function satisfying the conditions

\[
\zeta(t) = \begin{cases} 
0, & \text{if } t < 1; \\
1, & \text{if } t > 2.
\end{cases}
\]

Define \( \zeta_s \) and \( \zeta_s^+ \) on \( \mathbb{R}^d \) setting

\[
\zeta_s(x) = \zeta(|x| - \varepsilon s^{-1/\nu}), \quad x \in \mathbb{R}^d
\]

and

\[
\zeta_s^+(x) = \zeta(|x| - \varepsilon s^{-1/\nu} + 2), \quad x \in \mathbb{R}^d.
\]
Let also \( \eta = 1 - \zeta_s \) and \( \eta^+ = 1 - \zeta^+_s \). Then

\[
W_1(H - \lambda)^{-1}W_2 = W_1\eta(H - \lambda)^{-1}\zeta^+_s W_2,
\]
due to the fact that \( \eta W_1 = W_1 \) and \( \zeta^+_s W_2 = W_2 \). Consequently,

\[
W_1(H - \lambda)^{-1}W_2 = W_1\zeta^+_s (H - \lambda)^{-1}\eta W_2 + W_1(H - \lambda)^{-1}[\eta, H](H - \lambda)^{-1}W_2 + W_1(H - \lambda)^{-1}[H, \zeta^+_s](H - \lambda)^{-1}\eta W_2.
\]

Let us estimate the singular values of the middle operator:

\[
\sum_1^r s_k^p \left( W_1(H - \lambda)^{-1}[\eta, H](H - \lambda)^{-1}W_2 \right) \leq (3.18)
\]

\[
\sum_1^r s_k^p \left( W_1(H - \lambda)^{-1}\right) s_k^p \left( [\eta, H](H - \lambda)^{-1}W_2 \right),
\]

for any \( r \in \mathbb{N} \). Observe that

\[
s_k \left( [\eta, H](H - \lambda)^{-1}W_2 \right) \leq C||W_2||_\infty \cdot s_k \left( ||\nabla \eta|| + ||\Delta \eta||(-\Delta + I)^{-1/2} \right) \leq Ck^{-1/d} ||\nabla \eta|| + ||\Delta \eta|| \cdot \left( \frac{s}{\varepsilon^\nu} \right)^{1/2} \leq Ck^{-1/d} \left( \frac{s}{\varepsilon^\nu} \right)^{1/2 - (d-1)/d\nu}.
\]

Consequently,

\[
\sum_1^{r(s)} s_k^{2p} \left( [\eta, H](H - \lambda)^{-1}W_2 \right) \leq C \left( \frac{s}{\varepsilon^\nu} \right)^{p(1-2/\nu+2/d\nu)} \sum_1^{r(s)} k^{-2p/d} = (3.19)
\]

\[
O(s^{2p-d/2-2p(d-1)/d\nu}) = o(s^{p-d/\nu}), \quad \text{as } s \to 0.
\]

On the other hand,

\[
\sum_1^r s_k^{2p} \left( W_1(H - \lambda)^{-1} \right) \leq ||W_1(H - \lambda)^{-2}W_1||_p \leq C \| W_1 \|_{2p} \leq C \varepsilon^{d-\nu p} s^{-d/\nu} (3.20)
\]

It follows from (3.18), (3.19) and (3.20) that

\[
\sum_1^{r(s)} s_k^{2p} \left( W_1(H - \lambda)^{-1}[\eta, H](H - \lambda)^{-1}W_2 \right) = o(s^{p-d/\nu}),
\]

as \( s \to 0 \). One can prove similarly that

\[
\sum_1^{r(s)} s_k^{2p} \left( W_1(H - \lambda)^{-1}[H, \zeta^+_s](H - \lambda)^{-1}\eta W_2 \right) = o(s^{p-d/\nu}),
\]

as \( s \to 0 \). It remains to establish the relation

\[
\sum_1^{r(s)} s_k^{2p} \left( W_1\zeta^+_s (H - \lambda)^{-1}\eta W_2 \right) = o(s^{p-d/\nu}), (3.21)
\]
as $s \to 0$. For that purpose, we observe that for any $r \in \mathbb{N}$,
\[
\sum_{i=1}^{r} s_k^p \left( W_1 \zeta^+_s (H - \lambda)^{-1} \eta W_2 \right) \leq C \sum_{i=1}^{r} s_k^p \left( W_1 \zeta^+_s (-\Delta + I)^{-1/2} \right) s_k^p \left( W_2 \eta (-\Delta + I)^{-1/2} \right)
\]
with some constant $C > 0$. Consequently,
\[
\sum_{i=1}^{r} s_k^p \left( W_1 \zeta^+_s (H - \lambda)^{-1} \eta W_2 \right) \leq C \left\| W_1 \zeta^+_s \right\|_d^p \left\| W_2 \eta \right\|_d^p \sum_{i=1}^{r} \eta d^{-2p/d} \leq C \varepsilon_d^{1-2p/d} s_p^{p(2d-1)/d
u}.
\]

The relation (3.22) implies (3.21). \qed

Note that the restriction (3.16) implies that we consider all values $\nu > 2$ in $d = 1$ and $d = 2$. However, in $d = 3$, the largest possible value of $\nu$ we consider is $3 + 1/3$.

**Proposition 3.4.** Let $p \leq 1$ satisfy the condition $d/4 < p < d/\nu$. Then there is a constant $C > 0$ independent of $\varepsilon > 0$ and $s > 0$ such that
\[
\int_{s}^{\infty} \left| n(t, W_1 (H - \mu)^{-1} W_1) - n(t, W_1 (H - \lambda)^{-1} W_1) \right| t^{p-1} dt \leq C \varepsilon^{d-p\nu} s_p^{p-d/\nu},
\]

**Proof.** It turns out that one can show that
\[
\int_{0}^{\infty} \left| n(t, W_1 (H - \mu)^{-1} W_1) - n(t, W_1 (H - \lambda)^{-1} W_1) \right| t^{p-1} dt \leq C \varepsilon^{d-p\nu} s_p^{p-d/\nu},
\]
which is stronger than (3.23). In its turn, this inequality follows form combination of the fact that, for $0 < p \leq 1$,
\[
p \int_{0}^{\infty} \left( n(t, W_1 (H - \mu)^{-1} W_1) - n(t, W_1 (H - \lambda)^{-1} W_1) \right) t^{p-1} dt = \quad ,
\]
\[
\left\| W_1 (H - \lambda)^{-1} W_1 \right\|_{\mathcal{E}_p}^p - \left\| W_1 (H - \mu)^{-1} W_1 \right\|_{\mathcal{E}_p}^p \leq \quad ,
\]
\[
\left\| W_1 (H - \lambda)^{-1} W_1 - W_1 (H - \mu)^{-1} W_1 \right\|_{\mathcal{E}_p}^p = \left\| \lambda - \mu \right\| W_1 (H - \lambda)^{-1} (H - \mu)^{-1} W_1 \right\|_{\mathcal{E}_p}^p
\]
and the estimate
\[
p \int_{0}^{\infty} \left( n_-(t, W_1 (H - \mu)^{-1} W_1) + n_-(t, W_1 (H - \lambda)^{-1} W_1) \right) t^{p-1} dt \leq \quad ,
\]
\[
\left\| W_1 (H - \lambda)^{-1} W_1 \right\|_{\mathcal{E}_p}^p + \left\| W_1 (H - \mu)^{-1} W_1 \right\|_{\mathcal{E}_p}^p \leq \quad ,
\]
\[
\left\| S_\lambda \right\|^p \left\| W_1 (H - \lambda)^{-2} W_1 \right\|_{\mathcal{E}_p}^p + \left\| S_\mu \right\|^p \left\| W_1 (H - \mu)^{-2} W_1 \right\|_{\mathcal{E}_p}^p.
\]
Here
\[
S_\lambda = (H - \lambda)^2 [(H - \lambda)^{-1}], \quad \lambda \notin \sigma(H).
\]

Consequently,
\[
\int_{0}^{\infty} \left| n(t, W_1 (H - \mu)^{-1} W_1) - n(t, W_1 (H - \lambda)^{-1} W_1) \right| t^{p-1} dt \leq \quad ,
\]
\[
C_0 \left\| W_1 (-\Delta + I)^{-2} W_1 \right\|_{\mathcal{E}_p}^p \leq C \left\| W_1 \right\|_{2p}^{2p} \leq C \varepsilon^{d-p\nu} s_p^{p-d/\nu}.
\]
\qed
Proposition 3.5. Let $p < d/\nu$ and let

$$\omega = \int_1^\infty s^{p-d/\nu-1}ds = \frac{\nu}{d-p\nu}.$$ 

Then

$$\int_s^\infty n_-(t, W(H - \lambda)^{-1}W)t^{p-1}dt \sim s^{p-d/\nu}\omega \int_{\mathbb{R}^d} \left( \rho(\lambda) - \rho(\lambda - \Phi|x|^{-\nu}) \right) dx$$

as $s \to 0$.

Proof. We need to prove that

$$s^{-p+d/\nu}\int_s^\infty n_-(t, W(H - \lambda)^{-1}W)t^{p-1}dt \to \omega \int_{\mathbb{R}^d} \left( \rho(\lambda) - \rho(\lambda - \Phi|x|^{-\nu}) \right) dx,$$

as $s \to 0$, which is the same as

$$s^{d/\nu}\int_1^s n_-(st, W(H - \lambda)^{-1}W)t^{p-1}dt \to \omega \int_{\mathbb{R}^d} \left( \rho(\lambda) - \rho(\lambda - \Phi|x|^{-\nu}) \right) dx,$$

as $s \to 0$. For that purpose, we observe that, according to (3.2),

$$s^{d/\nu}n_-(st, W(H - \lambda)^{-1}W) \to t^{-d/\nu}\int_{\mathbb{R}^d} \left( \rho(\lambda) - \rho(\lambda - \Phi|x|^{-\nu}) \right) dx,$$

as $s \to 0$, for all $t > 0$. On the other hand,

$$n_-(st, W(H - \lambda)^{-1}W) \leq n_-(st, W(H - \lambda)^{-1}W) \leq n(st, CW(-\Delta + I)^{-2}W) \leq n(\sqrt{s/t}, CW(-\Delta + I)^{-1}).$$

Observe that $W \leq C(1 + |x|^2)^{-\nu/4}$ where $2 < \nu < 4$. Consequently, switching the roles of the momentum and the position variables (which are separated by the Fourier transform operator), we obtain that

$$n(\sqrt{s/t}, CW(-\Delta + I)^{-1}) \leq n(\sqrt{s/t}, C(-\Delta + I)^{-\nu/4}(|x|^2 + 1)^{-1}) \leq C_0(st)^{-d/\nu}.$$ (3.28)

The relation (3.25) follows now by the Lebesgue dominated convergence theorem from (3.26),(3.27) and (3.28). \qed

4. END OF THE PROOF OF THEOREM 1.3

Let

$$\tilde{X}_\lambda = W_1(H - \lambda)^{-1}W_1 + W_2(H - \lambda)^{-1}W_2.$$ (4.1)

Since $X_\lambda - \tilde{X}_\lambda = W_1(H - \lambda)^{-1}W_2 + W_2(H - \lambda)^{-1}W_1$, we can employ (2.9) and (3.17) to conclude that

$$p \int_s^\infty \left( n(t, X_\lambda) - n(t, \tilde{X}_\lambda) \right)t^{p-1}dt = o(s^{p-d/\nu}), \quad \text{as} \quad s \to 0.$$ (4.2)

Observe that the sum of the operators in (4.1) is orthogonal. Therefore,

$$n_\pm(t, \tilde{X}_\lambda) = n_\pm(t, W_1(H - \lambda)^{-1}W_1) + n_\pm(t, W_2(H - \lambda)^{-1}W_2), \quad \forall t > 0.$$
According to (3.23)
\begin{align*}
0 & \leq \int_s^\infty \left( n_+(t, \tilde{X}_\mu) - n_+(t, \tilde{X}_\lambda) \right) t^{p-1} dt - \\
& \quad \int_s^\infty \left( n_+(t, W_2(H - \mu)^{-1} W_2) - n_+(t, W_2(H - \lambda)^{-1} W_2) \right) t^{p-1} dt \\
& \leq C \varepsilon^{-d/p\nu} s^{p-d/p\nu}.
\end{align*}

According to (3.13), the relation above implies that
\begin{align*}
\int_1^\infty t^{p-1} \int_{|x|>\varepsilon} \left( \rho(\mu \pm t^{-1} \Phi|x|^{-\nu}) - \rho(\lambda \pm t^{-1} \Phi|x|^{-\nu}) \right) dx dt \\
& \leq \limsup_{s \to 0} s^{-\nu+d/\nu} \int_s^\infty \left( n_+(t, \tilde{X}_\mu) - n_+(t, \tilde{X}_\lambda) \right) t^{p-1} dt \\
& \leq C \varepsilon^{-d/p\nu} + \int_1^\infty t^{p-1} \int_{|x|>\varepsilon} \left( \rho(\mu \pm t^{-1} \Phi|x|^{-\nu}) - \rho(\lambda \pm t^{-1} \Phi|x|^{-\nu}) \right) dx dt.
\end{align*}

Therefore, the relation (4.2) leads to
\begin{align*}
\int_s^\infty \left( n(t, X_\mu) - n(t, X_\lambda) \right) t^{p-1} dt & \sim \\
& s^{p-d/\nu} \omega \int_{|x|>\varepsilon} \left( \rho(\mu + \Phi|x|^{-\nu}) - \rho(\lambda + \Phi|x|^{-\nu}) + \rho(\mu - \Phi|x|^{-\nu}) - \rho(\lambda - \Phi|x|^{-\nu}) \right) dx,
\end{align*}

as \( s \to 0 \).

It remains to use (3.24) and the equality \( n = n_+ + n_- \) to conclude that
\begin{align*}
\int_s^\infty \left( n_+(t, X_\mu) - n_+(t, X_\lambda) \right) t^{p-1} dt & \sim \\
& s^{p-d/\nu} \omega \int_{|x|>\varepsilon} \left( \rho(\mu + \Phi|x|^{-\nu}) - \rho(\lambda + \Phi|x|^{-\nu}) \right) dx,
\end{align*}

as \( s \to 0 \).

Due to (2.13) the relation (4.4) coincides with (1.5).

5. PROOF OF COROLLARY 1.2

We are going to use the Borel-Cantelli lemma. For any \( \varepsilon > 0 \), we consider the set
\[ S_n(\varepsilon) = \{ \theta \in [1, \tau] : N(\tau^n \theta) \geq \varepsilon \tau^p \}. \]

Observe now that according to Chebyshev’s inequality,
\[ |S_n(\varepsilon)| \leq \varepsilon^{-1} \tau^{-np} \int_1^\tau N(\tau^n \theta) d\theta \leq \varepsilon^{-1} \tau^{p+1} \int_{\tau^n}^{\tau^{n+1}} N(t) \frac{dt}{t^{p+1}}. \]

Consequently,
\[ \sum_n |S_n(\varepsilon)| < \infty. \]
Applying the Borel-Cantelli lemma, we obtain that, for any \( \varepsilon > 0 \) there is a set \( \Omega_\varepsilon \subset [1, \tau] \) of full measure \(|\Omega_\varepsilon| = \tau - 1 \) such that for almost every \( \theta \in \Omega_\varepsilon \), there is a number \( n_{\varepsilon, \theta} \in \mathbb{N} \) such that
\[
N(\tau^n \theta) < \varepsilon \tau^n, \quad \forall n > n_{\varepsilon, \theta}.
\]

Take now
\[
\Omega = \bigcap_{\varepsilon \in \mathbb{Q} \cap (0,\infty)} \Omega_\varepsilon.
\]

Then \( \Omega \) is a subset of \([1, \tau]\) of full measure \( \tau - 1 \), such that
\[
N(\tau^n \theta) = o(\tau^n), \quad \text{as} \quad \tau \to \infty, \quad \forall \theta \in \Omega.
\]

6. One dimensional Dirac operators

Another example of a self-adjoint operator whose spectrum has a bounded gap is the one dimensional Dirac operator on \( L^2(\mathbb{R}; \mathbb{C}^2) \) defined by
\[
\mathcal{D} = J \cdot i \frac{d}{dx} + J_0.
\]

Here, \( J \) and \( J_0 \) are two self-adjoint \( 2 \times 2 \)-matrices having the properties
\[
J^2 = J_0^2 = 1, \quad J_0J + JJ_0 = 0.
\]

This operator defined on the Sobolev space \( \mathcal{H}^1(\mathbb{R}; \mathbb{C}^2) \) is self-adjoint in \( L^2(\mathbb{R}; \mathbb{C}^2) \). The spectrum \( \sigma(\mathcal{D}) \) of the free Dirac operator is the complement of the interval \((-1, 1)\), i.e.
\[
\sigma(\mathcal{D}) = \mathbb{R} \setminus (-1, 1).
\]

Consider now the operator
\[
\mathcal{D}(\alpha) = \mathcal{D} - \alpha V, \quad \alpha > 0,
\]
where \( V \) is the operator of multiplication by a positive bounded function \( V \geq 0 \) defined on \( \mathbb{R} \). We will assume that
\[
\sum_n \left( \int_{n}^{n+1} V(x)dx \right)^p < \infty, \quad \text{for some} \quad 1/2 < p \leq 1.
\]

Under this condition, the spectrum of \( \mathcal{D}(\alpha) \) in \((-1, 1)\) is discrete. Therefore, one can study the number of eigenvalues of \( \mathcal{D}(\alpha) \) in any fixed subinterval \([\lambda, \mu]\) of \((-1, 1)\).

**Theorem 6.1.** Let \( \lambda < \mu \) be two points in \((-1, 1)\). Let \( N(\alpha) \) be the number of eigenvalues of the operator \( \mathcal{D}(\alpha) \) in \([\lambda, \mu]\). Assume that \( V \) satisfies (6.2) for some \( 1/2 < p \leq 1 \). Then there is a positive constant \( C \) independent of \( V \) such that
\[
\int_{0}^{\infty} N(\alpha) \alpha^{-p-1}d\alpha \leq C \sum_{n \in \mathbb{Z}} \left( \int_{n}^{n+1} V(x)dx \right)^p.
\]

To prove this theorem, we will need the following result.

**Proposition 6.2.** Let \( d = 1 \) and let \( 1/2 < p \leq 1 \). Then
\[
\|W(-\Delta + I)^{-1}W\|_{\mathcal{S}_p}^p \leq C \sum_n \left( \int_{n}^{n+1} V(x)dx \right)^p.
\]
Proof. According to Proposition 2.9, for $d = 1$,
\[
\int |W(-\Delta + I)^{-1/2}|_{\Sigma_1} \leq C\|W\|_{\ell^1(\mathbb{Z}; L^2([0,1]))}
\]
where
\[
\|W\|_{\ell^1(\mathbb{Z}; L^2([0,1]))} = \sum_n (\int_n^{n+1} W^2 dx)^{1/2}
\]
This implies that if $\chi_n$ is the characteristic function of the interval $[n, n+1)$, then for any $p > 1/2$,
\[
\|\chi_n W(-\Delta + I)^{-1/2}\|_{L_p} \leq C\left(\int_n^{n+1} W^2 dx\right)^{1/2}.
\]
(6.5)
Setting $V = W^2$, we can rewrite (6.5) in the form
\[
\|(-\Delta + I)^{-1/2}\chi_n V(-\Delta + I)^{-1/2}\|_{L_p} \leq C\left(\int_n^{n+1} V dx\right)^{1/2},
\]
which leads to
\[
\sum_n \|(-\Delta + I)^{-1/2}\chi_n V(-\Delta + I)^{-1/2}\|_{L_p} \leq C\sum_n \left(\int_n^{n+1} V dx\right)^{p}.
\]
for $p \leq 1$. It remains to note that, since the $s$-numbers of $W(-\Delta + I)^{-1/2}$ coincide with the $s$-numbers of $(-\Delta + I)^{-1/2}V(-\Delta + I)^{-1/2}$, the Schatten norms of these two operators are also the same. □

We remind the reader that the following assertion was proved in Section 2.

**Proposition 6.3.** Let $0 < p \leq 1$. Let $A$ and $B$ be two self-adjoint compact operators such that $B \leq A$. Then, for any $s > 0$,
\[
p \int_s^{\infty} \left(n_+(t, A) - n_+(t, B)\right) t^{p-1} dt \leq \sum_{k=1}^{n_+(s,A)+1} s_k^p (A - B).
\]

Let us now introduce the Birman-Schwinger operator
\[
\tilde{X}_\lambda = W(\mathcal{D} - \lambda)^{-1} W, \quad \lambda \in (-1, 1),
\]
(6.6)
where $W = \sqrt{V}$. The next statement follows from Corollary 6.3.

**Proposition 6.4.** Let $0 < p \leq 1$. Let $\tilde{X}_\lambda$ be the family of operators defined by (6.6). Then for any $\lambda < \mu$ and $s > 0$,
\[
p \int_s^{\infty} \left(n_+(s, \tilde{X}_\mu) - n_+(s, \tilde{X}_\lambda)\right) s^{p-1} ds \leq \sum_{k=1}^{n_+(s,\tilde{X}_\mu)+1} s_k^p (\tilde{X}_\mu - \tilde{X}_\lambda).
\]

The following proposition is a consequence of Corollary 2.7 and Proposition 6.4:
Proposition 6.5. Let $0 < p \leq 1$. Let $\tilde{X}_\lambda$ be the family of operators defined by (6.6). Let $N(\alpha)$ be the number of eigenvalues of the operator $\mathcal{D}(\alpha)$ in $[\lambda, \mu]$ contained in $(-1,1)$. Then for any $\alpha > 0$,
\[
p \int_0^\alpha N(t)t^{-p-1}dt \leq \sum_{k=1}^{n_+(s,\tilde{X}_\lambda)+1} s_k^p(\tilde{X}_\mu - \tilde{X}_\lambda), \quad \alpha s = 1.
\]

Now we are ready to prove the following result.

Theorem 6.6. Let $0 < p \leq 1$. Let $\lambda < \mu$ be two points in $(-1,1)$. Let $N(\alpha)$ be the number of eigenvalues of $\mathcal{D}(\alpha)$ in $[\lambda, \mu]$. Then there exists a positive constant $C$ which does not depend on the potential $V$ such that
\[
\int_0^\alpha N(t)\frac{dt}{t^{p+1}} \leq C \sum_{k=1}^{n_+(s,\tilde{X}_\mu)+1} s_k^p(W(-\Delta + I)^{-1}W), \quad s = \alpha^{-1}, \quad W = \sqrt{V}.
\]

Proof. It is enough to show that, for any $\lambda \in (-1,1)$ and $\mu \in (-1,1)$, there is a constant $C > 0$ such that
\[
s_k(X_\mu - X_\lambda) \leq Cs_k^2(W(-\Delta + I)^{-1/2}).
\]

This relation follows from
\[
s_k(X_\mu - X_\lambda) = |\mu - \lambda|s_k(W(\mathcal{D} - \mu)^{-1}(\mathcal{D} - \lambda)^{-1}W) \leq Cs_k^2(W(-\Delta + I)^{-1/2}).
\]

The proof is completed. \qed

The statement of Theorem 6.1 follows now from Proposition 6.2 and Theorem 6.6.

References


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