# EXISTENCE OF SOLUTIONS OF SOME QUADRATIC INTEGRAL EQUATIONS IN DIMENSIONS TWO AND THREE

VITALI VOUGALTER

Abstract. The article is devoted to the existence of solutions of a certain quadratic integral equation in $H^2(\mathbb{R}^d)$, $d = 2, 3$. The theory of quadratic integral equations has many important applications in the mathematical physics, economics, biology. It is crucial for describing the real world problems. The proof of the existence of solutions relies on a fixed point technique in the Sobolev space in dimensions two and three.

## 1. Introduction

The present work deals with the existence of solutions of the following integral equation

$$u(x) = u_0(x) + [Tu(x)] \int_{\mathbb{R}^d} K(x-y)g(u(y))dy, \quad x \in \mathbb{R}^d, \quad d = 2, 3. \tag{1.1}$$

We generalize the results of the preceding article [16], in which the solvability of the problem analogous to (1.1) was established in $H^1(\mathbb{R})$. The precise conditions on the functions $u_0(x)$, $g(u)$, the linear operator $T$ and the kernel $K(x)$ will be formulated below. The second term in the right side of (1.1) is a product of $Tu(x)$ and the integral operator acting on the function $g(u)$, for which the sublinear growth will be established in the proof of Theorem 1.3. further down. Thus, the integral equation of this kind is called *quadratic*. The theory of the integral equations has many important applications in describing the numerous events and problems of the real world. It is caused by the fact that this theory is frequently applicable in various branches of mathematics and in mathematical physics, economics, biology as well as for solving the real world problems. The quadratic integral equations appear in the theories of the radiative transfer, neutron transport, in the kinetic theory of gases, in the design of the bandlimited signals for the binary communication using the simple memoryless correlation detection, when the signals are disturbed by the additive white Gaussian noise (see e.g. [1], [5], [11] and the references therein). The article [1] is devoted to the solvability of a nonlinear quadratic integral equation in the Banach space

2010 *Mathematics Subject Classification.* 45G10, 47H09, 47H10.
*Key words and phrases.* quadratic integral equation, fixed point technique, Sobolev space.

of the real functions being defined and continuous on a bounded and closed interval using the fixed point technique. The works [2] and [4] deal with the studies of the existence of solutions for quadratic integral equations on unbounded intervals. The existence of solutions for quadratic integral inclusions was discussed in [3]. In the paper [10] the authors consider the nondecreasing solutions of a quadratic integral equation of Urysohn-Stieltjes type. The solvability of the quadratic integral equations in Orlicz spaces was treated in [7], [8], [9]. The integro-differential equations containing either Fredholm or non-Fredholm operators arise in the mathematical biology when studying the systems with the nonlocal consumption of resources and the intra-specific competition (see [12], [13], [17], [18] and the references therein). The contraction argument was applied in [15] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term were perturbed. The analogous ideas were used to demonstrate the persistence of pulses for certain reaction-diffusion type equations (see [6]). Let us suppose that the conditions below are satisfied.

**Assumption 1.1.** *Let the kernel $K(x) : \mathbb{R}^d \to \mathbb{R},\ d = 2, 3$ be nontrivial, so that $K(x), \Delta K(x) \ \in L^1(\mathbb{R}^d)$. The function $u_0(x) : \mathbb{R}^d \to \mathbb{R}$ does not vanish identically in $\mathbb{R}^d$ and $u_0(x) \in H^2(\mathbb{R}^d)$. We suppose also that the linear operator $T : H^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)$ is bounded, so that its norm $0 < \|T\| < \infty$.*

It can be trivially checked that for the operator

$$Tu(x) := (-\Delta + 1)^{-1}u(x), \quad u(x) \in H^2(\mathbb{R}^d) \tag{1.2}$$

the conditions above are fulfilled. We introduce the technical quantity

$$Q := \sqrt{\|K(x)\|^2_{L^1(\mathbb{R}^d)} + \|\Delta K(x)\|^2_{L^1(\mathbb{R}^d)}}. \tag{1.3}$$

Clearly, under the assumption above we have $0 < Q < \infty$. We will use the Sobolev space

$$H^2(\mathbb{R}^d) := \big\{u(x) : \mathbb{R}^d \to \mathbb{R} \mid u(x) \in L^2(\mathbb{R}^d),\ \Delta u(x) \in L^2(\mathbb{R}^d)\big\} \tag{1.4}$$

with $d = 2, 3$. It is equipped with the norm

$$\|u\|^2_{H^2(\mathbb{R}^d)} := \|u\|^2_{L^2(\mathbb{R}^d)} + \|\Delta u\|^2_{L^2(\mathbb{R}^d)}. \tag{1.5}$$

By virtue of the Sobolev embedding, we have

$$\|u(x)\|_{L^\infty(\mathbb{R}^d)} \le c_e \|u(x)\|_{H^2(\mathbb{R}^d)}, \quad d = 2, 3. \tag{1.6}$$

Here $c_e > 0$ is a constant. According to the algebra property for the Sobolev space, for any $u(x), v(x) \in H^2(\mathbb{R}^d),\ d = 2, 3$

$$\|u(x)v(x)\|_{H^2(\mathbb{R}^d)} \le c_a \|u(x)\|_{H^2(\mathbb{R}^d)} \|v(x)\|_{H^2(\mathbb{R}^d)}, \tag{1.7}$$

where $c_a > 0$ is a constant, such that $u(x)v(x) \in H^2(\mathbb{R}^d)$ as well.

The Young's inequality (see e.g. Section 4.2 of [14]) allows us to obtain the upper bound on the norm of the convolution as

$$\|u * v\|_{L^2(\mathbb{R}^d)} \leq \|u\|_{L^1(\mathbb{R}^d)}\|v\|_{L^2(\mathbb{R}^d)}. \tag{1.8}$$

Obviously, inequality (1.8) implies the estimate from above

$$\Big\|\Delta_x \int_{\mathbb{R}^d} u(x-y)v(y)dy\Big\|_{L^2(\mathbb{R}^d)} \leq \|\Delta u\|_{L^1(\mathbb{R}^d)}\|v\|_{L^2(\mathbb{R}^d)}. \tag{1.9}$$

Here and below $\Delta_x$ will denote the Laplace operator with respect to the $x$-variable.
Let us look fo the resulting solution of nonlinear problem (1.1) as

$$u(x) = u_0(x) + u_p(x). \tag{1.10}$$

Clearly, we derive the perturbative equation

$$u_p(x) = [T(u_0(x) + u_p(x))] \int_{\mathbb{R}^d} K(x-y)g(u_0(y) + u_p(y))dy, \tag{1.11}$$

where $d = 2, 3$. We introduce a closed ball in the Sobolev space

$$B_\rho := \{u(x) \in H^2(\mathbb{R}^d) \mid \|u\|_{H^2(\mathbb{R}^d)} \leq \rho\}, \quad 0 < \rho \leq 1. \tag{1.12}$$

Let us seek the solution of equation (1.11) as the fixed point of the auxiliary nonlinear problem

$$u(x) = [T(u_0(x) + v(x))] \int_{\mathbb{R}^d} K(x-y)g(u_0(y) + v(y))dy \tag{1.13}$$

in ball (1.12). We introduce the interval on the real line

$$I := [-c_e - c_e\|u_0\|_{H^2(\mathbb{R}^d)},\ c_e + c_e\|u_0\|_{H^2(\mathbb{R}^d)}] \tag{1.14}$$

along with the closed ball in the space of $C_1(I)$ functions, such that

$$D_M := \{g(z) \in C_1(I) \mid \|g\|_{C_1(I)} \leq M\}, \quad M > 0. \tag{1.15}$$

In this context the norm

$$\|g\|_{C_1(I)} := \|g\|_{C(I)} + \|g'\|_{C(I)}, \tag{1.16}$$

where $\|g\|_{C(I)} := \max_{z\in I}|g(z)|$.

**Assumption 1.2.** *Let $g(z) : \mathbb{R} \to \mathbb{R}$, so that $g(0) = 0$. We also assume that $g(z) \in D_M$ and it does not vanish identically on the interval $I$.*

Let us introduce the operator $\tau_g$, so that $u = \tau_g v$, where $u$ is a solution of problem (1.13). Our first main statement is as follows.

**Theorem 1.3.** *Let Assumptions 1.1 and 1.2 hold and*

$$c_a\|T\|(\|u_0\|_{H^2(\mathbb{R}^d)} + 1)^2 QM \leq \frac{\rho}{2}. \tag{1.17}$$

*Then problem (1.13) defines the map $\tau_g : B_\rho \to B_\rho$, which is a strict contraction. The unique fixed point $u_p(x)$ of this map $\tau_g$ is the only solution of equation (1.11) in $B_\rho$.*

Evidently, the cumulative solution of problem (1.1) given by (1.10) will be nontrivial in $\mathbb{R}^d$, $d = 2, 3$ since $g(0) = 0$, the operator $T$ is linear and the function $u_0(x)$ does not vanish identically in the whole space due to the given conditions.

Let us define the auxiliary quantity

$$\sigma := 2c_a(\|u_0\|_{H^2(\mathbb{R}^d)} + 1)\|T\|MQ > 0. \tag{1.18}$$

Our second major proposition is about the continuity of the resulting solution of equation (1.1) given by (1.10) with respect to the function $g$.

**Theorem 1.4.** *Let $j = 1, 2$, the assumptions of Theorem 1.3 hold, so that $u_{p,j}(x)$ is the unique fixed point of the map $\tau_{g_j} : B_\rho \to B_\rho$, which is a strict contraction because the upper bound (1.17) is valid and the cumulative solution of equation (1.1) with $g(z) = g_j(z)$ is given by*

$$u_j(x) = u_0(x) + u_{p,j}(x). \tag{1.19}$$

*Then the estimate from above*

$$\|u_1(x) - u_2(x)\|_{H^2(\mathbb{R}^d)} \leq \tag{1.20}$$

$$\leq \frac{\sigma}{2M(1-\sigma)}(\|u_0\|_{H^2(\mathbb{R}^d)} + 1)\|g_1(z) - g_2(z)\|_{C_1(I)}$$

*holds.*

We proceed to the proof of our first main result.

## 2. The existence of the perturbed solution

*Proof of Theorem 1.3.* We choose an arbitrary $v(x) \in B_\rho$. Using (1.13) along with (1.7), we derive the estimate from above

$$\|u\|_{H^2(\mathbb{R}^d)} \leq$$

$$\leq c_a\|T(u_0(x) + v(x))\|_{H^2(\mathbb{R}^d)}\left\|\int_{\mathbb{R}^d} K(x-y)g(u_0(y) + v(y))dy\right\|_{H^2(\mathbb{R}^d)}. \tag{2.1}$$

We obtain the upper bound on the right side of (2.1). Evidently,

$$\|T(u_0(x) + v(x))\|_{H^2(\mathbb{R}^d)} \leq \|T\|(\|u_0(x)\|_{H^2(\mathbb{R}^d)} + 1). \tag{2.2}$$

By virtue of inequality (1.8), we have

$$\left\|\int_{\mathbb{R}^d} K(x-y)g(u_0(y) + v(y))dy\right\|_{L^2(\mathbb{R}^d)} \leq \tag{2.3}$$

$$\leq \|K\|_{L^1(\mathbb{R}^d)}\|g(u_0(x) + v(x))\|_{L^2(\mathbb{R}^d)}.$$

Analogously, (1.9) gives us

$$\left\|\Delta_x \int_{\mathbb{R}^d} K(x-y)g(u_0(y) + v(y))dy\right\|_{L^2(\mathbb{R}^d)} \leq$$

$$\leq \|\Delta K\|_{L^1(\mathbb{R}^d)}\|g(u_0(x)+v(x))\|_{L^2(\mathbb{R}^d)}. \tag{2.4}$$

By means of bounds (2.3) and (2.4),

$$\Big\|\int_{\mathbb{R}^d} K(x-y)g(u_0(y)+v(y))dy\Big\|_{H^2(\mathbb{R}^d)} \leq$$

$$\leq Q\|g(u_0(x)+v(x))\|_{L^2(\mathbb{R}^d)}. \tag{2.5}$$

We can write

$$g(u_0(x)+v(x)) = \int_0^{u_0(x)+v(x)} g'(z)dz. \tag{2.6}$$

For $v(x)\in B_\rho$ by virtue of inequality (1.6) we easily arrive at

$$|u_0+v| \leq c_e(\|u_0\|_{H^2(\mathbb{R}^d)}+1). \tag{2.7}$$

Thus,

$$|g(u_0(x)+v(x))| \leq \max_{z\in I}|g'(z)||u_0(x)+v(x)| \leq M|u_0(x)+v(x)| \tag{2.8}$$

with the interval $I$ defined in (1.14). This yields

$$\|g(u_0(x)+v(x))\|_{L^2(\mathbb{R}^d)} \leq M(\|u_0\|_{H^2(\mathbb{R}^d)}+1). \tag{2.9}$$

Hence, we derive

$$\|u(x)\|_{H^2(\mathbb{R}^d)} \leq c_a\|T\|(\|u_0\|_{H^2(\mathbb{R}^d)}+1)^2 QM. \tag{2.10}$$

By means of (1.17), we have $\|u(x)\|_{H^2(\mathbb{R}^d)} \leq \rho$. Therefore, the function $u(x)$, which is uniquely determined by (1.13) is contained in $B_\rho$ as well. This means that equation (1.13) defines a map $\tau_g : B_\rho \to B_\rho$ under the stated assumptions.

Let us demonstrate that under the given conditions this map is a strict contraction. We choose arbitrarily $v_{1,2}(x)\in B_\rho$. By virtue of the argument above, $u_{1,2} := \tau_g v_{1,2} \in B_\rho$. By means of (1.13), we have

$$u_1(x) = [T(u_0(x)+v_1(x))]\int_{\mathbb{R}^d} K(x-y)g(u_0(y)+v_1(y))dy, \tag{2.11}$$

$$u_2(x) = [T(u_0(x)+v_2(x))]\int_{\mathbb{R}^d} K(x-y)g(u_0(y)+v_2(y))dy. \tag{2.12}$$

From system (2.11), (2.12) it easily follows that

$$u_1(x)-u_2(x) = [Tv_1(x)-Tv_2(x)]\int_{\mathbb{R}^d} K(x-y)g(u_0(y)+v_1(y))dy+ \tag{2.13}$$

$$+[T(u_0(x)+v_2(x))]\int_{\mathbb{R}^d} K(x-y)[g(u_0(y)+v_1(y))-g(u_0(y)+v_2(y))]dy.$$

From (2.13) using (1.7) we deduce that

$$\|u_1(x)-u_2(x)\|_{H^2(\mathbb{R}^d)} \leq c_a\|Tv_1(x)-Tv_2(x)\|_{H^2(\mathbb{R}^d)}\times$$

$$\times\Big\|\int_{\mathbb{R}^d} K(x-y)g(u_0(y)+v_1(y))dy\Big\|_{H^2(\mathbb{R}^d)} + c_a\|T(u_0(x)+v_2(x))\|_{H^2(\mathbb{R}^d)}\times$$

$$\times \Big\| \int_{\mathbb{R}^d} K(x-y)[g(u_0(y)+v_1(y)) - g(u_0(y)+v_2(y))]dy \Big\|_{H^2(\mathbb{R}^d)}. \tag{2.14}$$

We derive the estimate from above on the right side of (2.14). Clearly,

$$\|Tv_1(x) - Tv_2(x)\|_{H^2(\mathbb{R}^d)} \le \|T\| \|v_1(x) - v_2(x)\|_{H^2(\mathbb{R}^d)}. \tag{2.15}$$

By virtue of inequality (1.8), we have

$$\Big\| \int_{\mathbb{R}^d} K(x-y) g(u_0(y)+v_1(y)) dy \Big\|_{L^2(\mathbb{R}^d)} \le$$

$$\le \|K\|_{L^1(\mathbb{R}^d)} \|g(u_0(x)+v_1(x))\|_{L^2(\mathbb{R}^d)}. \tag{2.16}$$

Let us apply (1.9) to obtain

$$\Big\| \Delta_x \int_{\mathbb{R}^d} K(x-y) g(u_0(y)+v_1(y)) dy \Big\|_{L^2(\mathbb{R}^d)} \le$$

$$\le \|\Delta K\|_{L^1(\mathbb{R}^d)} \|g(u_0(x)+v_1(x))\|_{L^2(\mathbb{R}^d)}. \tag{2.17}$$

Upper bounds (2.16) and (2.17) imply that

$$\Big\| \int_{\mathbb{R}^d} K(x-y) g(u_0(y)+v_1(y)) dy \Big\|_{H^2(\mathbb{R}^d)} \le$$

$$\le Q \|g(u_0(x)+v_1(x))\|_{L^2(\mathbb{R}^d)}. \tag{2.18}$$

Evidently,

$$g(u_0(x)+v_1(x)) = \int_0^{u_0(x)+v_1(x)} g'(z) dz. \tag{2.19}$$

From (2.19) we derive that

$$|g(u_0(x)+v_1(x))| \le \max_{z \in I} |g'(z)| |u_0(x)+v_1(x)| \le M |u_0(x)+v_1(x)|, \tag{2.20}$$

so that

$$\|g(u_0(x)+v_1(x))\|_{L^2(\mathbb{R}^d)} \le M(\|u_0\|_{H^2(\mathbb{R}^d)} + 1). \tag{2.21}$$

Hence, the first term in the right side of bound (2.14) can be estimated from above by

$$c_a \|T\| \|v_1(x) - v_2(x)\|_{H^2(\mathbb{R}^d)} QM(\|u_0\|_{H^2(\mathbb{R}^d)} + 1). \tag{2.22}$$

Thus, it remains to obtain the upper bound on the second term in the right side of (2.14). Obviously,

$$\|T(u_0(x)+v_2(x))\|_{H^2(\mathbb{R}^d)} \le \|T\|(\|u_0\|_{H^2(\mathbb{R}^d)} + 1). \tag{2.23}$$

Using inequality (1.8), we arrive at

$$\Big\| \int_{\mathbb{R}^d} K(x-y)[g(u_0(y)+v_1(y)) - g(u_0(y)+v_2(y))]dy \Big\|_{L^2(\mathbb{R}^d)} \le$$

$$\le \|K\|_{L^1(\mathbb{R}^d)} \|g(u_0(x)+v_1(x)) - g(u_0(x)+v_2(x))\|_{L^2(\mathbb{R}^d)}. \tag{2.24}$$

Estimate (1.9) implies that

$$\Big\|\Delta_x \int_{\mathbb{R}^d} K(x-y)[g(u_0(y)+v_1(y)) - g(u_0(y)+v_2(y))]dy\Big\|_{L^2(\mathbb{R}^d)} \le$$

$$\le \|\Delta K\|_{L^1(\mathbb{R}^d)}\|g(u_0(x)+v_1(x)) - g(u_0(x)+v_2(x))\|_{L^2(\mathbb{R}^d)}. \tag{2.25}$$

Upper bounds (2.24) and (2.25) give us

$$\Big\|\int_{\mathbb{R}^d} K(x-y)[g(u_0(y)+v_1(y)) - g(u_0(y)+v_2(y))]dy\Big\|_{H^2(\mathbb{R}^d)} \le$$

$$\le Q\|g(u_0(x)+v_1(x)) - g(u_0(x)+v_2(x))\|_{L^2(\mathbb{R}^d)}. \tag{2.26}$$

Let us express

$$g(u_0(x)+v_1(x)) - g(u_0(x)+v_2(x)) = \int_{u_0(x)+v_2(x)}^{u_0(x)+v_1(x)} g'(z)dz. \tag{2.27}$$

By virtue of formula (2.27), we obtain

$$|g(u_0(x)+v_1(x)) - g(u_0(x)+v_2(x))| \le \max_{z\in I}|g'(z)||v_1(x)-v_1(x)| \le$$

$$\le M|v_1(x)-v_1(x)|, \tag{2.28}$$

so that

$$\|g(u_0(x)+v_1(x)) - g(u_0(x)+v_2(x))\|_{L^2(\mathbb{R}^d)} \le M\|v_1(x)-v_2(x)\|_{H^2(\mathbb{R}^d)}. \tag{2.29}$$

Hence, the second term in the right side of inequality (2.14) can be bounded from above by expression (2.22) as well. Therefore,

$$\|u_1(x)-u_2(x)\|_{H^2(\mathbb{R}^d)} \le$$

$$\le 2c_a(\|u_0\|_{H^2(\mathbb{R}^d)}+1)\|T\|MQ\|v_1(x)-v_2(x)\|_{H^2(\mathbb{R}^d)}. \tag{2.30}$$

By means of (2.30) along with definition (1.18), we arrive at

$$\|\tau_g v_1(x) - \tau_g v_2(x)\|_{H^2(\mathbb{R}^d)} \le \sigma\|v_1(x)-v_2(x)\|_{H^2(\mathbb{R}^d)}. \tag{2.31}$$

It can be trivially checked using (1.17) that the constant in the right side of inequality above

$$\sigma < 1. \tag{2.32}$$

This yields that the map $\tau_g : B_\rho \to B_\rho$ defined by equation (1.13) is a strict contraction under the stated assumptions. Its unique fixed point $u_p(x)$ is the only solution of problem (1.11) in the ball $B_\rho$. The cumulative $u(x)$ given by (1.10) solves equation (1.1). □

We conclude the work by establishing the validity of our second main proposition.

## 3. The continuity of the resulting solution with respect to the function $g$

*Proof of Theorem 1.4.* Evidently, under the given conditions, we have

$$u_{p,1}=\tau_{g_1}u_{p,1}, \quad u_{p,2}=\tau_{g_2}u_{p,2}. \tag{3.1}$$

Hence,

$$u_{p,1}-u_{p,2}=\tau_{g_1}u_{p,1}-\tau_{g_1}u_{p,2}+\tau_{g_1}u_{p,2}-\tau_{g_2}u_{p,2}. \tag{3.2}$$

Therefore,

$$\|u_{p,1}-u_{p,2}\|_{H^2(\mathbb{R}^d)}\le$$

$$\le \|\tau_{g_1}u_{p,1}-\tau_{g_1}u_{p,2}\|_{H^2(\mathbb{R}^d)}+\|\tau_{g_1}u_{p,2}-\tau_{g_2}u_{p,2}\|_{H^2(\mathbb{R}^d)}. \tag{3.3}$$

By virtue of (2.31), we have the estimate

$$\|\tau_{g_1}u_{p,1}-\tau_{g_1}u_{p,2}\|_{H^2(\mathbb{R}^d)}\le \sigma\|u_{p,1}-u_{p,2}\|_{H^2(\mathbb{R}^d)}, \tag{3.4}$$

where $\sigma$ is given by (1.18), so that (2.32) is valid. Thus, we arrive at

$$(1-\sigma)\|u_{p,1}-u_{p,2}\|_{H^2(\mathbb{R}^d)}\le \|\tau_{g_1}u_{p,2}-\tau_{g_2}u_{p,2}\|_{H^2(\mathbb{R}^d)}. \tag{3.5}$$

Clearly, for our fixed point $\tau_{g_2}u_{p,2}=u_{p,2}$. Let us introduce $\xi(x):=\tau_{g_1}u_{p,2}$ and arrive at

$$\xi(x)=[T(u_0(x)+u_{p,2}(x))]\int_{\mathbb{R}^d}K(x-y)g_1(u_0(y)+u_{p,2}(y))dy, \tag{3.6}$$

$$u_{p,2}(x)=[T(u_0(x)+u_{p,2}(x))]\int_{\mathbb{R}^d}K(x-y)g_2(u_0(y)+u_{p,2}(y))dy. \tag{3.7}$$

By means of system (3.6), (3.7),

$$\xi(x)-u_{p,2}(x)=[T(u_0(x)+u_{p,2}(x))]\times$$

$$\times\int_{\mathbb{R}^d}K(x-y)[g_1(u_0(y)+u_{p,2}(y))-g_2(u_0(y)+u_{p,2}(y))]dy. \tag{3.8}$$

Using (1.7), we derive

$$\|\xi(x)-u_{p,2}(x)\|_{H^2(\mathbb{R}^d)}\le c_a\|T(u_0(x)+u_{p,2}(x))\|_{H^2(\mathbb{R}^d)}\times$$

$$\times\Big\|\int_{\mathbb{R}^d}K(x-y)[g_1(u_0(y)+u_{p,2}(y))-g_2(u_0(y)+u_{p,2}(y))]dy\Big\|_{H^2(\mathbb{R}^d)}. \tag{3.9}$$

Obviously, the upper bound

$$\|T(u_0(x)+u_{p,2}(x))\|_{H^2(\mathbb{R}^d)}\le \|T\|(\|u_0\|_{H^2(\mathbb{R}^d)}+1) \tag{3.10}$$

holds. By virtue of inequality (1.8),

$$\Big\|\int_{\mathbb{R}^d}K(x-y)[g_1(u_0(y)+u_{p,2}(y))-g_2(u_0(y)+u_{p,2}(y))]dy\Big\|_{L^2(\mathbb{R}^d)}\le$$

$$\le \|K\|_{L^1(\mathbb{R}^d)}\|g_1(u_0(x)+u_{p,2}(x))-g_2(u_0(x)+u_{p,2}(x))\|_{L^2(\mathbb{R}^d)}. \tag{3.11}$$

Analogously, (1.9) yields

$$\Big\|\Delta_x \int_{\mathbb{R}^d} K(x-y)[g_1(u_0(y)+u_{p,2}(y)) - g_2(u_0(y)+u_{p,2}(y))]dy\Big\|_{L^2(\mathbb{R}^d)} \leq$$

$$\leq \|\Delta K\|_{L^1(\mathbb{R}^d)} \|g_1(u_0(x)+u_{p,2}(x)) - g_2(u_0(x)+u_{p,2}(x))\|_{L^2(\mathbb{R}^d)}. \tag{3.12}$$

By means of estimates (3.11) and (3.12),

$$\Big\|\int_{\mathbb{R}^d} K(x-y)[g_1(u_0(y)+u_{p,2}(y)) - g_2(u_0(y)+u_{p,2}(y))]dy\Big\|_{H^2(\mathbb{R}^d)} \leq$$

$$\leq Q\|g_1(u_0(x)+u_{p,2}(x)) - g_2(u_0(x)+u_{p,2}(x))\|_{L^2(\mathbb{R}^d)}. \tag{3.13}$$

Obviously,

$$g_1(u_0(x)+u_{p,2}(x)) - g_2(u_0(x)+u_{p,2}(x)) = \tag{3.14}$$

$$= \int_0^{u_0(x)+u_{p,2}(x)} [g_1'(z) - g_2'(z)]dz.$$

From (3.14) we derive

$$|g_1(u_0(x)+u_{p,2}(x)) - g_2(u_0(x)+u_{p,2}(x))| \leq$$

$$\leq max_{z\in I}|g_1'(z) - g_2'(z)||u_0(x)+u_{p,2}(x)| \leq$$

$$\leq \|g_1(z) - g_2(z)\|_{C_1(I)}|u_0(x)+u_{p,2}(x)|, \tag{3.15}$$

so that

$$\|g_1(u_0(x)+u_{p,2}(x)) - g_2(u_0(x)+u_{p,2}(x))\|_{L^2(\mathbb{R}^d)} \leq$$

$$\leq \|g_1(z) - g_2(z)\|_{C_1(I)}(\|u_0\|_{H^2(\mathbb{R}^d)} + 1). \tag{3.16}$$

By virtue of inequalities (3.9), (3.10), (3.13), (3.16) obtained above, we arrive at

$$\|\xi(x) - u_{p,2}(x)\|_{H^2(\mathbb{R}^d)} \leq$$

$$\leq c_a\|T\|(\|u_0\|_{H^2(\mathbb{R}^d)} + 1)^2 Q\|g_1(z) - g_2(z)\|_{C_1(I)}. \tag{3.17}$$

Upper bounds (3.5) and (3.17) yield

$$\|u_{p,1}(x) - u_{p,2}(x)\|_{H^2(\mathbb{R}^d)} \leq$$

$$\leq \frac{c_a}{1-\sigma}\|T\|(\|u_0\|_{H^2(\mathbb{R}^d)} + 1)^2 Q\|g_1(z) - g_2(z)\|_{C_1(I)}. \tag{3.18}$$

By means of formula (1.19) along with inequality (3.18) and definition (1.18), estimate (1.20) is valid. □

**Remark 3.1.** *The results of the present article will be generalized to the higher dimensions in the consecutive work.*

## 4. Acknowledgement

V. V. is grateful to Israel Michael Sigal for the partial support by the NSERC grant.


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(V. Vougalter) Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 2E4, Canada

*Email address*: `vitali@math.toronto.edu`