Coulomb Interaction Symmetries and the Mayer Series in the Two-Dimensional Dipole Gas

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Abstract

We study the Mayer series of the two-dimensional dipole gas in the high temperature-low density regime. Without performing any multiscale analysis, we obtain bounds showing that the Mayer coefficients are finite in the thermodynamic limit. These bounds are obtained by exploiting a particular partial symmetry of the interaction (which we name *θ-symmetry*), already used in some problems related to the two-dimensional Coulomb gas. By direct bounds on some Mayer graphs we also conjecture that any technique based uniquely on the θ-symmetry will not be sufficient to prove analyticity of the series.

Key words: Coulomb interaction symmetries; Mayer series; two-dimensional dipole gas

1 Introduction

The dipole gas and other gases of particles interacting through Coulomb forces are statistical systems studied for a long time. In particular, for the dipole gas, the lack of screening is well known [9], and the analyticity of the pressure in the high temperature and low activity region has been shown, in an indirect way, by mapping the system onto a field theory (sine-Gordon transformation) and using renormalization group methods (see [7] for the $d \geq 3$ lattice model, and [1] for the $d \geq 1$ continuum one). A direct proof of the analyticity of the pressure (*e.g.* for the two-dimensional case) based on bounds for the coefficients of the Mayer series is still an open problem. The close relationship between this model and the Coulomb gas in the Kosterlitz-Thouless phase ($\beta > 8\pi$), allied to the non-existence of any proof for the analyticity of the pressure in the Coulomb gas (indirect arguments are presented in [4], [2], [8]) makes this problem even more interesting.

The Mayer series for the case of the Coulomb gas with $\beta > 8\pi$ has already been studied directly in [5] and later in [10]. Both works use multiscale analysis and exploit one certain symmetry of the model in order to prove that the coefficients of the series are finite in the thermodynamic limit.

In the present paper, having in mind to understand the role played by symmetries in systems with Coulomb interactions, we study the two-dimensional dipole gas. Exploiting the same symmetry (already used in [5] and [10]), we prove, *without recourse to multiscale analysis*, finiteness of the Mayer series coefficients in the thermodynamic limit. We argue, however, that a richer symmetry structure (*i.e.* the total symmetry of the dipole gas) may be necessary to prove directly the analyticity of the pressure.

We introduce now the model, its properties, and make more precise some facts stated above. Throughout this article generic constants are denoted by C.

The dipole gas is a system of classical particles in a two-dimensional box Λ carrying vectorial moments q of unit length ($q \in \mathbf{R}^2, q^2 = 1$), interacting via the potential

$$V(\xi_1,\xi_2) \;=\; (q_1\cdot\partial_{x_1})(q_2\cdot\partial_{x_2})\,W(x_1-x_2) \;=\; \int \frac{dp}{(2\pi)^2}\,\frac{(q_1\cdot p)(q_2\cdot p)}{p^2}\,e^{-p^2}\,e^{ip\cdot(x_1-x_2)}\;, \quad (1.1)$$

where $\xi = (x,q)$, $x \in \Lambda \subset \mathbf{R}^2$ is the particle position, $q \in S^1$ its dipole moment and W the Coulomb potential

$$W(x_1-x_2) \;=\; \int \frac{dp}{(2\pi)^2}\,[e^{ip\cdot(x_1-x_2)}-1]\,\frac{e^{-p^2}}{p^2} \;=\; \frac{1}{4\pi}\int_1^\infty \frac{d\alpha}{\alpha}\,[e^{-(x_1-x_2)^2/4\alpha}-1]\;. \quad (1.2)$$

We are using in (1.2) the same UV cutoff of Ref. [5]. Introducing polar coordinates, (1.1) becomes

$$\begin{aligned} V(\xi_1,\xi_2) \;&=\; V(x_1-x_2,\theta_1,\theta_2) \quad (1.3)\\ &=\; \frac{1}{8\pi}\left[\cos(\theta_1-\theta_2)\,e^{-\frac{(x_1-x_2)^2}{4}} \;+\; \cos(\theta_1+\theta_2-2\phi_{12})f(x_1-x_2)\right], \end{aligned}$$

where θ_i is the angle between q_i and some fixed axis, ϕ_{12} is the angle between x_1-x_2 and the same axis and

$$f(x_1-x_2) \;=\; e^{-\frac{(x_1-x_2)^2}{4}} \;-\; \frac{1-e^{-\frac{(x_1-x_2)^2}{4}}}{(x_1-x_2)^2/4}\;. \quad (1.4)$$

Note that $V(\xi_1,\xi_2) \sim O(1/|x_1-x_2|^2)$ as $|x_1-x_2| \to \infty$, and also that $V(\xi,\xi) = \frac{1}{8\pi}$.

There are several symmetries in $V_{ij} \equiv V(\xi_i,\xi_j)$. For example, $V(x_i-x_j,\theta_i+\pi,\theta_j) = -V(x_i-x_j,\theta_i,\theta_j)$, which we name the *$\theta$-symmetry*, is the one used later to prove finiteness of the Mayer series coefficients. It implies that the interaction averaged over dipole orientations vanishes. A richer symmetry involving also ϕ_{ij}, the angular variable of x_i-x_j, makes finite the integral of the potential over a sphere: $\int dx_j V_{ij} < \infty$; the potential however is not absolutely integrable (*i.e.* it is not L_1). Note that the second term in (1.3), responsible for the slow decay of V (*i.e.* the delicate term), depends on $\cos(\theta_i+\theta_j-2\phi_{ij})$ with a rich symmetry.

Note also that V is a stable potential, which may be verified by using (1.1) and checking that $\sum_{i=1}^N\sum_{j=1}^N V_{ij} \geq 0$, $\forall N, \xi_1,\ldots,\xi_N$.

The partition function for the dipole gas in the grand canonical ensemble is

$$Z_\Lambda(\beta,\lambda) \;=\; \sum_{N=0}^{\infty}\frac{\lambda^N}{N!}\int d\xi_1\ldots\int d\xi_N\,\exp[-\beta\sum_{i<j}V_{ij}]\;, \quad (1.5)$$

where $\int d\xi_i = \int_0^{2\pi} d\theta_i \int_\Lambda dx_i$, λ is the activity and β the inverse temperature. The pressure is the thermodynamic limit $|\Lambda| \to \infty$ of

$$\beta p_\Lambda(\lambda, \beta) \;=\; \frac{1}{|\Lambda|} \log Z_\Lambda(\beta, \lambda) \;. \tag{1.6}$$

The Mayer series for the pressure is a formal expansion in powers of λ obtained by writing $\exp[-\beta V_{ij}] = [\exp(-\beta V_{ij}) - 1] + 1$, then expanding $\prod_{i<j} \exp(-\beta V_{ij})$ as a sum of terms labeled by Mayer graphs, and taking the logarithm (which eliminates the unconnected graphs) [6]. We have

$$\beta p_\Lambda(\lambda, \beta) \;=\; \sum_{N=1}^{\infty} C_{N,\Lambda}(\beta) \lambda^N \;, \tag{1.7}$$

with

$$C_{N,\Lambda}(\beta) \;=\; \frac{1}{N!|\Lambda|} \int d\xi_1 \ldots d\xi_N \sum_{G \in \mathcal{G}_N} \prod_{(i,j) \in G} [e^{-\beta V_{ij}} - 1]. \tag{1.8}$$

where $\mathcal{G}_N$ is the set of all connected Mayer graphs between N points.

A direct proof for the analyticity of the pressure as a function of λ in the thermodynamic limit means to find a bound $|C_{N,\Lambda}(\beta)| < [C(\beta)]^N$ uniform in Λ. As V_{ij} is not L_1, standard cluster expansion arguments fail [6], and even the proof of finiteness of the coefficients becomes non-trivial.

Finiteness of the Mayer series coefficients in the two-dimensional Coulomb gas above 8π has been directly proved in [5] and [10], as mentioned. A multiscale analysis is performed there, describing the gas as a system of clusters of charges hierarchically organized, and a symmetry property is used to solve problems in the thermodynamic limit arising in the analysis of the interaction between neutral clusters of charges (dipoles, basically). Precisely, in [5] a necessary gain in the decay of the interaction between neutral clusters is obtained by summing two neutral clusters with the charge distribution reversed (*i.e.* flipping dipoles), see figure 1.

Figure 1

In [10] the same gain is obtained by integrating over θ (*i.e.* rotating dipoles), see figure 2.

Figure 2

Notice that both procedures essentially involve the use of what we called the θ-symmetry of the dipole interaction.

In the next section, we use the θ-symmetry to prove that the coefficients of the Mayer series for the pressure of the bidimensional dipole gas are finite in the thermodynamic limit. Section 3 is devoted to arguing that the full symmetry of the dipole gas may be necessary in order to prove analyticity of the pressure. In section 4 we make final comments.

2 Finiteness of the Mayer series coefficients

As mentioned before, since the dipole potential (1.1) is not L_1, it is non-trivial to show that the N^{th}-order Mayer coefficient for the pressure, given by 1.8, is finite in the thermodynamic limit. Recall that it is expressed as a sum of terms labeled by connected Mayer graphs on N points, namely:

$$C_{N,\Lambda}(\beta) = \frac{1}{N!} \sum_{G \in \mathcal{G}_N} C^G_{N,\Lambda}(\beta) \ , \tag{2.1}$$

where

$$C^G_{N,\Lambda}(\beta) = \frac{1}{|\Lambda|} \int_\Lambda dx_1 \cdots \int_\Lambda dx_N \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_N \prod_{(i,j)\in G} [e^{-\beta\sigma_i\sigma_j V_{ij}} - 1] \ . \tag{2.2}$$

Therefore, we show now, by using only the θ-symmetry of the model, that we may obtain bounds for the thermodynamic limit $|\Lambda| \to \infty$ of $C^G_{N,\Lambda}(\beta)$.

Our results are stated as follows

Proposition 1 *Let $V_{ij} = V(x_i - x_j, \theta_i, \theta_j)$ be a two-dimensional potential, with $x_i - x_j \in \mathbf{R}^2$, $\theta_i, \theta_j \in [0, 2\pi]$, and such that*

1) $|V(x_i - x_j, \theta_i, \theta_j)| \leq C/[1 + (x_i - x_j)^2]$,

2) $V(x_i - x_j, \theta_i + \pi, \theta_j) = V(x_i - x_j, \theta_i, \theta_j + \pi) = -V(x_i - x_j, \theta_i, \theta_j)$.

Then, for all $G \in \mathcal{G}_N$, $C^G_{N,\Lambda}(\beta)$ as in (2.2) admits a bound uniform in Λ.

To establish the proposition, besides an adequate use of the θ-symmetry (property 2 above) we need the following lemma (whose proof is obtained following the demonstration of lemma 2, next section).

Lemma 1 *Let Σ be the ball of radius R centered in the origin (then $|\Sigma| = \pi R^2$). Let V satisfy part 1 of the proposition above. Then, $\forall x, y \in \mathbf{R}^2$*

$$\lim_{R\to\infty} \left\{ \int_\Sigma dx_1 \ldots \int_\Sigma dx_N |V(x-x_1)||V(x_1-x_2)| \ldots |V(x_{N-1}-x_N)||V(x_N-y)| \right\} < C_N < \infty \ ,$$

i.e., $\underbrace{|V| * \ldots * |V|}_{N+1}(x-y)$ (the $N+1$-fold convolution) is finite in the thermodynamic limit.

Note that the lemma alone takes care of all Mayer graphs which cannot be separated in two subgraphs by cutting a single line (this kind of graph is termed *one-particle irreducible*, Ref.[6]). In fact, recall that each line (i, j) in the graph represents a function $v_{ij} = \exp(-\beta V_{ij}) - 1 \approx \beta V_{ij}$ satisfying the hypothesis of the proposition. Thus we may apply repetitively the lemma in order to bound these graphs. For example, the first graph in figure 3 is bounded by $C|v| * |v| * |v|(0)$, whereas the second by noting that x_7 may be integrated out, $\int dx_7 |v(x_1 - x_7)v(x_7 - x_4)| = |v| * |v|(x_1 - x_4) < C$, and that the loop 1-2-3-4-5-6-1 is bounded as in graph 1.

Figure 3

Clearly there are graphs which are not controlled by lemma 1, *e.g.*

Figure 4

These graphs are precisely the ones which separate in two disconnected components by cutting just one line, *i.e.* the one-particle *reducible* ones. The special lines which disconnect the graph when cut are called *tree lines.* In order to control these graphs we make explicit the θ-symmetry by introducing new variables (the charges) $\sigma_i = \pm 1$ and noting that for any function F of the potential V satisfying condition 2 of the proposition we may write

$$\int_0^{2\pi}\int_0^{2\pi} F[V(x_i-x_j,\theta_i,\theta_j)]\; d\theta_i d\theta_j = \sum_{\sigma_i,\sigma_j=\pm 1}\int_0^{\pi}\int_0^{\pi} F[\sigma_i\sigma_j V(x_i-x_j,\theta_i,\theta_j)]\; d\theta_i d\theta_j \; .$$

Hence, for any Mayer graph G, (2.2) becomes

$$C^G_{N,\Lambda}(\beta) = \frac{1}{|\Lambda|}\sum_{\sigma_1,\cdots,\sigma_N=\pm 1}\int_\Lambda dx_1\cdots\int_\Lambda dx_N \int_0^\pi d\theta_1\cdots\int_0^\pi d\theta_N \prod_{(i,j)\in G}[e^{-\beta\sigma_i\sigma_j V_{ij}}-1]\; . \quad (2.3)$$

We proceed by writing $\exp(-\beta\sigma_i\sigma_j V_{ij}) - 1 = -\sigma_i\sigma_j\sinh(\beta V_{ij}) + \cosh(\beta V_{ij}) - 1$, graphically described as the split

Figure 5

The name "soft" comes from the fact that $\sinh(\beta V_{ij})$ is of order βV_{ij} (β and V_{ij} small), hence its volume integration is delicate; and hard because $\cosh(\beta V_{ij}) - 1$ is of order $(\beta V_{ij})^2$, a L_1 function.

With the split above, the graph G (and so $C^G_{N,\Lambda}(\beta)$ in (2.2)) becomes the sum of $2^{|G|}$ terms ($|G|$ is the number of links in G); each of these terms being represented by a graph G' with the same topological structure of G (*i.e.*, G' has a line between a pair (i,j) whenever G has this line), but now the links of G' are either soft or hard as in figure 5. The key point is that these new graphs G' have associated charges: each soft link reaching the vertex i carries a charge σ_i. When we sum over the charges, it follows that all graphs G' with an odd number of soft links reaching any vertex i vanish (since $\sum_{\sigma_i=\pm 1}(\sigma_i)^k = 0$ if k is odd). It is not difficult to see that only those G' survive in which the soft lines appear in one-particle irreducible subgraphs. In other words, any tree line in G' must be hard.

For example, among the graphs of figure 4 the only surviving G' graphs are

Figure 6

By using lemma 1, we may integrate safely soft-line subgraphs, the remaining hard links being easily taken into account by the fact that they represent an integrable factor. It follows that any G' is uniformly bounded in Λ, and hence G is uniformly bounded in Λ, proving the proposition.

Remark:

Although the splitting of figure 5 leads to a proliferation of terms, *i.e.* a single graph G proliferates into $2^{|G|}$ new terms G', it can be shown that, by using the θ-symmetry, it is possible to rewrite $C^G_{N,\Lambda}$ of (2.2) as one single term

$$C^G_{N,\Lambda}(\beta) = \frac{1}{|\Lambda|} \int_\Lambda dx_1 \cdots \int_\Lambda dx_N \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_N \prod_{(i,j)\in G} B_{ij} \; , \tag{2.4}$$

where

$$B_{ij} = \begin{cases} \cosh(\beta V_{ij}) - 1 & \text{if } (i,j) \text{ is a tree line} \\ e^{-\beta V_{ij}} - 1 & \text{otherwise} \end{cases}$$

Graphically, this means that in each graph G we can replace $e^{-\beta V_{ij}} - 1$ by $\cosh(\beta V_{ij}) - 1$ whenever (i,j) is a tree-line without affecting the value $C^G_{N,\Lambda}$, *e.g.*

Figure 7

Actually this symmetry seems to have an even greater range of applicability. For example, the second graph 2 in figure 3, already controlled in the thermodynamic limit by lemma 1 alone, has its bound improved by noting that vertices 1 and 4 cannot have three soft links (at least one must be hard).

3 The symmetries and the analyticity problem

As we have seen in the previous section, just considering the θ-symmetry present in the dipole gas potential we were able to show that the series coefficents remain finite in the thermodynamic limit. We point out again that this symmetry (toghether with a multiscale formalism) has been the mechanism used to prove also the existence of the formal Mayer series of the Coulomb gas above 8π [5], [10]. We believe however that using only the "θ-symmetry" we will not be able to show the analyticity of the dipole gas pressure: in this section we present some arguments which lead us to this conjecture.

To make explicit the extent of each symmetry, we compare Mayer graphs of two different potentials: the dipole gas potential (1.3) and another one with a similar decay ($1/x^2$ as $x \to \infty$), but containing only the "θ-symmetry". We take $W_{ij}^{(\alpha)} = \sigma_i \sigma_j V_{ij}^{(\alpha)}$, $\alpha = 1, 2$ with

$$V_{ij}^{(1)} = V(x_i - x_j, \theta_i, \theta_j) \text{ as in (1.3)} \tag{3.1}$$

and

$$V_{ij}^{(2)} = \frac{1}{1 + |x_i - x_j|^2}, \tag{3.2}$$

where $\sigma_i, \sigma_j = \pm 1$ as already defined; $x_i, x_j \in \Lambda$; $\theta_i, \theta_j \in [0, \pi]$. Remark that any technique using only the θ-symmetry means to take in account $|V^{(1)}|$ (instead of $V^{(1)}$) when bounding the integrals related to the Mayer graphs. The effect of the modulus is to turn off the angular symmetries of the dipole potential (apart from the θ-symmetry). Thus, actually, we should compare $V^{(1)}$ with $|V^{(1)}|$. However, we take $V^{(2)}$ above for sake of simplicity: it leads to simpler calculations (details ahead) and essentially behaves as $|V^{(1)}|$. More precisely. lemma 3 below clearly remains true for $|V^{(1)}|$ replacing $V^{(2)}$.

As described in the previous section, due to the sum over charges σ_i (*i.e.* the "θ-symmetry"), we expect Mayer series with finite coefficents for both potentials. In order to show analyticity of the pressure we must bound by C^N the Mayer coefficents, which are given by a sum of several positive and negative graphs. The control of these signs, related to the potential stability, will not be carried out here. We will only examine the behaviour of certain graphs with N to show that for the dipole potential $W_{ij}^{(1)}$ (with the full symmetry) they are bounded above by C^N, whereas for the simplified potential $W_{ij}^{(2)}$ these graphs have $N!C^N$ as a lower bound.

Our claim is stated in the following lemma (whose proof essentially works also for lemma 1, comments ahead).

Lemma 2 *Let G be the loop graph below with N lines and N vertices, and Λ be the ball of radius R centered in the origin (then $|\Sigma| = \pi R^2$).*

Figure 8

Then we have for $\mathcal{G}_N^{(\alpha)} \equiv \lim_{R\to\infty} \frac{2^N}{|\Sigma|}\int d\xi_1 \cdots \int d\xi_N \prod_{(i,j)\in G} V_{ij}^{(\alpha)}$, $\alpha = 1,2$; $V^{(1)}$, $V^{(2)}$ *as defined above,*

1) $|\mathcal{G}_N^{(1)}| \leq C^N$,

2) $|\mathcal{G}_N^{(2)}| \geq N! C^N$.

Proof:

Part 2: We note that $V^{(2)}(x) = 1/(1+x^2)$ is in $L^2(\mathbf{R}^2)$, and its its Fourier tranform $\tilde{V}^{(2)}(p)$ is asymptotically given by [6],

$$
\begin{aligned}
\tilde{V}^{(2)}(p) &\approx -\frac{1}{2\pi}\ln|p|, \quad \text{small } |p| \\
\tilde{V}^{(2)}(p) &\approx \left(\frac{1}{8\pi|p|}\right)^{1/2} \exp(-|p|), \quad \text{large } |p|
\end{aligned}
\tag{3.3}
$$

It is easy to check that $[\tilde{V}^{(2)}(p)]^N$ is also in $L^2(\mathbf{R}^2)$. ¿From the Parseval theorem for Fourier transform we have

$$
\int [\tilde{V}^{(2)}(p)]^2 e^{ip\cdot x} dp = \int V^{(2)}(y) V^{(2)}(x-y) dy \ ,
$$

and iterating it, we have

$$
\underbrace{V^{(2)} * V^{(2)} * \cdots V^{(2)}}_{N \text{ times}}(x) = \int [\tilde{V}^{(2)}(p)]^N e^{ip\cdot x} dp \tag{3.4}
$$

from which it follows that the convolution $V^{(2)} * V^{(2)} * \cdots V^{(2)}$ is in $L^2(\mathbf{R}^2)$. Using the dominated convergence theorem, $\mathcal{G}_N^{(2)}$ is given explicitly by

$$
\begin{aligned}
\mathcal{G}_N^{(2)} &= \lim_{R\to\infty} \frac{2\pi^N}{|\Lambda|} \int dx_1 \cdots dx_N \frac{1}{1+(x_1-x_2)^2} \frac{1}{1+(x_2-x_3)^2} \cdots \\
&\times \frac{1}{1+(x_{N-1}-x_N)^2} \frac{1}{1+(x_N-x_1)^2} = \underbrace{V^{(2)} * V^{(2)} * \cdots V^{(2)}}_{N \ times}(0) \ ,
\end{aligned}
\tag{3.5}
$$

and thus we have, for $x = 0$, $\mathcal{G}_N^{(2)} < \infty$. Note that with (3.4) we also prove lemma 1 . And from (3.3) above

$$
\begin{aligned}
\mathcal{G}_N^{(2)} &= \int_0^\infty \rho\, d\rho \int_0^{2\pi} d\varphi\, [V^{(2)}(\rho\cos\varphi, \rho\sin\varphi)]^N \\
&\geq C^N \int_0^1 d\rho\, \rho |\ln\rho|^N \geq C^N N!,
\end{aligned}
\tag{3.6}
$$

proving part 2.

Part 1 follows in a similar manner using

$$V^{(1)}(\theta_i, \theta_j, x_i - x_j) = \int dp \, \frac{(d_i \cdot p)(d_j \cdot p)}{p^2} \; e^{-p^2} \; e^{ip\cdot(x_i - x_j)} \tag{3.7}$$

where d_i, d_j are unit vectors describing the dipoles at x_i and x_j ($d_i \cdot e_1 = \cos\theta_i$) and noting that $\tilde{V}^{(1)} = \frac{(d_i \cdot p)(d_j \cdot p)}{p^2} \; e^{-p^2}$ is bounded in every neighborhood of $p = 0$.

We stress, once more, that $W^{(2)}$ is "morally" $W^{(1)}$ with the θ-symmetry only, i.e. $W^{(1)}$ with the other angular symmetries turned off; moreover the comparison between the dipole potential $W^{(1)}$ and the potential $W^{(2)}$ makes sense only in two dimensions. The expression (3.2) describes a potential with the θ-symmetry and with the *same* decay of the dipole gas only for $d = 2$. In one dimension, for example, the analogous dipole gas (3) would be a gaussian (i.e. a smooth function exponentially decaying at large distances) which is the smooth U.V. cut-off version of the delta funtion, the one dimensional dipole potential. This potential has a poorer symmetry (comparing with the two-dimensional case), but it is obviously absolutely integrable and so the Mayer series trivially converges.

4 Final Comments

Another argument for the necessity of using more than the θ-symmetry to prove the analyticity of the pressure appears when we turn to the well-known proof based on renormalization group techniques.

With the sine-Gordon transformation mapping the statistical system on a field theory problem, the dipole partition function becomes

$$\mathcal{Z}_\Lambda = \int P(d\phi) \exp\left\{\lambda \int_\Lambda dx \int_0^{2\pi} d\theta \, \exp[i\beta^{1/2}(q \cdot \partial_x)\phi(x)]\right\}$$

where, roughly, $P(d\phi)$ is a measure on ϕ with inverse Laplacian as covariance (see [1] or [5] for details). Due to the θ-symmetry we have $q(\theta + \pi) = -q(\theta)$, and so

$$\mathcal{Z}_\Lambda = \int P(d\phi) \exp\left\{2\lambda \int_\Lambda dx \int_0^{\pi} d\theta \, \cos[\beta^{1/2}(q \cdot \partial_x)\phi(x)]\right\}.$$

That is, the θ-symmetry is responsible for the even function cosine in the fields potential. It is important but not the main point: in [GK], for example, $\cos(\partial\phi)$ and $(\partial\phi)^4$ are treated at the same time as a class of functions leading to the same infrared behavior. The crucial fact in the field theory problem is that the action is given by Laplacian plus a perturbation depending only on $\partial\phi$ (which is related to an interaction such $\partial_i\partial_j/\partial^2$ as in (1.1)). Dependence only on $\partial\phi$ is due to the full symmetry and leads to a renormalization group with only marginal and irrelevant terms.

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