ON THE GIBBS STATES FOR ONE-DIMENSIONAL LATTICE BOSON SYSTEMS WITH A LONG-RANGE INTERACTION

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Abstract : We consider an infinite chain of interacting quantum (anharmonic) oscillators. The pair potential for the oscillators at lattice distance d is proportional to $(d^2(\log(d+1))F(d))^{-1}$ where $\Sigma_{r\in\mathbf{Z}}(rF(r))^{-1} < \infty$. We prove that for any value of the inverse temperature $\beta > 0$ there exists a limiting Gibbs state which is translationally invariant and ergodic. Furthermore, it is analytic in a natural sense. This shows the absence of phase transitions in the systems under consideration for any value of the thermodynamic parameters.

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0. INTRODUCTION

One-dimensional systems of statistical mechanics, both classical and quantum, are believed not to exhibit phase transitions provided that the interaction between particles decreases fast enough with the distance. A border case is the inverse square power interaction: classical one-dimensional systems with that type of interaction were investigated in [FS]. Quantum systems are more difficult to study; even for relatively simple classes of systems (spins on a one-dimensional lattice or one-dimensional particle systems with a fermion-type interaction) the rigorous proof of the absence of phase transitions requires sophisticated techniques.

In this paper we investigate a class of one-dimensional quantum lattice boson systems (chains of quantum anharmonic oscillators) with long-range interaction potentials that decrease slightly faster than the inverse square power of the lattice distance. The main technical tools to use are the Wiener integral representation [Gi] and the cluster expansions which, in the one-dimensional classical situation, was elaborated in [C.C.O.]. The absence of phase transitions is expressed here in the following terms: for any value of the inverse temperature $\beta > 0$ there exists a limiting Gibbs state which is translation-invariant and ergodic. Moreover, this state is analytic, in terms of the self-interaction and two-body interaction potentials, in the sense that the expectation values of certain observables admit an analytic continuation to a complex domain containing part of the real axis.

Our method may be considered as alternative to the one used in [P1] and [P2]. We aim to extend our results to one-dimensional continuous quantum systems in a separate publication.

The paper is organized as follows. In Section 1 we formulate our main results (Theorems 1,2) and introduce basic objects for future use. In Section 2 the proofs are accomplished. An Appendix contains the proof of superstability estimates which are not related to the one-dimensional structure of systems under consideration.

1. PRELIMINARIES, RESULTS AND TECHNICAL TOOLS

A Hilbert space $\mathcal{H}_j$ identified as $L_2(\mathbf{R})$ is associated with any site j of the one-dimensional lattice $\mathbf{Z}$. By $\mathcal{B}_j$ we denote the $C^\star$-algebra of the bounded operators in $\mathcal{H}_j$. Given a finite set $\Lambda \subset \mathbf{Z}$, we identify a Hilbert space $\mathcal{H}_\Lambda$ with $L_2(\mathbf{R}^J)$ (which is nothing but the tensor product $\otimes_{j\in\Lambda}\mathcal{H}_j$) and denote by $\mathcal{B}_\Lambda$ the $C^\star$-algebra of the bounded operators in $\mathcal{H}_\Lambda$. The inductive limit $\lim_{\Lambda\nearrow\mathbf{Z}}\mathcal{B}_\Lambda$ is denoted as $\mathcal{B}$; this is the *-algebra of local observables of our system. Its completion in the operator norm is the $C^\star$-algebra $\mathcal{B}$ of quasilocal observables. In the sequel we do not distinguish between the operators in $\mathcal{H}_\Lambda$ and the corresponding elements of $\mathcal{B}$.

The action of the space translation group $S_y, y \in \mathbf{Z}$, on $\mathcal{B}$ is defined in the standard way.

By q_j and p_j we denote the position and momentum operators in $\mathcal{H}_j$ (or the corresponding operators in $\mathcal{H}_\Lambda$ with $\Lambda \ni j$:)

$$q_j\ f(x_j) = x_j f(x_j), \qquad p_j\ f(x_j) = -i\frac{d}{dx_j}f(x_j).$$

The Hamiltonian (the operator of the energy) H_Λ of the system in a finite 'volume' Λ is the self-adjoint operator in $\mathcal{H}_\Lambda$

$$H_\Lambda \;=\; K_\Lambda \;+\; U_\Lambda. \tag{1.1}$$

Here K_Λ is the kinetic part and U_Λ the potential part:

$$K_\Lambda \;=\; \frac{1}{2}\sum_{j\in\Lambda} {p_j}^2, \tag{1.2}$$

and

$$U_\Lambda \;=\; U_{\Lambda,0} \;+\; U_{\Lambda,1}, \tag{1.3}$$

where $U_{\Lambda,0}$ is the self-interaction energy and $U_{\Lambda,1}$ is the two-body interaction energy

$$U_{\Lambda,0} \;=\; \sum_{j\in\Lambda}\Phi(\,q_j\,), \quad U_{\Lambda,1} \;=\; \frac{1}{2}\sum_{j,j'\in\Lambda:j\neq j'}\Psi_{|j-j'|}(\,q_j,q_{j'}\,). \tag{1.4}$$

Here, $\Phi:\mathbf{R}\to\mathbf{R}$ (a self-interaction potential) and $\Psi_d:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ (a two-body interaction potential, at distance d), $d\in\mathbf{Z}^+$, are C^2-functions, Ψ_d is symmetric. [We use the same symbols for denoting functions (of real variables) and the corresponding multiplication operators.] We list below the conditions that are imposed on the interaction potentials.

(I)The function $\Psi_d(x,y)$ obeys

$$|\Psi_d(x,y)| \;\leq \frac{(|x|+1)\;(|y|+1)}{d^2F(d)\log\;(d+1)},\;\; x,y\in\mathbf{R}, \tag{1.5}$$

where F is a monotone function $\mathbf{Z}^+\to\mathbf{R}^+$ with

$$\sum_{r\in\mathbf{Z}}(rF(r))^{-1}<\infty. \tag{1.6}$$

In addition, we suppose that

(II) there exists $r^0>0$ such that for any finite $\Lambda\subset\mathbf{Z}$, $x_\Lambda=(x_j,j\in\Lambda)\in\mathbf{R}^\Lambda$, and positive integer $r\geq r^0$

$$U_\Lambda^{(\leq r)}(x_\Lambda)\geq\;\sum_{j\in\Lambda}(c_1x_j^2-c_2), \tag{1.7}$$

where $c_1>0$ and $c_2\in\mathbf{R}$ are constants. Here

$$U_\Lambda^{(\leq r)}(x_\Lambda)=\frac{1}{2}\sum_{j,j'\in\Lambda:|j-j'|\leq r}\Psi_{|j-j'|}(x_j,x_{j'})+\sum_{j\in\Lambda}\Phi(x_j). \tag{1.8}$$

The bound (1.7), with r greater than the length of Λ, is usually called a superstability condition.

At a certain stage we shall need a similar condition for the derivatives:

(III) The functions $\Phi^{(\mu)}(x) = \dfrac{\partial^\mu}{\partial x^\mu}\Phi(x)$ and $\Psi_d^{(\mu)}(x,y) = \dfrac{\partial^\mu}{\partial x^\mu}\Psi_d(x,y)$, $\mu = 1,2$, satisfy the bounds

$$|\Psi_d^{(\mu)}(x,y)| \leq \frac{(|x|+1)\,(|y|+1)}{d^2 F(d)\log(d+1)},\ x,y \in \mathbf{R},\ \mu = 1,2, \tag{1.9}$$

and

$$|\Phi^{(\mu)}(x)| \leq \exp(\tilde{c}_1 x^2 - \tilde{c}_2)\ x \in \mathbf{R},\ \mu = 1,2, \tag{1.10}$$

where $\tilde{c}_1 > 0$ and $\tilde{c}_2 \in \mathbf{R}$ are some constants.

Examples of potentials Φ and Ψ_d satisfying the conditions stated are easily provided by 'polynomial' interactions.

A Gibbs state in a finite volume $\Lambda \subset \mathbf{Z}$ is defined by

$$\phi_\Lambda(a) \ = \ \mathrm{tr}\ (a\rho_\Lambda),\ a \in \mathcal{B}_\Lambda, \tag{1.11}$$

where ρ_Λ is the density matrix

$$\rho_\Lambda \ = \ \Xi_\Lambda^{-1}\exp(-\beta H_\Lambda), \tag{1.12}$$

Ξ_Λ is the partition function

$$\Xi_\Lambda \ = \ \mathrm{tr}\ \exp(-\beta H_\Lambda) \tag{1.13}$$

and $\beta > 0$ is the inverse temperature of the system. The existence of a state ϕ_Λ is guaranteed by

Proposition 1.1

Under the condition (II), for any $\beta > 0$ operator $\exp(-\beta H_\Lambda)$ is of trace class.

Being of trace class, the operator ρ_Λ is determined by its integral kernel $k_\Lambda(x_\Lambda, y_\Lambda)$:

$$\rho_\Lambda f(x_\Lambda) = \int_{\mathbf{R}^\Lambda} dy_\Lambda k_\Lambda(x_\Lambda, y_\Lambda) f(y_\Lambda),\ f \in L_2(\mathbf{R}^\Lambda),\ x_\Lambda, y_\Lambda \in \mathbf{R}^\Lambda, \tag{1.14}$$

where dy_Λ denotes the Lebesgue measure on $\mathbf{R}^\Lambda$.

The quantity $\lambda_\Lambda = -1/|\Lambda|\ \ln\ \mathrm{tr}\ \exp(-\beta H_\Lambda)$ gives the free energy per lattice site in the volume Λ ($|\Lambda|$ denotes the number of lattice sites in Λ).

Actually, it is of interest to consider $\phi_\Lambda(a)$ also for some unbounded operators a in $\mathcal{H}_\Lambda$ (this is possible when $a\rho_\Lambda$ is a trace class operator). Good examples are the position and the momentum operators q_j, p_j, $j \in \Lambda$ and functions of them such as the Hamiltonian H_Λ and its kinetic and potential parts, K_Λ and U_Λ. It is also interesting to consider the Hamiltonian H_J for a 'sub-system' in a smaller volume $J \subseteq \Lambda$ as well as its kinetic and potential parts. Finally, we can take a 'relative' potential energy operator $U_{J\,|\,\Lambda\setminus J} = U_\Lambda - U_J$.

In general, we consider operators of the form $\mathcal{F}_J = \mathcal{F}_J(q_j, j \in J)$ where $\mathcal{F}_J : \mathbf{R}^J \to \mathbf{R}$ is a measurable function such that

$$|\mathcal{F}_J(x_J)| \le \exp[\sum_{j\in J}(\bar{c}_1 x^2 - \bar{c}_2)], \tag{1.15}$$

where $\bar{c}_1 < c_1$ and $\bar{c}_2 \in \mathbf{R}$ (cf (1.7), (1.10)). We can also treat a more general case where $\mathcal{F}$ is a function of an infinite number of variables, but its 'essential' dependence is upon variables associated with a finite set $J \subseteq \mathbf{Z}$. An example of this kind is $U_{J\,|\,\mathbf{Z}\setminus J}$, the potential energy of a subsystem in J relative to the whole exterior $\mathbf{Z}\setminus J$. See Theorems 1 and 2 below. To avoid technically complicated constructions, we omit a formal general set-up related to functions of that type; the interested reader may reconstruct it by following the example mentioned.

The formalism introduced so far is indeed insensitive to the dimensionality of the system: we can replace the lattice $\mathbf{Z}$ by its multi-dimensional analogue $\mathbf{Z}^\nu$, $\nu \ge 1$, and the single-oscillator phase space $L_2(\mathbf{R})$ by $L_2(\mathbf{R}^k)$, $k \ge 1$. Also the potentials may not be translation-invariant, in which case the self-interaction will be described by a family $\{\Phi_j, j \in \mathbf{Z}\}$ and the two-body interaction by a family $\{\Psi_{j,j'}, j, j' \in \mathbf{Z}\}$. This type of models is called a general system of (quantum) oscillators (later on, we make this definition precise). As it was pointed out earlier, the main results of this paper are formulated for the one-dimensional case ($\nu = 1$) and translation-invariant interaction potentials. However, some auxiliary assertions (see e.g. Lemma 3 below) are of interest for purposes outside this paper and are stated in a general situation.

We are interested in studying the limits of ϕ_Λ and λ_Λ when $\Lambda \nearrow \mathbf{Z}$.

Theorem 1

Suppose that the conditions (I) and (II) are fulfilled. Then, for any $\beta > 0$,
(a) there exists the limit

$$\lambda = \lim_{\Lambda\nearrow\mathbf{Z}} \lambda_\Lambda \tag{1.16}$$

(the free energy of the infinite system),
(b) there exists the $w^\star-$limit

$$\phi = \lim_{\Lambda\nearrow\mathbf{Z}} \phi_\Lambda \tag{1.17}$$

which defines a state ϕ on $C^\star$-algebra $\mathcal{B}$.

Furthermore, the state ϕ is locally normal (i.e. is given by a family $(\rho^{(J)})$ of local density matrices), translationally invariant and has the following mixing property:

$$\lim_{u\nearrow\pm\infty} \phi(a_1\, S_u a_2) = \phi(a_1)\, \phi(a_2). \tag{1.18}$$

Hence, ϕ is ergodic, i.e., ϕ gives an extreme point of the set of translationally invariant states on the $C^\star$-algebra $\mathcal{B}$. For any finite $J \subset \mathbf{Z}$, the operator $\rho^{(J)}$ acts in $\mathcal{H}_J$; it is positively defined, of trace class and with $\mathrm{tr}\,\rho^{(J)} = 1$.

(c) The functionals $\phi(\mathcal{F}_J)$ (defined as tr $(\mathcal{F}_J)\rho^{(J)})$ *are finite for any finite* $J \subset \mathbf{Z}$ *and any measurable function* $\mathcal{F}_J : \mathbf{R}^J \to \mathbf{R}$ *which obeys (1.15). In addition, they coincide with the limits*

$$\lim_{\Lambda \nearrow \mathbf{Z}} \phi_\Lambda(\mathcal{F}_J). \tag{1.19}$$

In particular, this is true for $\mathcal{F}_J = U_J$. *Moreover, there exists a finite limit*

$$\phi(U_{J \mid \mathbf{Z}\setminus J}) = \lim_{\tilde{\Lambda} \nearrow \mathbf{Z}} \phi(U_{J \mid \tilde{\Lambda}\setminus J}) = \lim_{\Lambda \nearrow \mathbf{Z}} \phi_\Lambda(U_{J \mid \Lambda\setminus J}). \tag{1.20}$$

(d) If, in addition, condition (III) is valid, then, for any finite $J \subset \mathbf{Z}$, *the integral kernel* $k^{(J)}$ *of operator* $\rho^{(J)}$ *is a* C^2*-function of arguments* $x_J,\ y_J \in \mathbf{R}^J$. *Furthermore,* $\phi(p_l^2)$ *(defined as* tr $(p_l^2\rho^{(\{l\})})$ *) is finite and coincides with the limit*

$$\lim_{\Lambda \nearrow \mathbf{Z}} \phi_\Lambda(p_l^2). \tag{1.21}$$

Remarks. 1. The kernels $k^{(J)}$ are given by the formula (cf. (1.14))

$$\rho^{(J)} f(x_J) = \int_{\mathbf{R}^J} dy_J k^{(J)}(x_J, y_J) f(y_J),\ f \in L_2(\mathbf{R}^J),\ x_J, y_J \in \mathbf{R}^J, \tag{1.22}$$

where dy_J denotes, as before, the Lebesgue measure on $\mathbf{R}^J$; these kernels are formally determined almost everywhere with respect to this measure. Speaking of their smoothness property, we have in mind their variants that are determined everywhere on $\mathbf{R}^J$. The same is true for the analyticity property .

In fact, this remark holds for every kernel we deal with in the sequel, and for every property that is stated for any $x_J,\ y_J \in \mathbf{R}^J$.

2. In view of the translation-invariance property of ϕ , $\phi(p_l^2)$ does not depend on l and also $\phi(U_{S_u J})$ and $\phi(U_{S_u J|\mathbf{Z}\setminus S_u J})$ do not depend on u.

The following theorem expresses the analyticity properties of the limiting state ϕ constructed in Theorem 1.

Theorem 2

Let the functions $\Phi(\,\cdot\,, z_0)$ *and* $\Psi_d(\cdot\,, z_1)$ *depend on parameters* $z_l \in \mathbf{C}$, $l = 0, 1$, *in the following way:*

$$\Phi(\,\cdot\,, z_0\,) \;=\; z_0\,\Phi,\ \Psi_d(\,\cdot\,, z_1\,) \;=\; z_1\,\Psi_d, \tag{1.23}$$

where Φ *and* Ψ_d *satisfy conditions (I) and (II). Then, for any* $\beta > 0$,

(a) The free energy λ *is a real analytic function of variables* z_0, z_1 *in the region* $\mathcal{V} = \{z_0 \in \mathbf{R}^+, z_1 \in [0, z_0]\}$ *which has an analytic continuation in a complex domain in* $\mathbf{C}^2$ *containing* $\mathcal{V}$,

(b) For any finite J *and any* $x_J, y_J \in \mathbf{R}^J$, *the same assertion holds for the kernels* $k^{(J)}(x_J, y_J)$. *Furthermore the same is true for* $\phi(\mathcal{F}_J)$ *where* F_J *is as in Theorem 1 (in particular, for* $F_J = U_J$ *). Finally, the same is true for* $\phi(U_{J \mid \mathbf{Z}\setminus J})$.

(c) For any finite J and a Hilbert-Schmidt operator $a \in \mathcal{B}_J$, $\phi(a)$ admits an analytic continuation in a complex domain of $\mathbf{C}^2$ containing $\mathcal{V}$.

(d) If, in addition, the condition (III) is valid, then $\phi(p_l^2)$ is also a real-analytic function of z_0, $z_1 \in \mathcal{V}$ which admits an analytic continuation in a complex domain containing $\mathcal{V}$. Moreover, for any finite J and any x_J, $y_J \in \mathbf{R}^J$, $\frac{\partial^\mu}{\partial x^\mu} k^{(J)}(x_J, y_J)$ and $\frac{\partial^\mu}{\partial y^\mu} k^{(J)}(x_J, y_J)$, $\mu = 1, 2$, are real-analytic functions which again admit an analytic continuation in a complex domains of the same kind as before.

Remarks. 1. the variables z_0 and z_1 are subject to the restriction Re $z_0 > 0$, $0 \leq$ Re $z_1 \leq$ Re z_0 in order to preserve the superstability condition for Im z_0 =Im $z_1 = 0$.

2. Of course, one can admit a more general form of dependence of the potentials Φ and Ψ_d on the variables z_l (with the same kind of restrictions as in the previous remark). We have chosen the form of (1.23) for the sake of simplicity of the exposition.

3. Combining the results of this paper with those from [P2], one can also prove a theorem establishing a (weak) KMS property of the limiting state ϕ.

4. The complex domain of analyticity of $\phi(\cdot)$ in the assertions (b) and (c) of Theorem 2 depends on the operator in the argument of ϕ. The same is true for the assertion (d) where the domain of analyticity of $k^{(J)}(x_J, y_J)$ depends on x_J, y_J. However, under some extra conditions controlling the increasing of $\mathcal{F}_J$ or the decreassing of the kernel of a Hilbert-Schmidt operator a, this domain may be chosen independently on $\mathcal{F}_J$ or on a. Similarly, the analyticity domain of $k^{(J)}(x_J, y_J)$ may be chosen independently on x_J, y_J running over any given compact domain in $\mathbf{R}^J$. In any case, a 'width' (in 'imaginary directions') of the complex analyticity domain varies with z_0 and z_1 and in general tends to zero as $z_0 \to \infty$.

In the sequel, we write

$$z_0 = 1 + w_0,\ z_1 = 1 + w_1, \tag{1.24}$$

incorporating in the potentials Φ and Ψ_d 'unperturbed' values belonging to $\mathcal{V}$ and treating w_0 and w_1 as a small complex perturbations.

5. As it was noted before, the condition (III) involving derivatives of functions Φ and Ψ_d is used only for proving the assertions concerning the functional $\phi(p_l^\mu)$.

We now introduce some basic technical tools and mention preliminary facts which will be repeatedly used below. The main statement of Theorem 1, the existence of a limiting state ϕ (see assertion (b)), is a direct corollary of the following fact. For any finite $J \subset \mathbf{Z}$ the limit

$$\lim_{\Lambda \nearrow \mathbf{Z}} \rho_\Lambda^{(J)} \ = \ \rho^{(J)} \tag{1.25}$$

exits in the trace norm in $\mathcal{H}_J$. Here $\rho_\Lambda^{(J)}$ is the density matrix for the restriction of the state ϕ_Λ to the $C^\star$-algebra $\mathcal{B}_J$:

$$\rho_\Lambda^{(J)} \ = \ \mathrm{tr}_{\mathcal{H}_{\Lambda \setminus J}} \rho_\Lambda. \tag{1.26}$$

By using Lemma 1 from [S2], one reduces the problem to prove that the limit (1.25) holds in the Hilbert-Schmidt norm in $\mathcal{H}_J$. It is convenient to pass to the integral kernels $k_\Lambda^{(J)}(x_J, y_J)$, $x_J,\ y_J \in \mathbf{R}^J$, of the operators $\rho_\Lambda^{(J)}$, $J \subset \Lambda$, which are given by

$$\rho_\Lambda^{(J)} f(x_J) \ = \ \int_{\mathbf{R}^J} dy_J \; k_\Lambda^{(J)}(x_J, y_J)\; f(y_J). \tag{1.27}$$

In terms of the kernels $k_\Lambda^{(J)}(x_J, y_J)$ the Hilbert-Schmidt norm convergence means that

$$\lim_{\Lambda \nearrow \mathbf{Z}} \int_{\mathbf{R}^J \times \mathbf{R}^J} dx_J \times dy_J \; (\; k_\Lambda^{(J)}(x_J, y_J) \ - \ k^{(J)}(x_J, y_J)\;)^2 \ = \ 0. \tag{1.28}$$

Here $k^{(J)}(x_J, y_J)$ is a limiting kernel that defines the limiting density matrix $\rho^{(J)}$ in the same way as in (1.27). By the Lebesgue's dominated convergence theorem, it is enough to check that the kernels $k_\Lambda^{(J)}$ satisfy, uniformly in $\Lambda \supset J$, a bound

$$0 \ \leq \ k_\Lambda^{(J)}(x_J, y_J) \ \leq \ k_\star^{(J)}(x_J, y_J), \ x_J, y_J \ \in \mathbf{R}^J, \tag{1.29}$$

with $k_\star^{(J)} \in L_2(\mathbf{R}^J \times \mathbf{R}^J)$ and that the following pointwise convergence takes place:

$$\lim_{\Lambda \nearrow Z} k_\Lambda^{(J)}(x_J, y_J) \ = \ k^{(J)}(x_J, y_J), \ x_J, y_J \in \mathbf{R}^J. \tag{1.30}$$

The translation invariance of the limiting state ϕ follows from the equality for the kernels $k^{(J)}$:

$$k^{(J)}(x_J, y_J) \ = \ k^{(S_u J)}(S_u x_J, S_u y_J), \ x_J, y_J \in \mathbf{R}^J, \ u \in \mathbf{Z}, \tag{1.31}$$

where $S_u J \ = \ (j : \ j - u \ \in J)$ and $S_u z_J$ denotes, for $z_J \ = \ (z_j, \ j \in J) \ \in \ \mathbf{R}^J$, the element of $\mathbf{R}^{S_u J}$ given by

$$S_u z_J \ = \ (z'_{j'}, \ j' \in S_u J) \text{ with } z'_{j'} \ = \ z_{j'-u}.$$

The proof of the mixing property (1.18) proceeds in a similar way. Here the problem is reduced to proving the following relation for the limiting kernels $k^{(J)}$:

$$\lim_{u \to \infty} k^{(J^{(1)} \cup S_u J^{(2)})}(x_{J^{(1)}} \vee S_u x_{J^{(2)}}, \ y_{J^{(1)}} \vee S_u y_{J^{(2)}}) \ =$$

$$= \ k^{(J^{(1)})}(x_{J^{(1)}}, y_{J^{(1)}}) \; k^{(J^{(2)})}(x_{J^{(2)}}, y_{J^{(2)}}), \tag{1.32}$$

$$x_{J^{(1)}}, y_{J^{(1)}} \in \mathbf{R}^{J^{(1)}}, \ x_{J^{(2)}}, y_{J^{(2)}} \in \mathbf{R}^{J^{(2)}}.$$

Symbol $\vee$ indicates the operation of "glueing" configurations over non-intersecting volumes on $\mathbf{Z}$.

The ergodicity of the limiting state ϕ follows from the mixing property by virtue of general theorems (see e.g. [Ru1])

Let us now comment on the existence of $\phi(\mathcal{F}_J)$ (see assertion (c) of Theorem 1). Without loss of generality, we can assume that the function $\mathcal{F}_J$ is non-negative. We can write

$$\phi_\Lambda(\mathcal{F}_J) = \int_{\mathbf{R}^J} dx_J k_\Lambda^{(J)}(x_J, x_J)\mathcal{F}_J(x_J); \tag{1.33}$$

a similar equality holds for $\phi(\mathcal{F}_J)$. Under the condition (1.15) we will establish the estimate

$$k_\Lambda^{(J)}(x_J, x_J)\mathcal{F}_J(x_J) \leq \exp[-\sum_{j\in J}(\underline{c}_1 x_j^2 - \underline{c}_2)] \tag{1.34}$$

for some constants $\underline{c}_1 > 0$ and $\underline{c}_2 \in \mathbf{R}$ depending on $\mathcal{F}_J$, but not on $x_J, y_J \in \mathbf{R}^J$. The existence of the pointwise limit in (1.30) and the Lebesgue's dominated convergence Theorem will then imply the convergence to a finite limit in (1.19) and the coincidence of the limiting value with $\phi(\mathcal{F}_J)$.

In a similar way one can prove the existence of the limits in (1.20) and their coincidence with $\phi(U_{J\,|\,\mathbf{Z}\setminus J})$. We omit a detailed argument since it is the same as in the case of $\phi_\Lambda(p_l^2)$.

Finally, the smoothness of the limiting kernels $k^{(J)}$ and the existence of $\phi(p_l^2)$ (see the assertion (d) of Theorem 1) is established as follows. For any finite $\Lambda\subset\mathbf{Z}$ and $J\subseteq\Lambda$ the kernel $k_\Lambda^{(J)}$ is a C^2-function of the variables x_J, y_J and it indeed converges to a limit, as $\Lambda\nearrow\mathbf{Z}$, together with its derivatives

$$\frac{\partial^\mu}{\partial x_j^\mu} k_\Lambda^{(J)}(x_J, y_J) \quad\text{and}\quad \frac{\partial^\mu}{\partial y_j^\mu} k_\Lambda^{(J)}(x_J, y_J),\ j\in J,\ x_J, y_J\in\mathbf{R}^J,\ \mu = 1, 2. \tag{1.35}$$

Moreover, the convergence is uniform in x_J, y_J running over a compact set in $\mathbf{R}^J$. The Fubini's theorem then implies that the limiting kernel $k^{(J)}$ is a C^2-function of $x_J, y_J\in\mathbf{R}^J$ and that the limits of the derivatives coincide with the derivatives of the limiting kernel.

In addition, we establish a bound: for any finite $\Lambda\subset\mathbf{Z}$ and $J\subseteq\Lambda$,

$$\left|\frac{\partial^\mu}{\partial x^\mu} k_\Lambda^{(J)}(x_J, y_J)\lceil_{x_J=y_J}\right| \leq \exp[-\sum_{j\in J}(\tilde{c}_1 x_j^2 - \tilde{c}_2)], \tag{1.36}$$

where, for a fixed J, constants $\tilde{c}_1 > 0$ and $\tilde{c}_2\in\mathbf{R}$ do not depend on Λ and $x_J, y_J\in\mathbf{R}$. We then write $\phi_\Lambda(p_l^2)$ as an integral

$$\phi_\Lambda(p_l^2) \;=\; \int_{\mathbf{R}} dx_l\Big(\frac{\partial^2}{\partial x_l^2} k_\Lambda^{(\{l\})}(x_l, y_l)\lceil_{x_l=y_l}\Big); \tag{1.37}$$

a similar representation holds also for $\phi(p_l^2)$. After the estimate (1.36), the existence of a finite limit of the quantity (1.37) and its coincidence with $\phi_(p_l^2)$ follows, as before, from the convergence of the derivatives (1.35) and the Lebesgue's dominated convergence theorem.*

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* As a byproduct of this argument, we get that $\phi(p_l)$ is finite (and equals zero).

A key role in the proof of the assertions stated is played by the Wiener integral representations for the partition function Ξ_Λ and the kernels $k_\Lambda^{(J)}$, $J \subset \Lambda$, which follow from the Feynman-Kac formula for the integral kernel e_Λ of operator $\exp(-\beta H_\Lambda)$ (see [Gi]):

$$e_\Lambda(x_\Lambda, y_\Lambda) \;=\; \int_{\mathbf{W}^{(\beta)}_{x_\Lambda,y_\Lambda}} dP^{(\beta)}_{x_\Lambda,y_\Lambda}(\omega_\Lambda)\ \exp\left(-V_\Lambda(\omega_\Lambda)\right), \quad x_\Lambda, y_\Lambda \in \mathbf{R}^\Lambda. \tag{1.38}$$

Here, the space $\mathbf{W}^{(\beta)}_{x_\Lambda,y_\Lambda}$ is the Cartesian product

$$\times_{j\in\Lambda}\mathbf{W}^{(\beta)}_{x_j,y_j},$$

where $x_\Lambda = (x_j, j \in \Lambda)$, $y_\Lambda = (y_j, j \in \Lambda)$, and $\mathbf{W}^{(\beta)}_{x,y}$, $x, y \in \mathbf{R}$, is defined as the set of continuous functions (paths) $\omega : [0, \beta] \to \mathbf{R}$ with $\omega(0) = x$, $\omega(\beta) = y$, which is endowed with the standard Borel space structure. Furthermore, a measure $P^{(\beta)}_{x_\Lambda,y_\Lambda}$ is the product

$$\times_{j\in\Lambda}P^{(\beta)}_{x_j,y_j}$$

where $P^{(\beta)}_{x,y}$ is the non-normalized conditioned Wiener measure on $\mathbf{W}^{(\beta)}_{x,y}$ (this means that $P^{(\beta)}_{x,y}(\mathbf{W}^{(\beta)}_{x,y}) = (2\pi\beta)^{-1/2}\ \exp\left[-1/2\beta(x-y)^2)\right]$). Finally, $V(\omega_\Lambda)$ is the 'potential energy' of the path 'configuration' $\omega_\Lambda = (\omega_j,\ j \in \Lambda) \in \mathbf{W}^{(\beta)}_{x_\Lambda,y_\Lambda}$. In analogy with (1.2b,c) we set

$$V_\Lambda(\,\omega_\Lambda\,) \;=\; V_{\Lambda,0}(\,\omega_\Lambda\,) \;+\; V_{\Lambda,1}(\,\omega_\Lambda\,), \tag{1.39}$$

where

$$V_0(\,\omega_\Lambda\,) \;=\; \sum_{j\in\Lambda}\varphi(\,\omega_j\,), \qquad V_1(\,\omega_\Lambda\,) \;=\; \frac{1}{2}\sum_{j,j'\in\Lambda:j\neq j'}\psi_{|j-j'|}(\,\omega_j, \omega_{j'}\,), \tag{1.40}$$

$$\varphi(\,\omega\,) \;=\; \int_0^\beta \Phi(\,\omega(t)\,)dt, \qquad \psi_d(\omega, \omega') \;=\; \int_0^\beta \Psi_d(\omega(t), \omega'(t)\,)dt \tag{1.41}$$

For the sake of simplicity, we omit the index (β) from the notations. We also identify the spaces $\mathbf{W}_{x,x}$, $x \in \mathbf{R}$, with a single space $\mathbf{W} = \mathbf{W}_{0,0}$ by means of the mapping $\omega\omega + x$. Measures $P_{x,x}$ and $P = P_{0,0}$ are transformed thereby into each other. A measure space $(\mathbf{W}_{x^\Lambda,x^\Lambda}, P_{x^\Lambda,x^\Lambda})$ will be identified with the product-space $(\mathbf{W}^\Lambda, P^\Lambda)$. Sometimes it will also be convenient to use the map $\mathbf{W}_{x,y}\mathbf{W}_{0,0}$ given by $\omega\omega + L_{x,y}$ where $L_{x,y}$ is the linear function $L_{x,y}(t) = x + t\beta^{-1}(y - x)$. The measure $P_{x,y}$ is transformed thereby into $\exp\left(-1/2\beta(x-y)^2\right)P_{0,0}$. The product-space $\mathbf{W}_{x_J,y_J}$ is transformed into $\mathbf{W}^J$ by a vector analogue of this construction where the function $\bar{L}_{x_J,y_J}(t) = x_J + t\beta^{-1}(y_J - x_J)$ is used.

It is easy to check that, under our conditions on functions Φ and Ψ_d, the kernel e_Λ is a continuous function of variables x_Λ and y_Λ. According to the Mercer's theorem and our previous arguments, we can write the following formulas for the partition function Ξ_Λ and the kernels $k_\Lambda^{(J)}$:

$$\Xi_\Lambda \; = \; \int_{\mathbf{R}^\Lambda \times \mathbf{W}^\Lambda} du_\Lambda dP^\Lambda(\omega_\Lambda) \exp(-V(\omega_\Lambda + u_\Lambda)) \tag{1.42}$$

and

$$k_\Lambda^{(J)}(x_J, y_J) \; = \; (\Xi_\Lambda)^{-1} \, \Xi_\Lambda^{(J)}(x_J, y_J), \quad x_J, y_J \in \mathbf{R}^J, \tag{1.43}$$

where

$$\Xi_\Lambda^{(J)}(x_J, y_J) \; = \; \exp[-1/2\beta \sum_{j \in J} (x_j - y_j)^2] \int_{\mathbf{R}^{\Lambda \setminus J} \times \mathbf{W}^{\Lambda \setminus J}} du_{\Lambda \setminus J} dP^{\Lambda \setminus J}(\omega_{\Lambda \setminus J}) \times$$

$$\int_{\mathbf{W}^J} dP^J(\omega_J) \exp(-V_\Lambda((\omega_{\Lambda \setminus J} + u_{\Lambda \setminus J}) \vee (\omega_J + \bar{L}_{x_J, y_J})). \tag{1.44}$$

Here $\omega_\Lambda + u_\Lambda$ is the collection of the shifted trajectories $(\omega_j + u_j, j \in \Lambda)$, and du_Λ is the Lebesgue measure on $\mathbf{R}^{|\Lambda|}$. The symbol $\vee$ has the same meaning as in (1.32).

We give at this point a scheme of the proof of Proposition 1.1 and Theorems 1 and 2 (taking into account the intermediary assertions stated so far in the course of exposition).

Using formula (1.38), we reduce the problem of proving Proposition 1.1 to check that the integral in the RHS is finite for any value of $\beta > 0$. This is a straightforward (though tedious) calculation based on the superstability condition (1.7). Cf. the bound (2.32) in Section 2 of this paper.

The above arguments show that the part (b) of Theorem 1 follows from Lemma 1 (see below). Furthermore, the proof of the part (a) is contained in the proof of this lemma.

Lemma 1

Assume that the interaction potentials Φ and Ψ_d satisfy the conditions (I) and (II). Let the kernels $k_\Lambda^{(J)}$ be given by (1.43). Then the pointwise limit (1.30) exits and the limiting functions $k^{(J)}$ obey (1.31), (1.32). Moreover, the kernels $k_\Lambda^{(J)}$ (and hence also the limiting kernels $k^{(J)}$) satisfy the bound (1.29) uniformly in Λ with a function $k_*^{(J)}$ that has the properties listed above.

For any function $\mathcal{F}_J$ obeying (1.15) the bound (1.34) is fulfilled.

If, in addition, the condition (III) is valid, then the kernels $k_\Lambda^{(J)}$ are of class C^2 in the variables x_J, y_J and they converge to limits together with their derivatives (1.35). Moreover, this convergence is uniform in x_J, y_J running over a compact set in $\mathbf{R}^J$. Finally, the bound (1.36) holds.

Theorem 2 follows from Theorem 1 and Lemma 2:

Lemma 2

Assume that the potentials Φ and Ψ_d, $d \geq 1$, depend on the parameters z_0, z_1 as indicated in (1.23). Under the conditions (I) and (II), for any β there exist neighborhoods

$\mathcal{O}_0$,$\mathcal{O}_1$ of the origin in $\mathbf{C}$ such that, for any finite $\Lambda \subset \mathbf{Z}$, λ_Λ admits an analytic continuation in $w_\ell \in \mathcal{O}_\ell$ and $|\lambda_\Lambda|$ is bounded uniformly in Λ and $w_\ell \in \mathcal{O}_\ell$, $\ell = 0, 1$. Furthermore, as $\Lambda \nearrow \mathbf{Z}$, the analytic functions λ_Λ converge, uniformly in $\mathcal{O}_0 \times \mathcal{O}_1$, to a limit which is again an analytic function in $w_\ell \in \mathcal{O}_\ell$, $\ell = 0, 1$.

Similarly, for any $\beta > 0$, any finite $J \subset \mathbf{Z}$ and any $x_J, y_J \in \mathbf{R}^J$, there exist neighborhoods $\mathcal{O}_0$ and $\mathcal{O}_1$ of the origin in $\mathbf{C}$ such that for any finite $\Lambda \supseteq J$ the kernel $k_\Lambda^{(J)}(x_J, y_J)$ admits an analytic continuation in a domain $\mathcal{O}_0 \times \mathcal{O}_1$ and $|k_\Lambda^{(J)}(x_J, y_J)|$ is bounded uniformly in Λ and $w_\ell \in \mathcal{O}_\ell$, $\ell = 0, 1$. Furthermore, as $\Lambda \nearrow \mathbf{Z}$, the analytic functions $k_\Lambda^{(J)}(x_J, y_J)$ converge, uniformly in $\mathcal{O}_0 \times \mathcal{O}_1$, to a limit which is again an analytic function in $w_\ell \in \mathcal{O}_\ell$, $\ell = 0, 1$. If x_J, y_J run over a compact set in $\mathbf{R}^J$, then the neighborhoods $\mathcal{O}_\ell$ may be chosen independently on x_J, y_J and the convergence is uniform in x_J, y_J.

Moreover, similar assertions hold, for any $\beta > 0$ and finite $J \subset \mathbf{Z}$, for $\phi_\Lambda(a)$ where a is an operator in $\mathcal{H}_J$ of the kind considered in assertions (b) and (c) of Theorem 2.

If, in addition, condition (III) is valid, then the same assertions hold for the derivatives (1.35) and $\phi_\Lambda(p_l^2)$ given by (1.37).

The proof of Lemmas 1 and 2 are similar and are given in Section 2.

In the following we shall use the notation

$$\mathbf{S} = \mathbf{R} \times \mathbf{W} \quad \text{and} \quad s = (x, \omega) \in \mathbf{S} \tag{1.45}$$

as well as

$$\bar{\mathbf{S}} = \mathbf{R} \times \mathbf{R} \times \mathbf{W} \quad \text{and} \quad \bar{s} = (x, y, \omega) \in \bar{\mathbf{S}}. \tag{1.46}$$

Of course, $\mathbf{S}$ may be identified as a 'diagonal part' of $\bar{\mathbf{S}}$; in the sequel we use this fact without mentioning it. The spaces $\mathbf{S}$ and $\bar{\mathbf{S}}$ are provided with the norms

$$\parallel s \parallel_r = \left(\int_0^\beta |s(t)|^r dt\right)^{\frac{1}{r}},\ r = 1, 2, \tag{1.47}$$

and

$$\parallel \bar{s} \parallel_r = \left(\int_0^\beta |\bar{s}(t)|^r dt\right)^{\frac{1}{r}},\ r = 1, 2, \tag{1.48}$$

where

$$s(t) = x + \omega(t),\ \bar{s}(t) = L_{x,y}(t) + \omega(t).$$

Given a finite $J \subset \mathbf{Z}$, we denote

$$s_J = (s_j, j \in J) \in \mathbf{S}^J (= \mathbf{R}^J \times \mathbf{W}^J),\ s_j = (x_j; \omega_j) \in \mathbf{S}, \tag{1.49}$$

or, equivalently,

$$s_J = (x_J; \omega_J),\ x_J \in \mathbf{R}^J,\ \omega_J \in \mathbf{W}^J,$$

and

$$\bar{s}_J = (\bar{s}_j, j \in J) \in \bar{\mathbf{S}}^J (= \mathbf{R}^J \times \mathbf{R}^J \times \mathbf{W}^J),\ \bar{s}_j = (x_j, y_j; \omega_j) \in \bar{\mathbf{S}}, \tag{1.50}$$

or, equivalently,

$$\bar{s}_J = (x_J, y_J; \omega_J),\ x_J, y_J \in \mathbf{R}^J,\ \omega_J \in \mathbf{W}^J.$$

As before, s_J and $\bar{s}_J$ are called path configurations over J.

We also denote by ds_J the measure $du_J dP(\omega_J)$ on $\mathbf{S}^J$ and use, as before, the notation $s_J \vee s_{J'}$ (and also $s_J \vee \bar{s}_{J'}$, $\bar{s}_J \vee s_{J'}$, etc.), J and J' being non-intersecting finite subsets of $\mathbf{Z}$, for the operation of "glueing" path configurations. The space-translations S_u, $u \in \mathbf{Z}$, act on path configurations in a natural way: they map $\mathbf{S}^J$ onto $\mathbf{S}^{S_u J}$ and transform the measure $du_J dP(\omega_J)$ to $du_{S_u J} dP(\omega_{S_u J})$. We can then use the notation $\varphi(s)$, $\psi_{|j-j'|}(s_j, s'_j)$, $V_\Lambda(s_\Lambda)$ and $V_\Lambda(s_{\Lambda\setminus J} \vee \bar{s}_J)$ and define:

$$k_\Lambda^{(J)}(\bar{s}_J) = (\Xi_\Lambda)^{-1} \Xi_\Lambda^{(J)}(\bar{s}_J) \tag{1.51}$$

where

$$\Xi_\Lambda^{(J)}(\bar{s}_J) = \int_{\mathbf{S}^{\Lambda\setminus J}} ds_{\Lambda\setminus J}\ \exp\left(-V_\Lambda(\bar{s}_J \vee s_{\Lambda\setminus J})\right). \tag{1.52}$$

This allows us to write:

$$k_\Lambda^{(J)}(x_J, y_J) = \exp[-1/2\beta \sum_{j\in J}(x_j - y_j)^2] \int_{\mathbf{W}^J} dP^J(\omega_J) k_\Lambda^{(J)}(\bar{s}_J) \tag{1.53}$$

with $\omega_J = (\omega_j)_{j\in J}$ and $\bar{s}_J = (\bar{s}_j)_{j\in J}$ where $\bar{s}_j = (x_j, y_j; \omega_j)$.

We are now going to write down formulas for the derivatives (1.35). For the sake of brevity, we restrict ourselves to the case of the derivatives $\dfrac{\partial^\mu}{\partial x_j^\mu}$; obvious modifications needed to cover the case of $\dfrac{\partial^\mu}{\partial y_j^\mu}$ may easily be done by the reader. Introducing the notation

$$\varphi^{(\mu)}(\bar{s}) = \int_0^\beta dt (1 - \beta^{-1} t)^\mu \Phi^{(\mu)}(\bar{s}(t)),\ \mu = 1, 2,\ \bar{s} \in \bar{\mathbf{S}}, \tag{1.54}$$

$$\psi_d^{(\mu)}(\bar{s}, \bar{s}') = \int_0^\beta dt (1 - \beta^{-1} t)^\mu \Psi_d(\bar{s}(t), \bar{s}'(t)),\ \mu = 1, 2,\ \bar{s}, \bar{s}' \in \bar{\mathbf{S}}, \tag{1.55}$$

and

$$(V_\Lambda)_j^{(\mu)}(\bar{s}_\Lambda) = \varphi^{(\mu)}(\bar{s}_j) + \sum_{j'\in\Lambda: j'\neq j} \psi_{|j-j'|}^{(\mu)}(\bar{s}_j, \bar{s}_{j'}),\ \bar{s}_\Lambda = (\bar{s}_{\tilde{j}}, \tilde{j} \in \Lambda), \tag{1.56}$$

we can write, by using (1.42) - (1.44) and (1.51) - (1.53),

$$\frac{\partial}{\partial x_j} k_\Lambda^{(J)}(x_J, y_J) = -(\Xi_\Lambda)^{-1} [\frac{1}{\beta}(x_j - y_j)\ \Xi_\Lambda(x_J, y_J) + (\Xi_\Lambda^{(J)}(x_J, y_J))_j^{(1)}], \tag{1.57}$$

where

$$(\Xi_\Lambda^{(J)}(x_J, y_J))_j^{(1)} = \exp[-1/2\beta \sum_{\tilde{j}\in J}(x_{\tilde{j}} - y_{\tilde{j}})^2] \int_{\mathbf{W}^J} dP^J(\omega_J) \times$$

$$\int_{\mathbf{S}^{\Lambda\setminus J}} ds_{\Lambda\setminus J} (V_\Lambda)_j^{(\mu)}(\bar{s}_J \vee s_{\Lambda\setminus J}) \exp[-V_\Lambda(\bar{s}_J \vee s_{\Lambda\setminus J})],\ \bar{s}_J = (x_J, y_J; \omega_J), \tag{1.58}$$

and

$$\frac{\partial^2}{\partial x_j^2} k_\Lambda^{(J)}(x_J, y_J) = (\Xi_\Lambda)^{-1}[(\frac{1}{\beta^2}(x_j - y_j)^2 - \frac{1}{\beta})\Xi_\Lambda(x_J, y_J) + + \frac{2}{\beta}(x_j - y_j)\ (\Xi_\Lambda^{(J)}(x_J, y_J))_j^{(1)} - (\Xi_\Lambda^{(J)}(x_J, y_J))_j^{(2)}], \qquad (1.59)$$

where

$$(\Xi_\Lambda^{(J)}(x_J, y_J))_j^{(2)} = \exp[-1/2\beta \sum_{\tilde{j}\in J}(x_{\tilde{j}} - y_{\tilde{j}})^2] \times \int_{\mathbf{W}^J} dP^J(\omega_J) \int_{\mathbf{S}^{\Lambda\setminus J}} ds_{\Lambda\setminus J}\{[-(V_\Lambda)_j^{(1)}(\bar{s}_J \vee s_{\Lambda\setminus J})]^2 + +(V_\Lambda)_j^{(2)}(\bar{s}_J \vee s_{\Lambda\setminus J})\} \exp[-V_\Lambda(\bar{s}_J \vee s_{\Lambda\setminus J})],\ \bar{s}_J = (x_J, y_J; \omega_J). \qquad (1.60)$$

The idea that we follow in the sequel (see the end of Section 2) is to treat separately the addends in the square brackets $[\ldots]$ in (1.57) and (1.59) and, in the case of (1.60), the single addends in the braces $\{\ldots\}$ in the integrand. Furthermore, those addends are decomposed into series, according to (1.56), and we deal with a single term in such a series. In fact, at a certain point we need to consider separately the 'positive' and 'negative' parts of these terms meaning merely the integrals of the positive and the negative parts of the corresponding integrand. For example, the positive part of a term in the RHS of (1.60), which, after decomposing quantity $(V_\Lambda)_j^{(2)}(\bar{s}_J \vee s_{\Lambda\setminus J})$, corresponds to $\psi_{|j-j'|}(\bar{s}_j, \bar{s}_{j'})$, is

$$\exp[-1/2\beta \sum_{\tilde{j}\in J}(x_{\tilde{j}} - y_{\tilde{j}})^2] \int_{\mathbf{W}^J} dP^J(\omega_J) \int_{\mathbf{S}^{\Lambda\setminus J}} ds_{\Lambda\setminus J}[(V_\Lambda)_j^{(1)}(\bar{s}_J \vee s_{\Lambda\setminus J})]^2 \times \exp[-V_\Lambda(\bar{s}_J \vee s_{\Lambda\setminus J})],\ \bar{s}_J = (x_J, y_J; \omega_J). \qquad (1.61)$$

We use, for the positive and negative parts of such a term, a conventional notation $[(\Xi_\Lambda^{(J)}(x_J, y_J))_j^{(\mu)}]_\pm^{\text{single}}$.

We conclude this section with a lemma containing the basic probability estimate for a general system of oscillators. As it was noted before, a general system of oscillators may be considered on a multi-dimensional lattice $\mathbf{Z}^\nu$ and have $L_2(\mathbf{R}^l$ as a single-particle phase space, ν, $l \geq 1$.We now make this concept precise. In Lemma 3 below, by a general system of quantum oscillators (in a finite volume $\Lambda \subset \mathbf{Z}^\nu$) we mean a probability distribution on the path configuration space $\bar{\mathbf{S}}^\Lambda$ (or on its subset such as $\mathbf{S}^\Lambda$ or $\mathbf{W}^J \times \mathbf{S}^{\Lambda\setminus J}$ where $J \subseteq \Lambda$). By a path we now mean a multi-dimensional Wiener trajectory $\omega : [0, \beta] \to \mathbf{R}^l$; all objects introduced so far are extended to this case without difficulties.

The structure of a probability distribution is motivated for example by the formula (1.51). More precisely, such a measure is determined by a normalizing denominator which has the form of an integral, over a subset of $\bar{\mathbf{S}}^\Lambda$, with a non-negative integrand. In other words, we follow the definition of a Gibbs measure in classical Statistical Mechanics, in a situation where the role of a 'spin' is played by a path.

In these terms, the denominator Ξ_Λ determines the 'original' measure on $\mathbf{S}^\Lambda$:

$$\frac{1}{\Xi_\Lambda} du_\Lambda \, dP^\Lambda(\omega_\Lambda) \exp[-V(\omega_\Lambda + u_\Lambda)]$$

Other examples of interest are measures on $\mathbf{W}^J \times \mathbf{S}^{\Lambda\setminus J}$ determined by the denominators $\Xi_\Lambda^{(J)}(x_J, y_J)$ and $\pm[(\Xi_\Lambda^{(J)}(x_J, y_J))_j^{(\mu)}]_\pm^{\rm single}$, $x_j, y_J \in \mathbf{R}^J$, $\mu = 1, 2$, provided that they do not vanish, of course (in the case $l > 1$ where derivatives are replaced with gradients, we treat separately different components of the corresponding vector- and tensor-functions).

Furthermore, given a non-negative function $\mathcal{E}_J : (\mathbf{S}^l)^J \to \mathbf{R}^+$ and a Hilbert-Schmidt operator a in $\mathcal{H}_J (= L_2((\mathbf{R}^l)^J))$ with a non-negative kernel $\mathcal{A} : (\mathbf{R}^l)^J \times (\mathbf{R}^l)^J \to \mathbf{R}^+$, we can speak of measures on $\mathbf{S}^\Lambda$ determined by the denominators $\Xi_\Lambda(\mathcal{E}_J)$ and $\Xi_\Lambda(a)$ given by

$$\Xi_\Lambda(\mathcal{E}_J) = \int_{\mathbf{S}^\Lambda} ds_\Lambda \mathcal{F}(s_\Lambda) \exp[-V_\Lambda(s_\Lambda)] \tag{1.62}$$

and

$$\Xi_\Lambda(a) = \int_{\mathbf{S}^{\Lambda\setminus J}} ds_{\Lambda\setminus J} \int_{\bar{\mathbf{S}}^J} d\bar{s}_J \mathcal{A}(y_J, x_J) \exp[-V_\Lambda(\bar{s}_J \vee s_{\Lambda\setminus J})], \; \bar{s}_J = (x_J, y_J; \omega_J). \tag{1.63}$$

In a general case (of complex-valued functions and kernels), we can deal with the positive and negative restrictions (of both positive and negative parts) and consider the corresponding probability measures on $\mathbf{S}^\Lambda$.

Furthermore, an 'energy' $V_\Lambda(\bar{s}_\Lambda)$ of a path configuration $\bar{s}_\Lambda \in \mathbf{S}^\Lambda$ may be generated by a non-translation-invariant, multi-body interaction which is described by a family $\{\Phi_j, j \in \mathbf{Z}^\nu\}$ of the self-interaction potentials with

$$\sup_{j\in\mathbf{Z}^\nu} \sup_{x\in\mathbf{R}^l : ||x|| \le 1} |\Phi_j(x)| \le \underline{c} < \infty, \tag{1.64}$$

and a family $\{\Psi_B, B \subset \mathbf{Z}^\nu\}$ of the interaction potentials. Here, Ψ_B describes the contribution, into the potential energy, of a subsystem of oscillators over a finite set $B \subset \mathbf{Z}^\nu$. We assume that Ψ_B is a function $(\mathbf{R}^l)^B \to \mathbf{R}$. Actually, for our purposes it is sufficient to assume that Ψ_B is non-zero for a finite collection of sets B with $|B| \ge 2$. As for B with $|B| = 2$, we assume that

$$|\Psi_{\{j,j'\}}(x_j, x_{j'})| \le \hat{c}(||x_j|| + 1) \, (||x_{j'}|| + 1) \, |j - j'|^{-(\nu+\eta)}, \tag{1.65}$$

where $\hat{c}$ and η are positive constants and $|j - j'|$ is the Euclidean distance between j and j'. The condition (1.7) is preserved (with obvious modifications in the corresponding definitions).

We use the notation $\mathbf{m}_\Lambda$ for a measure of one of the types previously considered. In analogy with (1.51), the 'local' density in a volume $J \subset \Lambda$ is denoted by $k_{\mathbf{m}_\Lambda}^{(J)}(\bar{s}_J)$, $\bar{s}_J \in \bar{\mathbf{S}}^J$.

Lemma 3

Consider a general system of quantum oscillators, as specified above. Then for any $c_1^\star \in (0, c_1)$ (see (1.7)) there exists a constant $c_2^\star \in \mathbf{R}$ (depending on a measure $\mathbf{m}_\Lambda$) such that for any finite $\Lambda \subset \mathbf{Z}$ and $J \subset \Lambda$ and any $\bar{s}_J = (\bar{s}_j, j \in J) \in \bar{\mathbf{S}}^J$

$$k^{(J)}_{\mathbf{m}_\Lambda}(\bar{s}_J) \ \leq \ \exp\,[-\sum_{j\in J}(c_1^\star \parallel \bar{s}_j \parallel_2^2 - c_2^\star)]. \tag{1.66}$$

The constant $c_2^\star$ may be chosen to be uniform for measures with the same form of the denominator provided that, in the RHS of (1.7), (1.64) and (1.65), c_1 and η are separated from zero and c_2, $\underline{c}$ and $\hat{c}$ are varying in compact sets. In the case of the measures on $\mathbf{W}^J \times \mathbf{S}^{\Lambda\setminus J}$ with the denominators $\Xi^{(J)}_\Lambda(x_J, y_J)$ and $\pm[(\Xi^{(J)}_\Lambda(x_J, y_J))^{(\mu)}_j]^{\rm single}_\pm$, $x_j, y_J \in \mathbf{R}^J$, $\mu = 1, 2$, (and fixed interaction potentials) constants $c_1^\star$ and $c_2^\star$ may be chosen uniformly for $x_J, y_J \in \mathbf{O}$ where $\mathbf{O} \subset (\mathbf{R}^k)^J$ is a compact set.

The proof of Lemma 3 is carried out in the Appendix.

2. PROOF OF LEMMAS 1 AND 2

The proof is based on methods developed in [C.C.O.]. For completeness we reproduce a construction used in [C.C.O.] to transform our "ensemble" of interacting paths into a polymer system. We start by treating the partition function Ξ_Λ for the unperturbed Hamiltonian H (that is, for $w_0 = w_1 = 0$ in (1.24)). Let L, n, p be positive integers and $\Lambda (= \Lambda_{L,n,p}) \subset \mathbf{Z}$ be an interval of length $|\Lambda| = (2p+1)L + 2pnL$, centred at the origin.* Interval Λ is decomposed into pair-wise disjoint consecutive intervals, or blocks, A_i and B_i (alternatively called blocks of type A and B, or briefly, A- and B-blocks):

$$\Lambda_p = A_{-p} \cup B_{-p} \cup A_{-p+1} \ldots \cup B_{-1} \cup A_0 \cup B_0 \ldots \cup B_{p-1} \cup A_p$$

where $|A_i| = L$, $|B_i| = nL$. Furthermore, for any $i = -p, \ldots, p-1$ we decompose the B-block B_i into n consecutive intervals (blocks) of length L :

$$B_i = \bigcup_{k=1}^n B_{i,k}, \qquad |B_{i,k}| = L.$$

The blocks A_j and $B_{j,k}$ are sometimes called elementary.

The block partition of volume Λ allows us to write a path configuration $s_\Lambda \in \mathbf{S}^\Lambda$ as a sequence

$$(s_{A_{-p}}, s_{B_{-p}}, ..., s_{B_p}, s_{A_p})$$

and furthermore a path configuration $s_{B_j} \in \mathbf{S}^{B_j}$ as

$$(s_{B_{j,1}}, ..., s_{B_{j,n}})$$

* The term 'interval' and notation $[\alpha, \alpha']$ are used here and below for intervals on the lattice $\mathbf{Z}$. By centred at the origin we mean that the origin coincides with the rightmost point of the A-block A_0 (see below).

and to use the notation introduced so far. In particular, given an integer $L > 0$, the potential energy of a collection s_Λ may be written as the sum

$$V^{\leq L}(s_\Lambda) + V^{>L}(s_\Lambda) \tag{2.1a}$$

(subscript Λ is omitted for the sake of simplicity). Here $V^{\leq L}$ includes the self-interaction energy and the energy of two-body interaction on distances $\leq L$:

$$V^{\leq L}(s_\Lambda) = V_0(s_\Lambda) + V_1^{\leq L}(s_\Lambda), \tag{2.1b}$$

$$V_1^{\leq L}(s_\Lambda) = \frac{1}{2} \sum_{i,i' \in \Lambda: i \neq i'} \psi^{\leq L}_{|i-i'|}(s_i, s_{i'}) \tag{2.1c}$$

and $V^{>L}(s_\Lambda)$ is the remaining part of the energy containing the long-range two-body interaction terms:

$$V^{>L}(s_\Lambda) = \frac{1}{2} \sum_{i,i' \in \Lambda} \psi^{>L}_{|i-i'|}(s_i, s_{i'}), \tag{2.1d}$$

where (cf. (1.41)), for $s, s' \in \mathbf{S}$,

$$\psi_d^{\leq L}(s, s') = \psi_d(s, s'), \text{ if } d \leq L, \tag{2.1e}$$

$$\psi_d^{\leq L}(s, s') = 0, \text{ if } d > L, \tag{2.1f}$$

and

$$\psi_d^{>L}(s, s') = \psi_d(s, s') - \psi(s, s'). \tag{2.1g}$$

We call $V^{\leq L}$ a short-range part of the interaction energy. Notice that in the cut-off interaction picture the non-adjacent elementary blocks do not interact.

Let us now consider the long-range part of the interaction. Given two blocks D and D' (not necessarily distinct), we set:

$$W^{>L}(s_D, s_{D'}) = \sum_{i \in D} \sum_{i' \in D'} \psi^{>L}_{|i-i'|}(s_i, s_{i'}) \tag{2.2a}$$

For a pair $C = \{D, D'\}$ we then write

$$W^{>L}(s_C) \; = \; W^{>L}(s_D, s_{D'}) \tag{2.2b}$$

We denote by $\mathcal{C}_\Lambda$ the collection of block pairs C of the following form:

$$\begin{aligned}
&C = \{A_j, A_{j'}\}, \qquad -p \leq j, j' \leq p, \quad j \neq j-1, j, j+1, \\
&C = \{A_j, B_{j'}\}, \qquad -p \leq j \leq p, \quad -p \leq j' \leq p-1, \quad j' \neq j, j-1, \\
&C = \{B_j, B_{j'}\}, \qquad -p \leq j, j' \leq p.
\end{aligned}$$

Furthermore, given a block 'triplet' $\{A_j, B_j, A_{j+1}\}$, we set

$$W^{>L}(s_{A_j}, s_{B_j}, s_{A_{j+1}}) \;=\; W^{>L}(A_j, A_{j+1}) +$$

$$W^{>L}(s_{A_j}, s_{B_j}) + W^{>L}(s_{B_j}, s_{B_j}) + W^{>L}(s_{B_j}, s_{A_{j+1}}). \tag{2.2c}$$

We then denote by $\mathcal{P}_\Lambda$ the collection of triplets of the form $C = \{A_j, B_j, A_{j+1}\}$, $-p \leq j < p$.

With this notation we can write

$$V^{>L}(s_\Lambda) = \sum_{C \in \mathcal{C}_\Lambda \cup \mathcal{P}_\Lambda} W^{>L}(s_C) \tag{2.3}$$

Therefore, calling $\varrho_C = \exp\,[-W^{>L}(s_C)] - 1$, we get

$$\exp\,[-V^{>L}(s_\Lambda)] \;=\; \prod_{C \in \mathcal{C}_\Lambda \cup \mathcal{P}_\Lambda} (1 + \varrho_C(s_C)) \;=\; 1 + \sum_{\Gamma \subset \mathcal{C}_\Lambda \cup \mathcal{P}_\Lambda} \prod_{C \in \Gamma} \varrho_C(s_C). \tag{2.4}$$

To give an idea of the proof, let us consider the term corresponding to unity in (2.4). We first introduce a cut-off on the path configurations s_k for $k \in A_j$, $j \in -p, ..., p$. That is, we write:

$$\begin{aligned}
1 &= \prod_{j=-p}^{p} \left(\left[\chi_M(s_{A_j})\chi_M(s_{A_{j+1}})\right] + \left[1 - \chi_M(s_{A_j})\chi_M(s_{A_{j+1}})\right] \right) \\
&= \prod_{j=-p}^{p} \chi_M(s_{A_j})\chi_M(s_{A_{j+1}}) \\
&\quad + \sum_{Y \subset \{-p,\ldots,p\}} \prod_{k \in Y^c} \chi_M(s_{A_k})\chi_M(s_{A_{k+1}}) \prod_{k' \in Y} \Big(1 - \chi_M(s_{A_{k'}})\chi_M(s_{A_{k'+1}})\Big)
\end{aligned} \tag{2.5}$$

where χ_M denotes an indicator function of a set of path configuration that will be defined later.

Let us consider the following quantity :

$$\exp\,[-V^{\leq L}(s_\Lambda)] \prod_{j=-p}^{p} \chi_M(s_{A_j})\chi_M(s_{A_{j+1}}). \tag{2.6}$$

it can be written as

$$\exp[-\frac{1}{2}V^{\leq L}_{A_{-p}}(s_{A_{-p}})] \prod_{j=-p}^{p} \chi_M(s_{A_j})\mathbf{T}(s_{A_j}, s_{B_j}, s_{A_{j+1}})\chi_M(s_{A_{j+1}}) \tag{2.7}$$

$$\exp[-\frac{1}{2}V^{\leq L}_{A_p}(s_{A_p})],$$

where

$$\mathbf{T}(s_{A_j}; s_{B_j}; s_{A_{j+1}}) =$$
$$T(s_{A_j}; s_{B_{j,1}}) \prod_{k=1}^{n-1} T(s_{B_{j,k}}; s_{B_{j,k+1}}) T(s_{B_{j,n}}; s_{A_{j+1}}) \tag{2.8}$$

and

$$T(s_D; s_{D'}) = \exp\,[\frac{1}{2}V^{\leq L}(s_D) - V^{\leq L}(s_D \vee s_{D'}) + \frac{1}{2}V^{\leq L}(s_{D'})]. \tag{2.9}$$

Let $(={}_L)$ denote the linear integral operator generated by the kernel T . This operator acts in a space of functions on $\mathbf{S}^{J_L}$ (this may be $C(\mathbf{S}^{J_L})$ or $L_1(\mathbf{S}^{J_L}, ds_{J_L})$, or $L_2(\mathbf{S}^{J_L}, ds_{J_L})$) where J_L is the single lattice interval of length L (it is convenient to set $J_L = [0, L-1]$ and write $\mathbf{S}^L$ instead of $\mathbf{S}^{J_L}$ and s^L instead of s_{J_L}). The operator transforms a non-negative function into a positive one and is compact. According to the Krein-Rutman Theorem [K.Ru.] (in either version), it has a unique positive eigenfunction $v(= v_L)$. The corresponding eigenvalue $\gamma(= \gamma_L)$ is not degenerate and gives the maximal point of the spectrum of . Finally, the width of the gap between γ and the remainder of the spectrum is positive.

A similar assertion holds for the adjoint operator *; its extremal eigenvector is denoted by $v^\star(= v_L^\star)$ (the corresponding eigenvalue $\gamma^\star$ coincides with γ). We normalize v and $v^\star$ in such a way that

$$\langle v, v^\star \rangle = 1. \tag{2.10}$$

Here and below $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L_2(\mathbf{S}^L, ds^L)$:

$$\langle v, v^\star \rangle = \int_{\mathbf{S}^L} ds^L v(s^L) v^\star(s^L).$$

By using space-translations S_u we can define the "shifted" functions $v(s_{A_j})$ and $v^*(s_{A_j})$, $j = -p, \cdots, p$; for these functions the relation (2.10) will hold for any j.

Now let $\mathcal{G}_\Lambda$ denote the family of pairs $\{A_i, A_{i+1}\}$, $-p \leq i \leq p-1$. If $C = \{A_i, A_{i+1}\} \in \mathcal{G}_\Lambda$, we define

$$\rho_C^2(s_C) = \frac{T^{(n)}(s_{A_j}; s_{A_{j+1}})}{\gamma^n v(s_{A_j}) v^\star(s_{A_{j+1}})} - 1. \tag{2.11}$$

Here, for $m \geq 2$, $T^{(m)}(s, s')$ is defined iteratively by :

$$T^{(m)}(s; s') = \int ds''\ T^{(m-1)}(s; s'') T(s'';\ s')$$

where s, s' and s'' stand for path configurations over appropriate intervals (e.g. for $s_{[0,L-1]}$, $s_{[mL,(m+1)L-1]}$ and $s_{[(m-1)L+1,mL]}$, respectively).

Returning to (2.6), we can write

$$\int ds_\Lambda \exp\,[-V^{\leq L}(s_\Lambda)]\ \prod_{j=-p}^{p} \chi_M(s_{A_j})\ =$$

$$\gamma^{2pn}\int ds_{A_{-p}}\times\ldots\times ds_{A_p}\prod_{j=-p}^{p-1}[v(s_{A_j})v^\star(s_{A_{j+1}})\times$$

$$\chi_M(s_{A_j})\chi_M(s_{A_{j+1}})]\exp[-\frac{1}{2}V_{A_{-p}}(s_{A_{-p}})-\frac{1}{2}V_{A_p}(s_{A_p})]\times$$

$$(1\ +\sum_{\substack{\Delta\subset\mathcal{G}_\Lambda\\ \Delta\neq\emptyset}}\ \prod_{C\in\Delta}\varrho_C^2(s_C)). \tag{2.12}$$

We can now specify the reference system around which we perform a perturbative expansion. This system is formed by a family of independent paths configurations over A-blocks. The partition function of this system is precisely the term corresponding to unity in the RHS of (2.12).

Let us now explain how we make this expansion. For $\Gamma\subset\mathcal{C}_\Lambda\cup\mathcal{P}_\Lambda$ we define

$$\mathbf{B}(\Gamma)=\{i: B_i\in\bigcup_{C\in\Gamma}C\} \tag{2.13}$$

and

$$\mathbf{B}^c(\Gamma)=[-p,p]\backslash\mathbf{B}(\Gamma).$$

We start the argument by writing:

$$\int ds_\Lambda\ \exp\left[-V(s_\Lambda)\right]=\gamma^{2np}\sum_{\Gamma\subset\mathcal{C}_\Lambda\cup\mathcal{P}_\Lambda}$$

$$\int ds_{\Lambda\backslash\cup_{i\in\mathbf{B}^c(\Gamma)}B_i}\prod_{i\in\mathbf{B}(\Gamma)}\frac{\mathbf{T}(s_{A_i};s_{B_i};s_{A_{i+1}})}{\gamma^n}\prod_{C\in\Gamma}\varrho_C(s_C)\times$$

$$\prod_{j\in\mathbf{B}^c(\Gamma)}[\frac{T^{(n)}(s_{A_j};s_{A_{j+1}})}{\gamma^n}]\exp[-\frac{1}{2}V_{A_{-p}}(s_{A_{-p}})]\exp[-\frac{1}{2}V_{A_p}(s_{A_p})] \tag{2.14}$$

We have used here the fact that, if there is no coupling between a B-block and anything else coming from the ϱ_C-terms, we can perform the path configuration integration over this block. We finally get the region $\bigcup_{i\in\mathbf{B}^c(\Gamma)}B_i$ where we deal with the nth iterates ^{n} of operator .

The next step is to analyse the terms $T^{(n)}(s_{A_j};s_{A_{j+1}})$ for $j\in\mathbf{B}^c(\Gamma)$ according to whether or not the path configuration s_{A_j} belongs to a subset of $\mathbf{S}^{A_j}$ where $\chi_M=1$. As before, we write:

$$\begin{aligned}1&=\prod_{j\in\mathbf{B}^c(\Gamma)}\left([\chi_M(s_{A_j})\chi_M(s_{A_{j+1}})]+[1-\chi_M(s_{A_j})\chi_M(s_{A_{j+1}})]\right)\\&=\sum_{I\subset\mathcal{B}^c(\Gamma)}\prod_{i\in I}\chi_M(s_{A_i})\chi_M(s_{A_{i+1}})\prod_{j\in\mathcal{B}^c(\Gamma)\backslash I}(1-\chi_M(s_{A_j})\chi_M(s_{A_{j+1}}))\end{aligned} \tag{2.15}$$

and then

$$\prod_{j\in\mathbf{B}^c(\Gamma)} \frac{T^{(n)}(s_{A_j};s_{A_{j+1}})}{\gamma^n} =$$

$$= \sum_{I\subset\mathbf{B}^c(\Gamma)} \prod_{i\in I} \frac{T^{(n)}(s_{A_i};s_{A_{i+1}})}{\gamma^n} \chi_M(s_{A_i})\chi_M(s_{A_{i+1}})\times$$

$$\prod_{j\in\mathbf{B}^c(\Gamma)\setminus I} \frac{T^{(n)}(s_{A_j};s_{A_{j+1}})}{\gamma^n}[1-\chi_M(s_{A_j})\chi_M(s_{A_{j+1}})] \tag{2.16}$$

For the terms with $i\in I$ we write :

$$\frac{T^{(n)}(s_{A_i};s_{A_{i+1}})}{\gamma^n} = \left(\frac{T^{(n)}(s_{A_i};s_{A_{i+1}})}{\gamma^n v(s_{A_i})v^\star(s_{A_{i+1}})} - 1 + 1\right) v(s_{A_i})v^\star(s_{A_{i+1}})$$

$$= (1+\varrho^2_C(s_C))v(s_{A_i})v^\star(s_{A_{i+1}}) \tag{2.17}$$

where $C=\{A_i,A_{i+1}\}$. Therefore, if we identify our pair $C=\{A_i,A_{i+1}\}$ with site i, we get:

$$\prod_{i\in I} \frac{T^{(n)}(s_{A_i};s_{A_{i+1}})}{\gamma^n} \chi_M(s_{A_i})\chi_M(s_{A_{i+1}}) =$$

$$\sum_{Y\subset I}\prod_{C\in Y}\varrho^2_C(s_C)\prod_{i\in I} v(s_{A_i})\chi_M(s_{A_i})v^\star(s_{A_{i+1}})\chi_M(s_{A_{i+1}})$$

We then collect the product-terms

$$\prod_{i\in\mathbf{B}^c\setminus I}(1-\chi_M(s_{A_j})\chi_M(s_{A_{j+1}}))$$

calling them

$$\prod_{C\in\mathbf{B}^c(\Gamma)\setminus I}\varrho^1_C(s_C).$$

As a result, we get

$$\prod_{j\in\mathbf{B}^c(\Gamma)} \frac{T^{(n)}(s_{A_j};s_{A_{j+1}})}{\gamma^n} = \sum_{I\subset\mathbf{B}^c(\Gamma)}\sum_{Y\subset I}\prod_{C\in Y}\varrho^2_C(s_C)$$

$$\prod_{C\in\mathbf{B}^c(\Gamma)\setminus I}\varrho^1_C(s_C)\prod_{i\in I} v(s_{A_i})\chi_M(s_{A_i})v^\star(s_{A_{i+1}})\chi_M(s_{A_{i+1}})$$

$$\prod_{j\in\mathbf{B}^c(\Gamma)\setminus I}[\frac{T^{(n)}(s_{A_j};s_{A_{j+1}})}{\gamma^n}] \tag{2.18}$$

Let us note that, given $I \subset \mathbf{B}^c(\Gamma)$ and $Y \subset I$, if a pair $C = \{A_j, A_{j+1}\}$ appears as an index in a term $\varrho^2_C(s_C)$ with $C \in Y \subset I$, it cannot appear simultaneously as an index of ϱ^1 (in which case C would have to belong to $\mathbf{B}^c(\Gamma) \backslash I$).

Collecting all the previous expressions, we get:

$$\int ds_\Lambda \exp[-V(s_\Lambda)] = \gamma^{2np} \sum_{\Gamma \subset \mathcal{C}_\Lambda \cup \mathcal{P}_\Lambda} \sum_{I \subset \mathbf{B}^c(\Gamma)} \sum_{Y \subset I}$$

$$\int ds_{\Lambda \backslash \bigcup_{i \in \mathbf{B}^c(\Gamma)} B_i} \exp[-\frac{1}{2} V_{A_{-p}}(s_{A_{-p}}) - \frac{1}{2} V_{A_p}(s_{A_p})] \times \tag{2.19}$$

$$\prod_{i \in \mathbf{B}^c(\Gamma)} \frac{\mathbf{T}(s_{A_i}; s_{B_i}; s_{A_{i+1}})}{\gamma^n} \prod_{C \in \Gamma} \varrho_C(s_C) \prod_{C \in Y} \varrho^2_C(s_C) \prod_{C \in \mathbf{B}^c(\Gamma) \backslash I} \varrho^1_C(s_C) \times$$

$$\prod_{j \in \mathbf{B}^c(\Gamma) \backslash I} \frac{T^n(s_{A_j}; s_{A_{j+1}})}{\gamma^n} \prod_{i \in I} v(s_{A_i}) \chi_M(s_{A_i}) v^\star(s_{A_{i+1}}) \chi_M(s_{A_{i+1}}).$$

The term $\prod_{C \in Y} \varrho^2_C(s_C) \prod_{C \in \mathbf{B}^c(\Gamma) \backslash I} \varrho^1_C(s_C)$ can be seen as associated to a pair (Γ_3, Γ_4) where $\Gamma_3, \Gamma_4 \subset \mathcal{G}_\Lambda$, $\Gamma_3 \cap \Gamma_4 = \phi$:

$$\prod_{C \in \Gamma_3} \varrho^1_C(s_C) \prod_{C \in \Gamma_4} \varrho^2_C(s_C) \tag{2.20}$$

(the principle of the notation will be clear below).

We now want to express the partition function of the original system as the one of a polymer gas where the interaction is a hard-core exclusion. Given $C \in \mathcal{C}_\Lambda \cup \mathcal{P}_\Lambda \cup \mathcal{G}_\Lambda$ (that is, C is either a pair of blocks of one of the types indicated before or a triplet (A_i, B_i, A_{i+1})), we define the support of C, denoted by $\widehat{C}$, as:

$$\widehat{C} = (\bigcup_{i: A_i \in C} A_i) \cup (\bigcup_{i: B_i \subset C} B_i) \cup (\bigcup_{i: B_i \text{ or } B_{i-1} \subset C} A_i) \tag{2.21}$$

Note that in this way we have added the two neighbor A-blocks to any B-block which appears in C.

Now let us consider a quadruple $R = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ with $\Gamma_1 \subset \mathcal{C}_\Lambda$, $\Gamma_2 \subset \mathcal{P}_\Lambda$ and $\Gamma_3, \Gamma_4 \subset \mathcal{G}_\Lambda$. A quadruple R is called admissible (which means that it could appear in the expression (2.19) as a particular term in the sum $\sum_\Gamma \sum_I \sum_Y$), if $\Gamma_3 \cap \Gamma_4 = \phi$ and moreover if, for any block B_i that enters some pair or triplet $C \in \Gamma_1 \cup \Gamma_2$, the pair of the two neighbor-blocks A_i, A_{i+1} does not belong to $\Gamma_3 \cup \Gamma_4$. The last condition comes from the fact that, by construction, a pair $\{A_i, A_{i+1}\}$ appears only in association with the intermediary B-block B_i. Hence, if $B_i \in C$ where C is from $\Gamma_1 \cup \Gamma_2$ then pair $\{A_i, A_{i+1}\}$ can appear as an index neither in a term $\varrho^2_C(S_C)$ with $C \in Y \subset I \subset \mathbf{B}^c(\Gamma)$ nor in a term $\varrho^1_C(S_C)$ with $C \in \mathbf{B}^c(\Gamma) \backslash I$ (recall that we identify $\{A_i, A_{i+1}\}$ with site i).

If the block pairs or triplets, C_1 and C_2, belong to $\cup_{i=1}^4 \Gamma_i$, we say that C_1 and C_2 are connected if $\widehat{C}_1 \cap \widehat{C}_2 \neq \emptyset$. An admissible $R = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ is called a polymer if, for any C and $C' \in \cup_{i=1}^4 \Gamma_i$, there exists a sequence C_j, $j = 1, \ldots, k$, such that any $C_j \in \cup_{i=1}^4 \Gamma_i$, $C_1 = C$, $C_k = C'$ and C_j and C_{j+1} are connected for $1 \leq j \leq k-1$.

If $R = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ is a polymer, we let $\mathbf{I}(R)$ be the set of those values i for which either the corresponding B-block B_i enters some $C \in \Gamma_1 \cup \Gamma_2$, or a pair $\{A_i, A_{i+1}\} \in \Gamma_3$. Furthermore, $\mathbf{J}(R)$ denotes the set of those values i for which the corresponding A-block A_i enters some $C \in \Gamma_1 \cup \Gamma_4$. We define the support of R by

$$\widehat{R} = \Big(\bigcup_{i \in \mathbf{I}(R)} (A_i \cup B_i \cup A_{i+1}) \Big) \cup \Big(\bigcup_{j \in \mathbf{J}(R)} A_j \Big). \tag{2.22}$$

For any polymer R it is easy to see that the support $\widehat{R}$ can be decomposed into the union of pair-wise disjoint intervals:

$$\widehat{R} = \bigcup_{i=1}^{K} G_i.$$

Here $K = K(R)$ and G_i may be either an A-block A_j (in which case $j \in \mathbf{J}(R)$ and A_j does not enter any triplet $\{A_i, B_i, A_{i+1}\}$ with $i \in \mathbf{I}(R)$) or a union corresponding to a chain of consecutive triplets:

$$G_i = A_{l_i} \cup B_{l_i} \cup A_{l_i+1} \cup \ldots \cup B_{l_i+m_i} \cup A_{l_i+m_i+1}.$$

Let us define the probability measures on $\mathbf{S}^{\widehat{R}}$:

$$\mu_R(ds_{\widehat{R}}) = \prod_{i=1}^{K} \mu_{G_i}(ds_{G_i}). \tag{2.23}$$

Here, if $G = A_l$, then

$$\frac{d\mu_G(s_G)}{ds_G} = \frac{u_l^\star(s_{A_l}) u_l(s_{A_l})}{\mathcal{N}_G}, \tag{2.24}$$

and, if $G = A_l \cup B_l \cup A_{l+1} \cup \ldots \cup B_{l+m} \cup A_{l+m+1}$, then

$$\frac{d\mu_G(s_G)}{ds_G} =$$

$$\frac{u_l^\star(s_{A_l}) \mathbf{T}(s_{A_l}; s_{B_l}; s_{A_{l+1}}) \cdots \mathbf{T}(s_{A_{l+m}}; s_{B_{l+m}}; s_{A_{l+m+1}}) u_{l+m+1}(s_{A_{l+m+1}})}{\gamma^{m(n+1)} \mathcal{N}_G} \tag{2.25}$$

where

$$u_\ell^\star = v^\star \chi_M, \quad \text{if} \quad -p+1 \leq \ell \leq p,$$

$$u_\ell = v \chi_M, \quad \text{if} \quad -p \leq \ell \leq p-1,$$

$$u^\star_\ell(s_{A_\ell}) = \exp[-\frac{1}{2}V_{A_\ell}(s_{A_\ell})] = u_\ell(s_{A_\ell}), \quad \text{if } \ \ell = -p \text{ or } p,$$

and $\mathcal{N}_G$ is the normalization to have a probability measure.

Next, we assign to a polymer R its (not necessary positive) "fugacity". First, we set:

$$\widetilde{\zeta}(R) = \int \mu_R(ds_{\widehat{R}}) \prod_{C\in\Gamma_1\cup\Gamma_2} \varrho^0_C(s_C) \prod_{C\in\Gamma_3} \varrho^1_C(s_C) \prod_{C\in\Gamma_4} \varrho^2_C(s_C). \tag{2.26}$$

where $\varrho^0_C \equiv \varrho_C$ see eq. (2.4). One can then check that the partition function Ξ_Λ may be written in the following way:

$$\Xi_\Lambda = \gamma^{2pn}\langle v \exp[-\frac{1}{2}V_{A_{-p}}], \chi_M\rangle\langle v^\star \exp[-\frac{1}{2}V_{A_p}], \chi_M\rangle(\langle v^\star v, \chi_M\rangle)^{2p-1}\times$$

$$\left[1 + \sum_{k\geq 1} \sum_{\substack{R_1,\dots,R_k:\\ \widehat{R}_i\subset\Lambda, 1\leq i\leq k,\\ \widehat{R}_i\cap\widehat{R}_{i'}=\emptyset, 1\leq i<i'\leq k}} \prod_{i=1}^{k} \bar{\zeta}(R_j)\right]. \tag{2.27}$$

Here the internal sum is taken over all (unordered) collections of polymers for which the conditions indicated are fulfilled and

$$\bar{\zeta}(R) = \frac{\tilde{\zeta}(R)\mathcal{N}_R}{(\langle vv^\star, \chi_M\rangle)^{\sharp\{i\ :\ i\neq -p,+p;\ A_i\subseteq\widehat{R}\}}}\times$$

$$\frac{1}{\left[(\langle v\exp[-\frac{1}{2}V_{A_{-p}}], \chi_M\rangle)^{\widehat{\delta}(-p,\widehat{R})}(\langle v^\star \exp[-\frac{1}{2}V_{A_p}], \chi_M\rangle)^{\widehat{\delta}(p,\widehat{R})}\right]} \tag{2.28}$$

where $\widehat{\delta}(l,\widehat{R}) = 1$ if $A_l \subseteq \widehat{R}$ and $\widehat{\delta}(l,\widehat{R}) = 0$, otherwise.

The term in the square bracket in the RHS of (2.27) can be interpreted as the partition function of a polymer system with a hard-core interaction and fugacity $\bar{\zeta}$. The product in front of this term is the partition function of the system of non-interacting A-blocks.

Notice that the fugacity $\bar{\zeta}(R)$ depends in general on p (this is the case of those polymers that include the border blocks $A_{\pm p}$). However, this dependence is rather weak and does not affect the argument used.

Furthermore, introducing parameters w_0 and w_1 as specified in (1.24), one can produce a 'full' polymer expansion for Ξ_Λ where the complex terms are taken into account. This leads to a more complicated definition of a polymer, which nevertheless is based on the same kind of ideas . The rigorous scheme repeats the one from [C.C.O.] and we will not go into technical details. The corresponding complex fugacity of a polymer is again denoted by $\bar{\zeta}(R)$; it contains, apart from the factors ϱ^i, $i = 0, 1, 2,$ some new terms ϱ^3 that are complex analogues of ϱ^0. See [C.C.O.] for further details.

In order to control our cluster expansion we need to estimate the fugacity of a polymer. We start by studying the contribution coming from the terms $\varrho^2_C(s_C)$. Let us first specify the indicator functions χ_M. The parameter M runs over $\mathbf{R}_+$, the positive half-axis. The function $\chi_M(s_{A_i})$, $i = -p, \dots, p$, is defined as the space-translation of a function $\chi_M(s^L)$, $s^L \in \mathbf{S}^L$. The latter is the indicator of the set

$$\mathbf{S}^L_M \ = \ \{s^L = (s_j, j \in [0, L-1]) \in \mathbf{S}^L \ : \ \|s_j\|_2 \ \leq \ M(1 + r(j))\} \tag{2.29}$$

where the norm $\|\cdot\|_2$ is defined in (1.48) and

$$r(j) \ = \ \min\,[\,\log(1+j), \log(L-j)\,].$$

Given a positive integer N, let $E (= E_N)$ be the probability density on $\mathbf{S}^{[0,NL-1]}$ (with respect to the measure $ds_{[0,NL-1]}$) of the form

$$E(s_{[0,NL-1]}) = \frac{u^\star(s^{(1)})T(s^{(1)}; s^{(2)}) \dots T(s^{(N-1)}; s^{(N)})u(s^{(N)})}{\mathcal{N}_{[0,NL-1]}}. \tag{2.30}$$

Here, the path configuration $s^{(i)} \in \mathbf{S}^{[(i-1)L, iL-1]}$ is the restriction of $s_{[0,NL-1]}$ to the interval $[(i-1)L, iL-1]$, $i = 1, \dots, N$. Furthermore, the function $u^\star(s^{(1)})$ is either $v^\star(s^{(1)})$ or $\exp[-\frac{1}{2}V_{[0,L-1]}(s^{(1)})]$ and $u(s^{(N)})$ is either $v(s^{(N)})$ or $\exp[-\frac{1}{2}V_{[(N-1)L,NL-1]}(s^{(N)})]$ (all the combinations are possible), cf. (2.23). Finally, $\mathcal{N}_{[0,NL-1]}$ is, as before, the normalization to have a probability measure.

Given $J \subseteq [0, NL-1]$, we set, in analogy with (1.51):

$$k_E^{(J)}(s_J) = \int_{\mathbf{S}^{[0,NL-1]\setminus J}} ds_{[0,NL-1]\setminus J} E(s_{[0,NL-1]\setminus J} \vee s_J). \tag{2.31}$$

By Lemma 3, we get an estimate: for any $J \subset [0, NL-1]$,

$$k_E^{(J)}(s_J) \leq \exp\,[-\sum_{j\in\Delta}(c^\star\|s_j\|_2^2 - \delta)]. \tag{2.32}$$

In fact, the probability density $E_{[0,NL-1]}$ is either of a form assumed in Lemma 3 or the limit of those densities. The constants $c^\star > 0$ and δ do not depend on N and L.

Moreover, for the probability measure μ_E on $\mathbf{S}^{[0,NL-1]}$ with a density E of the form (2.30) and for any $j = 0, \dots N-1$, we have:

$$\mu_E(\{s_{[0,NL-1]} : \ \chi_M(s^{(j)}) = 0\}) \leq \bar{c}_1 e^{-\bar{c}_2 M^2}; \tag{2.33}$$

the constants $\bar{c}_1$ and $\bar{c}_2$ again do not depend on N and L.

Indeed, the bound (2.33) follows easily from the definition (2.29) - (2.31),the bound (2.32) and the estimate:

$$\int_{\{s\in\mathbf{S}:\|s\|_2\geq y\}} ds \exp[-c^\star\|s\|_2^2] \leq \exp[-(c^\star - \epsilon)y^2] \int_{\mathbf{S}} ds \exp[-\epsilon\|s\|_2^2]$$

provided that we are able to prove that the integral in the RHS is finite for any $\epsilon > 0$.

The last assertion may be proved in the following way. Given $s = (x, \omega)$, we use the definition of the norm $\|s\|_2^2$ and the Schwartz inequality to write the bound

$$\|s\|_2^2 \ \geq \ \beta x^2 + 2x \int_0^\beta dt\omega(t) + \beta^{-1}(\int_0^\beta dt\omega(t))^2$$

which implies

$$\int_{\mathbf{S}} ds \exp(-\epsilon \|s\|_2^2) \leq \int_{\mathbf{R}} dx \int_{\mathbf{W}} P(d\omega) \exp[-\epsilon\beta^{-1}(\beta x + \int_0^\beta dt\omega(t))^2].$$

Performing in the RHS the integration in the variable x, we get the desired result.

The following theorem is the crucial ingredient to control the terms $\varrho_C^2(s_C)$:

Theorem 2.1

There exist positive integers L_0 *and positive constants* $\bar{c}_3$, $\bar{c}_4$ *such that for any* $L > L_0$ *and* $M > 1$ *and for any* s, $s' \in \mathbf{S}_M$

$$\left| \frac{T^n(s;s')}{\gamma^n v(s) v^\star(s')} - 1 \right| \leq \exp(\bar{C}_1 M^2 - n\ e^{-\bar{C}_2 M^2}) \tag{2.34}$$

Since the proof of Theorem 2.1 is similar to the one of Theorem 2.1 of [C.C.O], the estimate (2.33) playing the role of the bound (2.4) of [C.C.O], we omit it. For estimating the terms ϱ^0 and, in a complex domain, the terms ϱ^3, one again proceeds in a way similar to [C.C.O.]. It is convenient to summarize all the bounds in Proposition 2.2 below. In this assertion, the condition that M, n and L are large enough, w_0 and w_1 are small enough means the following. First, we impose the restriction $M \geq M^0$ and $n \geq n^0$ where M^0 and n^0 depend only on β and the potentials Φ and Ψ_d, $d \geq 1$. Then, for M and n satisfying these conditions, we take $L \geq L^0$ where $L^0 = L^0(M,n)$. Finally, for M, n and L that satisfy the above restrictions, we consider w_0 and w_1 with max $[|w_0|, |w_1|] \leq w^0$ where $w^0 = w^0(M,n,L)$. The claim is that we can guarantee a proper choice of the thresholds M^0, n^0, L^0 and w^0 (in bounds (2.40) and (2.44) - (2.46) below these thresholds depend on a parameter σ). Note that all the estimates that follow are claimed to be uniform in p.

Proposition 2.2

Given M, n *and* L *large enough and* w_0 *and* w_1 *small enough, the (complex) polymer fugacity* $\bar{\zeta}(R)$ *satisfies the bound*

$$|\ \bar{\zeta}(R)\ | \leq \prod_{C \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4} \widehat{g}_C. \tag{2.35}$$

Here, for $C = \{D, D'\} \in \mathcal{C}_\Lambda$,

$$\widehat{g}_C = \max\{\frac{12M^2n^2(1+w^2)\log(nL+1)}{r_C F(r_C L)},$$

$$6\bar{c}\exp\left[-c^\star \frac{M^2}{8}\log(r_C+1)\right]\}, \tag{2.36}$$

where r_C *is the total number of blocks of the both types,* A *and* B*, situated between* D *and* D' *, whereas for* $C \in \mathcal{P}_\Lambda \cup \mathcal{G}_\Lambda$,

$$\widehat{g}_C(M,n,L,w) = \max\{100M^2\log(nL+1)n\left[\frac{1}{\log(L+1)F(L)} + wL\right] +$$

$$100nLw,\;\; 6\bar{c}\exp(-\frac{c^\star}{48}M^2),\;\; \exp(\bar{c}_3M^2 - ne^{-\bar{c}_4M^2})\}. \tag{2.37}$$

Constant $c^\star$ *comes from Lemma 3, and* $\bar{c}_3$, $\bar{c}_4$ *from Theorem 2.1 and* $w = \max[w_0, w_1]$.

The proof of Proposition 2.2 is similar to the one of Lemma 3.1 of [C.C.O.]. In the proof one uses Theorem 2.1 together with (2.33) and the following fact:

$$|\,\psi_d(s_i,s_j)\,| \le \frac{\langle |\,s_i\,|,\;|\,s_j\,|\rangle}{(i-j)^2\log(1+|\,i-j\,|)F(|\,i-j\,|)} \tag{2.38}$$

where

$$\begin{aligned}\langle |\,s_i\,|,\;|\,s_j\,|\rangle &\equiv \int_0^\beta |x_i+w_i(t)|\;|x_j+w_j(t)|dt\\ &\le \frac{1}{2}(\|s_i\|_2^2 + \|s_j\|_2^2)\end{aligned} \tag{2.39}$$

As before, we omit the details referring the reader to [C.C.O.].

It now can be checked (cf. [C.C.O.], Proposition 2.2) that for M and L large enough we have, for some constant $\mathcal{N} > 0$:

$$\begin{aligned}&\frac{3}{4} < (v\tilde{v}^\star, \chi_{\bar{M}}) \le (vv^\star, \chi_M) \le 1,\\ &\frac{3}{4}\mathcal{N}^{-1} < \left(\frac{v\;e^{-h/2}}{\sqrt{\gamma}}, \chi_M\right) \le \mathcal{N},\\ &\mathcal{N}_R \le \mathcal{N}^2.\end{aligned}$$

Furthermore, for every polymer R the following inequality holds:

$$|\,R\,| \le 3\sharp(\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4)$$

where $|\,R\,|$ denotes the number of blocks of type A or B contained in $\widehat{R}$. It then follows that, given $\sigma \in [\frac{1}{2}, 1)$, we can choose n, M and L large enough and w_0 and w_1 small enough so that, for every polymer $R = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$, the complex fugacity $\bar{\zeta}(R)$ admits the bound:

$$|\,\bar{\zeta}(R)\,| \le \sigma^{|R|}\prod_{C\in\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4} g_c \tag{2.40}$$

where

$$g_c = 2^3 \mathcal{N}^4 \hat{g}_c. \tag{2.41}$$

The bound (2.40) is the basic ingredient for proving the convergence of the cluster expansion. More precisely, proceeding as in [C.C.O.] Lemma 1, we get

Proposition 2.3

Let $\tilde{\Xi}_\Lambda$ *denote the polymer partition function figuring in the square brackets in the RHS of (2.27) (for the complex perturbation of the Hamiltonian):*

$$\tilde{\Xi}_\Lambda = 1 + \sum_{k\geq 1} \sum_{\substack{R_1,\ldots,R_k: \\ \widehat{R}_i \subseteq \Lambda, 1\leq i\leq k, \\ \widehat{R}_i \cap \widehat{R}_{i'} = \emptyset, 1\leq i<i'\leq k}} \prod_{j=1}^{k} \bar{\zeta}(R_j) \tag{2.42}$$

and κ *be given by*

$$\kappa \;=\; 4\hat{g}_{\{A_0,A_1\}} + \sum_{C\in\mathcal{C}_A\cup\mathcal{P}_\Lambda} g_C \tag{2.43}$$

(see (2.36), (2.37) and (2.40)). Then, given $\sigma \in [\frac{1}{2}, 1)$, *we can choose* n, L *and* M *large enough and* w_0 *and* w_1 *small enough so that estimate (2.40) is fulfilled and the following bounds hold:*

$$\exp\kappa < \frac{1}{\sqrt{\sigma}(2-\sqrt{\sigma})}, \tag{2.44}$$

$$\sup_{\substack{V\subset\{A_i, i\in[-p_1,\ldots,p]\} \\ \cup\{B_i, i\in[-p_1,\ldots,p]\}}} \sum_{R: V\subseteq\hat{R}\subseteq\Lambda} |\, \bar{\zeta}(R) \,| \leq \sigma\kappa \frac{[1-(e^\kappa-1)]}{1+\sigma^2 e^\kappa - 2\sigma e^\kappa} \equiv G(\kappa,\sigma), \tag{2.45}$$

and for any polymer R

$$\sum_{k\geq 1} \sum_{\substack{R_1,\ldots,R_k \\ R_1=R}} |\, \varrho(R_1,\ldots,R_k) \,| \prod_{i=2}^{k} |\, \bar{\zeta}(R_i) \,|$$
$$\leq \bar{\zeta}(R) \frac{\exp\left[G(\kappa,\sqrt{\sigma}) \mid R \parallel\right]}{1-\sqrt{\sigma}\exp[G(\kappa,\sqrt{\sigma})]} \tag{2.46}$$

Here ϱ *denotes the standard Möbius function:*

$$\varrho(R_1,\ldots,R_k) = \frac{1}{k!} \sum_{\mathbf{g}\in\mathbf{G}(R_1,\ldots,R_k)} (-1)^{\sharp(\text{edges in } \mathbf{g})} \tag{2.47}$$

where $\mathbf{G}(R_1,\dots,R_k)$ *stands for the set of all connected subgraphs of the graph with* k *vertices* $\{1, \dots, k\}$ *and with the edges corresponding to those pairs* (i,j) *for which* $\widehat{R}_i \cap \widehat{R}_j \neq \emptyset$ *(the sum in (2.47) equals zero if* $\mathbf{G}(R_1,\dots,R_k)$ *is empty and one if* $k=1$*).*

By using Proposition 2.3, we can control the standard cluster representation

$$\widetilde{\widetilde{\Xi}}_\Lambda = \exp\,[\sum_{k\geq 1}\ \sum_{\substack{R_1,\dots,R_k:\\ \widehat{R}_i\subseteq\Lambda,1\leq i\leq k}} \varrho(R_1,\dots,R_k)\prod_{j=1}^{k}\bar{\zeta}(R_j)] \tag{2.48}$$

which follows from (2.42).

Now we can give the proof of Lemmas 1 and 2. The expansion argument provided so far allows us to prove the assertion of Lemma 2 about the analytic continuation of λ_Λ and the boundedness of $|\lambda_\Lambda|$. Furthermore, we are able to prove the uniform convergence of the analytic functions

$$\lim_{p\to\infty}\lambda_\Lambda = \lambda,$$

while parameters M, n, L and w_0, w_1 are kept fixed (but chosen either large enough or small enough, respectively, in the sense that was specified above).

The analysis of the kernels $k_\Lambda^{(J)}$ and $\phi_\Lambda(\cdot)$ for the operators that are specified in the formulation of Lemma 2 is based on similar expansions of quantities $\Xi_\Lambda^{(J)}(x_J, y_J)$, $x_J, y_J \in \mathbf{R}^J$.

Let us start by discussing the limit relation (1.30). We again assume that Λ is of the form specified above. [This assumption will soon be dropped.] Let us suppose for definiteness that J is a subset of B-block B_0. Proceeding in the same way as before, we can write the following representation for the quantity under consideration:

$$\begin{gathered}\Xi_\Lambda^{(J)}(x_J,y_J) = \gamma^{(2p-1)n}\langle v\exp[-\frac{1}{2}V_{A_{-p}}],\chi_M\rangle\langle v^\star\exp[-\frac{1}{2}V_{A_p}],\chi_M\rangle(\langle v^\star v,\chi_M\rangle)^{2p-2}\times\\ \int ds_{A_0}\,ds_{A_1}\,ds_{B_0\setminus J}d\omega_J\mathbf{T}(s_{A_0};s_{B_0\setminus J}\vee\bar{s}_J;s_{A_1})\times\\ \left[1+\sum_{k\geq 1}\ \sum_{\substack{R_1,\dots,R_k:\\ \widehat{R}_i\subseteq\Lambda,1\leq i\leq k,\\ \widehat{R}_i\cap\widehat{R}_{i'}=\emptyset,1\leq i<i'\leq k}}\ \prod_{j=1}^{k}\bar{\zeta}(R_j;x_J,y_J)\right].\end{gathered} \tag{2.49}$$

Here $\bar{s}_J = (x_J, y_J; \omega_J)$ and the quantity $\mathbf{T}(s_{A_0}; s_{B_0\setminus J}\vee\bar{s}_J; s_{A_1})$ is defined by formulas similar to (2.8), (2.9). The sum in the square brackets has the same nature as before (cf. (2.27)). The definition of a polymer is again the same and the definition of the polymer fugacity $\bar{\zeta}(R; x_J, y_J)$ follows a similar idea (cf. (2.23) - (2.26) and (2.28)). This fugacity now depends, in general, not only on p, but also on x_J, y_J, but as before, this dependence is rather weak: it affects only the polymers R for which $\widehat{R}\cap(A_{-p}\cup A_0\cup B_0\cup A_1\cup A_p)\neq\emptyset$.

Finally, introducing the parameters w_0 and w_1 as specified in (1.24), we are able to produce a full polymer expansion for $\Xi_\Lambda(x_J, y_J)$ with complex terms. We again denote the corresponding complex fugacity by $\bar{\zeta}(R; x_J, y_J)$.

Our final aim is the same as above: we want to control the cluster representation

$$\widetilde{\Xi}_\Lambda(x_J, y_J) \;=\; \exp\,[\sum_{k\geq 1} \sum_{\substack{R_1,\dots,R_k:\\ \widehat{R_i}\subseteq\Lambda, 1\leq i\leq k}} \varrho(R_1,\dots,R_k)) \prod_{j=1}^{k} \bar{\zeta}(R_j; x_J, y_J)] \tag{2.50}$$

for a polymer partition function $\widetilde{\Xi}_\Lambda(x_J, y_J)$ given by the term in the square brackets in the RHS of (2.49).

An analysis of the scheme of estimating the various terms contributing to $\bar{\zeta}(R; x_J, y_J)$ shows no major differences with the above construction. There are two changes worth to note. First, the threshold values M_0, n_0, L_0 and w_0, w_1 depend in general on x_J and y_J. Second, we use the assertion of Lemma 3 for both cases, as for the denominator Ξ_Λ as for $\Xi_\Lambda^{(J)}(x_J, y_J)$, because the measures μ_G figuring in the definition of the polymer fugacity either have denominators of this type or are limits of those measures.

After performing the necessary estimates we arrive at assertions that are similar to Propositions 2.2 and 2.3 and give a desired control of the convergence in representation (2.50). Having the control of both expansions, (2.48) and (2.50) we can proceed in a standard way and guarantee the existence of the limit

$$\lim_{p\to\infty} k_\Lambda^{(J)}(x_J, y_J) = k^{(J)}(x_J, y_J), \tag{2.51}$$

while the parameters M, n, L and w_0, w_1 are, as before, kept fixed (but again chosen large enough and small enough, respectively). More precisely, we choose values of these parameters so that the assertions of Propositions 2.2 and 2.3 and their analogues for $\bar{\zeta}(R, x_J, y_J)$ hold and then perform the limit $p \to \infty$. The same scheme is used in an argument that follows.

In addition to (2.51), we get a representation of the limiting kernel $k^{(J)}(x_J, y_J)$ in the form

$$k^{(J)}(x_J, y_J) = \exp\,[\sum_{k\geq 1} \sum_{\substack{R_1,\dots,R_k:\\ \hat{R}_i\cap(A_0\cup B_0\cup A_1)\neq\emptyset\\ \text{for some } i\in\{1,\dots k\}}} \rho(R_1, \dots R_k)\times$$

$$\times(\prod_{j=1}^{k} \bar{\zeta}(R_j; x_J, y_J) - \prod_{j=1}^{k} \bar{\zeta}(R_j))]. \tag{2.52}$$

Note that, for the polymers figuring in the expansion in the RHS of (2.52), both fugacities $\bar{\zeta}$ and $\bar{\zeta}(R; x_J, y_J)$ do not depend on p.

Let us now comment on how to extend the limiting relation (2.51) to a general case $\Lambda \nearrow \mathbf{Z}$. Given an interval $\Lambda \subset \mathbf{Z}$, we can 'fill' it with our A- and B-blocks as indicated (with $p = p_\Lambda = [\dfrac{|\Lambda| - L}{2(n+1)}]$), with the proviso that the 'border' blocks $A_{\pm p}$ have, in general, a greater length (for example, we can include, in each of these blocks, half of the rest length $|\Lambda| - L(2p(n+1)+1)$. Then we can proceed in the same way as before, estimating the difference between $k_\Lambda^{(J)}(x_J, y_J)$ and $k^{(J)}(x_J, y_J)$ in terms of p_Λ. While $\Lambda \nearrow \mathbf{Z}$, p_Λ tends to infinity, which guarantees the convergence in (1.30).

In fact, we are able to do more. By using the argument developed, we can prove that, given a finite $J \subset \mathbf{Z}$ and $x_J, y_J \in \mathbf{R}^J$, the kernel $k_\Lambda^{(J)}(x_J, y_J)$ admits an analytic continuation, in the variables w_0, w_1, to $\mathcal{O}_0 \times \mathcal{O}_1$ where $\mathcal{O}_\ell$, $\ell = 0, 1$, is a neighborhood of the origin in $\mathbf{C}$ (which depends, in general, on x_J, y_J, but not on $\Lambda \subseteq J$. Furthermore, these analytic functions converge, uniformly in $(w_0, w_1) \in \mathcal{O}_0 \times \mathcal{O}_1$, to a limit, as $\Lambda \nearrow \mathbf{Z}$. The limiting function is nothing but the analytic continuation of $k^{(J)}(x_J, y_J)$.

Moreover, the formulas determining fugacities $\bar{\zeta}(R; x_J, y_J)$ show that $\mathcal{O}_0$ and $\mathcal{O}_1$ may be chosen independently on x_J, y_J and the series under consideration converge uniformly in x_J, y_J provided that these variables run over a compact set in $\mathbf{R}^J$.

The same argument is used for proving the analyticity, boundedness and convergence of $\phi_\Lambda(a)$ where a is an operator of the kind considered in the assertions (b) and (c) of Theorem 2. To avoid a repetition, we omit the detailed argument. The reader can reconstruct it from the corresponding discussion below related to $\phi_\Lambda(p_l^2)$ (which is, apparently, the most difficult case from technical point of view).

A cluster expansion argument is used also for proving the mixing property (1.32). Here, the main construction has to be slightly modified. Namely, for u large enough, $(x_{J^{(1)}}, y_{J^{(1)}})$ and $(x_{S_u J^{(2)}}, y_{S_u J^{(2)}})$ are associated with non-adjacent blocks. If $J^{(1)}$ and $S_u J^{(2)}$ belong to, say, blocks B_{j_1} and B_{j_2}, respectively, then, in a cluster representation for

$$k^{(J^{(1)} \cup S_u J^{(2)})}(x_{J^{(1)}} \vee S_u x_{J^{(2)}}, y_{J^{(1)}} \vee S_u y_{J^{(2)}})$$

(which is similar to (2.51)), the condition

$$\hat{R}_i \cap (A_{j_1} \cup B_{j_1} \cup A_{j_1+1} \cup A_{j_2} \cup B_{j_2} \cup A_{j_2+1}) \neq \emptyset$$

becomes 'almost equivalent' to one of the mutually disjoint conditions

$$\widehat{R}_i \cap (A_{j_1} \cup B_{j_1} \cup A_{j_1+1}) \neq \emptyset, \ \text{ but } \ \widehat{R}_i \cap (A_{j_2} \cup B_{j_2} \cup A_{j_2+1}) = \emptyset,$$

or

$$\widehat{R}_i \cap (A_{j_2} \cup B_{j_2} \cup A_{j_2+1}) \neq \emptyset, \ \text{ but } \ \widehat{R}_i \cap (A_{j_1} \cup B_{j_1} \cup A_{j_1+1}) = \emptyset.$$

More precisely, the polymers R_i whose supports have a non-empty intersection with both $A_{j_1} \cup B_{j_1} \cup A_{j_1+1}$ and $A_{j_2} \cup B_{j_2} \cup A_{j_2+1}$ give a contribution that vanishes as $|j_1 - j_2| \to \infty$. This leads to relation (1.32).

The bound (1.29) (in the real domain) follows from the assertion of Lemma 3 (see (1.66)), for the case of a measure $\mathbf{m}_\Lambda$ with the denominator Ξ_Λ. Indeed, by integrating the RHS of (1.66) in $P^J(d\omega_J)$, while x_J, y_J remain fixed, gives a function $k_*^{(J)}(x_J, y_J)$ with the desired properties. The last fact may be proved by repeating the argument used for establishing (2.33).

To prove the bound (1.34), we proceed as follows. First, we write

$$k_\Lambda^{(J)}(x_J,x_J)\mathcal{F}_J(x_J)=\left[\frac{\Xi_\Lambda(\mathcal{F}_J)}{\Xi_\Lambda}\right]\left[\frac{\Xi_\Lambda^{(J)}(x_J,x_J)\mathcal{F}_J(x_J)}{\Xi_\Lambda(\mathcal{F}_J)}\right]$$

(cf. (1.62)). [Recall that the function $\mathcal{F}_J$ is assumed to be non-negative.] Repeating the scheme elaborated above, we can prove the convergence to a finite limit, as $\Lambda\nearrow\mathbf{Z}$, of the ratio $\dfrac{\Xi_\Lambda(\mathcal{F}_J)}{\Xi_\Lambda}$. Hence. this ratio is bounded uniformly in Λ. The remaining ratio

$$\frac{\Xi^{(J)}(x_J,x_J)\mathcal{F}_J(x_J)}{\Xi_\Lambda(\mathcal{F}_J)}$$

does not exceed, in view of Lemma 3, the RHS of (1.66). This gives the desired result.

A bit more sophisticated reasoning is used for proving the rest parts of Lemmas 1 and Lemma 2. We begin the discussion with proving the analyticity, boundedness and convergence of the derivatives (1.35). Again to avoid a repetition, let us consider the second derivative only.

The starting point is formula (1.59). We see that the problem is reduced to prove that the ratio of a single addend in the parentheses and the denominator Ξ_Λ possesses the aforementioned properties. For definiteness, consider the ratio $\dfrac{\Xi_\Lambda^{(J)}(x_J,y_J))_j^{(2)}}{\Xi_\Lambda}$. As we noted before, we can consider separately each addend arising in the braces $\{\ldots\}$ in the RHS; we again confine the consideration to one of them, say, the quantity

$$\begin{gathered}\int_{\mathbf{W}^J}dP^J(\omega_J)\int_{\mathbf{S}^{\Lambda\setminus J}}ds_{\Lambda\setminus J}(V_\Lambda)^{(2)}(\bar{s}_J\vee s_{\Lambda\setminus J})\exp[-V_\Lambda(\bar{s}_J\vee s_{\Lambda\setminus J})],\\ \bar{s}_J=(x_J,y_J;\omega_J)\end{gathered}\tag{2.53}$$

(we omitted non-essential factor $\exp[-1/2\beta\sum\limits_{i\in J}(x_j-y_j)^2]$). Furthermore, we expand (2.53) into the sum

$$\begin{aligned}&(\Xi_\Lambda)^{-1}\int_{\mathbf{W}^J}dP^J(\omega_J)\int_{\mathbf{S}^{\Lambda\setminus J}}ds_{\Lambda\setminus J}\varphi^{(2)}(s_j)\exp[-V_\Lambda(\bar{s}_J\vee s_{\Lambda\setminus J})]+\\&(\Xi_\Lambda)^{-1}\sum_{j'\in\Lambda:j'\neq j}\int_{\mathbf{W}^J}dP^J(\omega_J)\int_{\mathbf{S}^{\Lambda\setminus J}}ds_{\Lambda\setminus J}\psi^{(2)}_{|j'-j|}(\bar{s}_j,\bar{s}_{j'})\exp[-V_\Lambda(\bar{s}_J\vee s_{\Lambda\setminus J})],\end{aligned}\tag{2.54}$$

where $\bar{s}_j=(x_j,y_j;\omega_j)$, and $\bar{s}_{j'}=(x_{j'},y_{j'}\omega_{j'})$ for $j'\in J$ and $\bar{s}_{j'}=s_{j'}=(x_{j'};\omega_{j'})$ for $j'\in\Lambda\setminus J$.

We claim that a single addend in (2.54) admits an analytic continuation, in the variables w_0, w_1, to a complex domain, of the same type as before, and

(i) this domain may be chosen independently of an addend, and also of $\Lambda\supseteq J$ and of x_J, y_J running over a compact set $\mathbf{O}\subset\mathbf{R}^J$,

(ii) the whole series (2.54) converges uniformly in $w_0,\, w_1$ varying within this domain and uniformly in $\Lambda \supseteq J$ and in the variables $x_J,\, y_J \in \mathbf{O}$.
Furthermore, as $\Lambda \nearrow \mathbf{Z}$, each addend tends to a limit, uniformly in $w_0,\, w_1,\, x_J$ and y_J in the same sense as above, and the limiting (analytic) functions form a series that converges, in the complex domain under consideration, again uniformly in the same sense.

To verify these statements, it is convenient to take a single addend in the 'positive' and 'negative' parts as was explained before and treat them separately. Again for definiteness, let us consider the term corresponding to a fixed $j' \notin J$ and take its positive part only:

$$\frac{[(\Xi_\Lambda^{(J)}(x_J, y_J))^{(2)}_{j,j'}]_+}{\Xi_\Lambda} \tag{2.55}$$

where

$$[(\Xi_\Lambda^{(J)}(x_J, y_J))^{(2)}_{j,j'}]_+ = \int_{\mathbf{W}^J} dP^J(\omega_J) \int_{\mathbf{S}^{\Lambda\setminus J}} ds_{\Lambda\setminus J}[\psi^{(2)}_{|j'-j|}(\bar{s}_j, s_{j'})]_+ \times$$
$$\exp[-V_\Lambda(\bar{s}_J \vee s_{\Lambda\setminus J})],\ \ \bar{s}_J = (x_J, y_J; \omega_J) \tag{2.56}$$

For the sake of simplicity, we can assume that $j = 0,\, J = \{0\}$ and omit the subscripts $j,\, J$ and $+$ and the superscripts (2) and (J) from the notation, writing $\Xi_L(x, y)_{j'}$ instead of $[(\Xi_\Lambda^{(J)}(x_J, y_J)^{(2)}_{j,j'}]_+$.

The analysis of the term (2.55) proceeds in a way similar to that for $\dfrac{\Xi_\Lambda^{(J)}(x_J, y_J)}{\Xi_\Lambda}$. We can write for $\Xi_\Lambda(x, y)_{j'}$ a representation similar to (2.49). The point is that the threshold values $M^0,\, N^0,\, L^0$ and $w_0,\, w_1$ can be chosen independently on j' and on $x_J,\, y_J$ running over $\mathbf{O}$. Therefore, the series that arises have a radius of convergence that does not depend on $x_J, y_J \in \mathbf{O}$. Furthermore, by virtue of (1.5), the sum may be estimated by a quantity

$$const\ (d^2 \log(d+1) F(d))^{-1}$$

which guarantees the uniform convergence of the series (2.54).

Similar reasoning is used for establishing the bound (1.36). Let us again discuss the case $\mu = 2$ and consider the contribution from $(\Xi_\Lambda^{(J)}(x_J, y_J))^{(2)}_j$. As before, we decompose it according to (1.60). For diversity, we will consider the term $[(V_\Lambda)^{(1)}_j(s_J \vee s_{\Lambda\setminus J})]^2$. A single term in the corresponding series is related either to $\varphi^{(1)}(s_j)^2$, or $\varphi^{(1)}(s_j)\psi^{(1)}_{|j-j'|}(s_j, s_{j'})$, or $\psi^{(1)}_{|j-j'|}(s_j, s_{j'})\psi^{(1)}_{|j-j''|}(s_j, s_{j''})$, $j', j'' \in \Lambda \setminus \{j\}$. For definiteness, consider a term that corresponds to $\psi^{(1)}_{|j-j'|}(s_j, s_{j'})\psi^{(1)}_{|j-j''|}(s_j, s_{j''})$, $j', j'' \in \Lambda \setminus J$.

Such a term obviously does not exceed the ratio

$$\frac{[(\Xi_\Lambda^{(J)}(x_J, x_J))^{(1)}_{j;j',j''}]^+}{\Xi_\Lambda}$$

where

$$[(\Xi_\Lambda^{(J)}(x_J, x_J))^{(1)}_{j;j',j''}]^+ = \int_{\mathbf{W}^J} dP^J(\omega_J) \int_{\mathbf{S}^{\Lambda\setminus J}} ds_{\Lambda\setminus J} |\psi^{(1)}_{|j-j'|}(s_j, s_{j'})| \times$$
$$|\psi_{|j-j''|}(s_j, s_{j''})| \exp[-V_\Lambda(s_J \vee s_{\Lambda\setminus J})],\ s_J = (x_J; \omega_J) \tag{2.57}$$

As before, we assume for simplicity that $j = 0$ and $J = \{0\}$ and omit excessive indeces from the notation. According to (1.5) and (1.9), (2.57) is less than or equal to

$$[|j'|^2 \log (|j'|+1) F(|j'|)]^{-1} [|j''|^2 \log (|j''|+1) F(|j''|)]^{-1} \times$$
$$\int_{\mathbf{W}} dP(\omega_J) \int_{\mathbf{S}^{\Lambda \setminus \{0\}}} ds_{\Lambda \setminus \{0\}} (||s_0||_1 + 1)^2 (||s_{j'}||_1 + 1)$$
$$(||s_{j''}||_1 + 1) \exp[-V_\Lambda (s_0 \vee s_{\Lambda \setminus \{0\}})], \; s_0 = (x_0; \omega_0) \tag{2.58}$$

Hence, all we need is to prove the bound

$$\frac{\tilde{\Xi}_\Lambda(x)_{j',j''}}{\Xi_\Lambda} \leq \exp\left[-\tilde{c}_1 (x^2 - \tilde{c}_2)\right] \tag{2.59}$$

with constants $\tilde{c}_1 > 0$ and $\tilde{c}_2 \in \mathbf{R}$ independent on j', j'' and Λ. Here

$$\frac{\tilde{\Xi}_\Lambda(x)_{j',j''}}{\Xi_\Lambda} = \int_{\mathbf{W}} dP(\omega_J) \int_{\mathbf{S}^{\Lambda \setminus \{0\}}} ds_{\Lambda \setminus \{0\}} (||s_0||_1 + 1)^2 \times$$
$$(||s_{j'}||_1 + 1)(||s_{j''}||_1 + 1) \exp[-V_\Lambda (s_0 \vee s_{\Lambda \setminus \{0\}})]. \tag{2.60}$$

This may be achieved by using the assertion of Lemma 3 (with the integration in $dP(\omega_0)$ over $\mathbf{W}$) for a measure with the denominator $\Xi_\Lambda(\mathcal{E}_{\widetilde{J}})$ where $\widetilde{J} = \{0, j', j''\}$ and

$$\mathcal{E}_{\widetilde{J}} = (||s_0||_1 + 1)^2 (||s_{j'}||_1 + 1)(||s_{j''}||_1 + 1).$$

APPENDIX

We prove Lemma 3 by using an argument which is similar to the one from [R2] and [R3]. To make the exposition easier we will use, wherever possible, the notations from [R3], or a close one. Let us start by proving the assertion of Lemma 3 in the simplest case of the measure with the denominator Ξ_Λ. As in [R2] and [R3], we deal with a sequence of volumes (cubes) $[\ q\ ]$, $q = 1, 2,...,$ where

$$[\ q\ ] \ = \ \{j = (j_1, ..., j_\nu) \in \mathbf{Z}^\nu : \ |j_i| \leq l_q\}$$

and a sequence of positive integers l_q is chosen so that $|l_{q+1}/l_q - (1 + 2\alpha)| < \alpha$ where $\alpha > 0$ is a constant. The volume of $[\ q\ ]$ is denoted by v_q : $v_q = (2l_q + 1)^\nu$. We denote by $||\bar{s}||$ the norm (1.48) with $r = 2$. The key technical assertion is the following proposition (cf. Proposition 2.1 from [R2] and [R3]):

Proposition A.1 *Under the conditions of Lemma 3, for any* $\epsilon > 0$ *and* $C \geq 0$, *there exists* $\alpha^0 > 0$ *such that for any* $\alpha \in (0, \alpha^0)$ *one can find* $P \geq 1$ *and a monotone increasing sequence* ϖ_q, $q = P, P+1,...,$ *with* $\varpi_q \geq 1$ *and* $\lim\limits_{q\to\infty} \varpi_q = \infty$ *such that the following holds: Let* $\Lambda \subset \mathbf{Z}^\nu$ *be a finite set and* $\bar{s}_\Lambda = (\bar{s}_j)$ *be a path configuration from* $\bar{\mathbf{S}}^\Lambda$. *Suppose that* $q \geq P$ *is the largest integer for which* $\sum\limits_{j\in[\ q\ ]\cap\Lambda} ||\bar{s}_j||^2 \geq \varpi_q v_q$. *Then*

$$\sum_{j\in[\ q+1\ ]\cap\Lambda} C + \sum_{j\in[\ q+1\ ]\cap\Lambda}\ \sum_{j'\in\Lambda\setminus[\ q+1\ ]} |\Psi_{\{j,j'\}}|\frac{1}{2}(||\bar{s}_j||^2 + ||\bar{s}_{j'}||^2) \leq$$
$$\leq \epsilon \sum_{j\in[\ q+1\ ]\cap\Lambda} ||s_j||^2. \qquad (A.1)$$

Moreover, if ϵ *and* C *and* c_1, c_2, $\underline{c}$, $\hat{c}$ *and* η *figuring in (1.7), (1.64) and (1.65) are varying within compact sets (in the case of* ϵ, c_1, $\hat{c}$ *and* η *— separated from* 0 *), then* α^0 *and — for any* $\alpha \in (0, \alpha^0)$ *—* P *and* $\{\varpi_q, q \geq P\}$ *may be chosen independently on these values.*

We omit the proof of Proposition A.1: it repeats that of Proposition 2.1 of[R2]. In what follows we fix $\epsilon \in (0, c_1/3)$, $C = c_2$ and fix $\alpha \in (0, \alpha^0)$. Given $\Lambda \subset \mathbf{Z}^\nu$, denote by $\Re_0$ the set of the path configurations $\bar{s}_\Lambda \in \bar{\mathbf{S}}^\Lambda$ for which $\sum\limits_{j\in[\ q\ ]\cap\Lambda} ||\bar{s}_j||^2 \leq \varpi_q v_q$ for any $q \geq P$ and by $\Re_q$, $q \geq P$, the set of the path configurations $\bar{s}_\Lambda$ for which q is the largest integer $\geq P$ with $\sum\limits_{j\in[\ q\ ]\cap\Lambda} ||\bar{s}_j||^2 > \varpi_q v_q$.

An important corollary of Proposition A.1 is

Proposition A.2 (a) *Let a path configuration* $\bar{s}_\Lambda \in \Re_q$ *where* $q \geq P$. *Then*

$$- V_\Lambda(\bar{s}_\Lambda) + V_{\Lambda\cap[\ q+1\ ]}(\bar{s}_{\Lambda\cap[\ q+1\ ]}) \leq$$
$$\leq (-c_1 + 3\epsilon) \sum_{j\in[\ q+1\ ]} ||s_j||^2 - C'\varpi_{q+1}v_{q+1} \qquad (A.2)$$

where the constant $C' > 0$ *does not depend on* Λ.

(b) *Let* $\bar{s}_\Lambda \in \Re_0$. *Then, for any* $j \in \Lambda$,

$$-V_\Lambda(\bar{s}_\Lambda) + \varphi_j(\bar{s}_j) \le D', \tag{A.3}$$

where $\varphi_j(\bar{s}) = \int_0^\beta dt \Phi_j(\omega(t) + L_{x,y}(t))$, $\bar{s} = (x, y; \omega)$ *(cf. (1.41)) and the constant* D' *does not depend on* Λ.

The constants, C' *and* D', *possess the uniformity property stated in Proposition A.1.*

The proof of Proposition A.2 is similar to the one of Proposition 2.5 of [R2] (combined with the proof of the bound (2.29) from [R2]) and we again omit it . Notice that all constants figuring in the various estimates below possess the uniformity property.

The partition of $\bar{\mathbf{S}}^\Lambda$ into sets $\Re_0$ and $\bigcup_{q \ge P} \Re_q$ generates, for any $J \subset \Lambda$ and $\bar{s}_J = (\bar{s}_j,\, j \in J) \in \bar{\mathbf{S}}^J$, the corresponding expansion $k^{(J)}_{\mathbf{m}_\Lambda}(\bar{s}_J) = k'(\bar{s}_J) + k''(\bar{s}_J)$. As in [R3], we are going to prove later that

$$k'(\bar{s}_J) \le C'' \exp\left[\, (\bar{\Psi} - c_1)||\bar{s}_j||^2 \,\right] k^{(J\setminus\{j\})}_{\mathbf{m}_\Lambda}(\bar{s}_{J\setminus\{j\}}) \tag{A.4}$$

for any $j \in \Lambda$ and

$$\begin{aligned} k''(\bar{s}_J) \;\le\; \sum_{q \ge P} \exp\Big[-C' \varpi_{q+1} v_{q+1} + D'' v_{q+1} - (c_1 - 3\epsilon) \sum_{j \in [\,q+1\,] \cap J} ||\bar{s}_j||^2 \,\Big] \times \\ \times\, k^{(J \setminus [\,q+1\,])}_{\mathbf{m}_\Lambda}(\bar{s}_{J\setminus[\,q+1\,]}), \end{aligned} \tag{A.5}$$

where $C'' > 0$ and $D'' \in \mathbf{R}$ are constants independent on Λ and $J \subseteq \Lambda$ and

$$\bar{\Psi} \;=\; \sup_{j \in \mathbf{Z}^\nu} \sum_{j' \in \mathbf{Z}^\nu : j' \ne j} |\Psi_{\{j,j'\}}|$$

(cf. (2) and (3) in [R3]).

Having proved (A.4) and (A.5), we can establish, by using an induction on the card of J (cf. [R3]), that

$$k^{(J)}_{\mathbf{m}_\Lambda}(\bar{s}_J) \;\le\; \exp\Big[\sum_{j \in J} (E||\bar{s}_j||^2 + F) \Big] \tag{A.4}$$

for some constants $E, F \in \mathbf{R}$ (cf. (4) in [R3]). The next step is to check that indeed a stronger inequality holds:

$$k^{(J)}_{\mathbf{m}_\Lambda}(\bar{s}_J) \;\le\; \exp\Big(\sum_{j \in J} \left[(-c_1 + 3\epsilon)||\bar{s}_j||^2 + \delta \,\right] \Big), \tag{A.7}$$

where $\delta > 0$ again does not depend on Λ, $J \subset \Lambda$ (cf. (5) in [R3]). The assertion of Lemma 3 then follows with $c_1^\star = c_1 - 3\epsilon$, $c_2^\star = \delta$.

The choice of δ is: $\delta = (E + c_1 - 3\epsilon)\varpi_P v_P + F$ (precisely as in [R3]). If $||\bar{s}_j||^2 \leq \varpi_P v_P$ for any $j \in J$, (A.7) follows from (A.4). Therefore, we can assume that $||\bar{s}_j||^2 > \varpi_P v_P$ for some $j \in J$. Then $k'(\bar{s}_J) = 0$ and $k^{(J)}_{\mathbf{m}_\Lambda}(\bar{s}_J) = k''(\bar{s}_J)$.

By using (A.5) and an induction on the card of J, we can write in this case (cf. [R3]):

$$
\begin{aligned}
k^{(J)}_{\mathbf{m}_\Lambda}(\bar{s}_J) \ \leq \ \exp & \left[\sum_{j \in J} (-c_1 + 3\epsilon) ||\bar{s}_j||^2 \right] \times \\
& \times \sum_{q \geq P} \exp \left[-C''' \varpi_{q+1} v_{q+1} + D'' v_{q+1} + \delta \text{ card } (J \setminus [\, q+1 \,]) \,\right] \leq \\
& \leq \ \exp \left[\, - \sum_{j \in J} (c_1 - 3\epsilon) ||\bar{s}_j||^2 + \delta \text{ card } (J \setminus [\, P+1 \,]) \,\right] \leq \\
& \leq \ \exp \left[\, - \sum_{j \in J} (c_1 - 3\epsilon) ||\bar{s}_j||^2 + \delta \text{ card } J \,\right]
\end{aligned}
$$

which finishes the proof of Lemma 3.

It remains to check the bounds (A.4) and (A.5). The reasoning is again similar to [R3] (cf. Appendix in [R3]). We write

$$
\begin{aligned}
k'(\bar{s}_J) = & \Xi_\Lambda^{-1} \int_{\mathbf{S}^{\Lambda \setminus J}} ds_{\Lambda \setminus J} \chi_{\Re_0}(\bar{s}_J \vee s_{\Lambda \setminus J}) \ \exp \left[-V_\Lambda (\bar{s}_J \vee s_{\Lambda \setminus J}) \,\right] \leq \\
& \leq \exp \left[-V_J(\bar{s}_J) \right] \, \Xi_\Lambda^{-1} \int_{\mathbf{S}^{\Lambda \setminus J}} ds_{\Lambda \setminus J} \chi_{\Re_0}(\bar{s}_J \vee s_{\Lambda \setminus J}) \ \exp \left[-V_\Lambda (s'_J \vee s_{\Lambda \setminus J}) \right] \times \\
& \quad \times \exp \left[\frac{1}{2} (||\bar{s}_j||^2 + ||s'_j||^2) \bar{\Psi} + 2D' \right] .
\end{aligned}
\tag{A.8}
$$

Bound (A.8) follows from Proposition A.2 (b).

Now pick a subset $\mathbf{S}_0 = \{ s \in \mathbf{S} : ||s||^2 \leq 1 \}$, then for any finite $\widetilde{\Lambda} \subset \mathbf{Z}^\nu$

$$
\int_{\mathbf{S}_0^{\widetilde{\Lambda}}} ds_{\widetilde{\Lambda}} \exp \left[-V_{\widetilde{\Lambda}}(s_{\widetilde{\Lambda}}) \right] \geq \lambda^{-\text{card } \widetilde{\Lambda}},
\tag{A.9}
$$

where $\lambda > 0$ (see below). Then the RHS of (A.8) is

$$
\begin{aligned}
& \leq \lambda e^{2D'} \exp \left[-c_1 ||\bar{s}_j||^2 + c_2 + \frac{1}{2} \bar{\Psi} ||\bar{s}_j||^2 + \frac{1}{2} \bar{\Psi} \right] \times \\
& \quad \times \Xi_\Lambda^{-1} \int_{\mathbf{S}^{(\Lambda \setminus J) \cup \{j\}}} ds'_{(\Lambda \setminus J) \cup \{j\}} \exp \left[-V_\Lambda (\bar{s}_{J \setminus \{j\}} \vee s'_{(\Lambda \setminus J) \cup \{j\}}) \right] \leq \\
& \leq C'' \exp \left[(\bar{\Psi} - c_1) ||\bar{s}_j||^2 \right] \, k^{(J \setminus \{j\})}_{\mathbf{m}_\Lambda}(\bar{s}_{J \setminus \{j\}}),
\end{aligned}
\tag{A.10}
$$

which proves (A.4).

Furthermore,

$$k''(\bar{s}_J) = \sum_{q \geq P} \Xi_\Lambda^{-1} \int_{\mathbf{S}^{\Lambda\setminus J}} ds_{\Lambda\setminus J} \chi_{\Re_q}(\bar{s}_J \vee s_{\Lambda\setminus J}) \ \exp\left[-V_\Lambda(\bar{s}_J \vee s_{\Lambda\setminus J})\ \right] \leq$$

$$\leq \sum_{q \geq P} \Xi_\Lambda^{-1} \int_{\mathbf{S}^{\Lambda\setminus J}} ds_{\Lambda\setminus J} \chi_{\Re_q}(\bar{s}_J \vee s_{\Lambda\setminus J}) \ \exp\left[\sum_{j \in [\ q+1\ ]\cap\Lambda} (-c_1||\bar{s}_j||^2 + c_2)\right] \times$$

$$\times \exp\left[\sum_{j \in [\ q+1\ ]\cap\Lambda} \ \sum_{j' \in \Lambda\setminus[\ q+1\ ]} |\Psi_{\{j,j'\}}| \frac{1}{2}(||\bar{s}_j||^2 + ||\bar{s}_{j'}||^2)\right] \times$$

$$\times \exp\left[\sum_{j \in [\ q+1\ ]\cap\Lambda} \ \sum_{j' \in \Lambda\setminus[\ q+1\ ]} |\Psi_{\{j,j'\}}| \frac{1}{2}(||s'_j||^2 + ||\bar{s}_{j'}||^2)\right] \times$$

$$\times \exp\left[-V_\Lambda(\bar{s}_{J\setminus[\ q+1\ ]} \vee s_{(\Lambda\setminus J)\setminus[\ q+1\ ]} \vee s'_{[\ q+1\ ]\cap\Lambda}) + V_{[\ q+1\ ]\cap\Lambda}(s'_{[\ q+1\ ]\cap\Lambda})\ \right]; \qquad (A.11)$$

as before, bound (A.11) holds for any (sequence of) path configurations $s'_{[\ q+1\ ]\cap\Lambda} \in \mathbf{S}^{[\ q+1\ ]\cap\Lambda}$, $q \geq P$. By using Proposition A.2 (a), we continue (A.11) by estimating the RHS as

$$\leq \sum_{q \geq P} \exp\left[\sum_{j \in [\ q+1\ ]\cap J} (-c_1 + 3\epsilon)||\bar{s}_j||^2 - C'\varpi_{q+1}v_{q+1}\right] \times$$

$$\times \left[\lambda \ \exp\left(\frac{1}{2}\bar{\Psi}\right)\right]^{\mathrm{card}\ ([\ q+1\ ]\cap\Lambda)} \times$$

$$\times \Xi_\Lambda^{-1} \int_{\mathbf{S}^{\Lambda\setminus(J\setminus[\ q+1\ ])}} ds'_{\Lambda\setminus(J\setminus[\ q+1\ ])} \exp\left[-V_\Lambda(\bar{s}_{J\setminus[\ q+1\ ]} \vee s'_{\Lambda\setminus(J\setminus[\ q+1\ ])})\right] \leq$$

$$\leq \sum_{q \geq P} \exp\left[\sum_{j \in [\ q+1\ ]\cap J} (-c_1 + 3\epsilon)||\bar{s}_j||^2 - C'\varpi_{q+1}v_{q+1} + D''v_{q+1}\right] \times$$

$$\times\ k_{\mathbf{m}_\Lambda}^{(J\setminus[\ q+1\ ])}(\bar{s}_{J\setminus[\ q+1\ ]}), \qquad (A.12)$$

which proves (A.5).

To prove (A.9), we observe (as in [R3]) that

$$V_{\widetilde{\Lambda}}(\bar{s}_{\widetilde{\Lambda}}) \leq \sum_{j \in \widetilde{\Lambda}} \varphi_j(\bar{s}_j) + \bar{\Psi} \sum_{j \in \widetilde{\Lambda}} ||\bar{s}_j||^2$$

and therefore

$$\int_{\mathbf{S}_0^{\widetilde{\Lambda}}} ds_{\widetilde{\Lambda}} \exp\left[-V_{\widetilde{\Lambda}}(s_{\widetilde{\Lambda}})\right] \geq$$

$$\leq \prod_{j \in \widetilde{\Lambda}} \int_{\mathbf{S}_0} ds \exp\left[-\varphi_j(s) - \bar{\Psi}||s||^2\right] \geq$$

$$\geq \lambda^{\mathrm{card}\ \widetilde{\Lambda}}$$

where

$$\lambda = \inf_j \int_{\mathbf{S}_0} ds \exp\left[-\varphi_j(s) - \bar{\Psi}||s||^2 > 0;\right.$$

the last estimate follows directly from the properties of the Wiener measure and the conditions imposed on potentials Φ_j.

The proof of the assertion of Lemma 3 for other types of measures does not differ from the case of the measure with the denominator Ξ_Λ. In fact what matters is the system of bounds (1.7), (1.64) and (1.65). The extension of the proof provided to other cases is immediate and we leave it to the reader.

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