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Irreversible weak limits of classical dynamical systems

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Abstract. Limits of classical dynamical systems depending on a parameter are considered, leading in general to irreversible time evolutions and stochastic maps. The appearence of irreversibility is characterized in terms of convergence properties of the corresponding automorphisms of the observable algebra. The resulting maps are shown to be automatically "bistochastic". Necessary and sufficient conditions are given for the limits to define a time independent Markov process. A model is studied for a particle interacting with fixed configurations of external obstacles.

KEY WORDS: irreversibility, weak limits, classical dynamical systems.

1. Introduction

Irreversible time evolution plays a major and generic rôle in the phenomenological description of macroscopic systems. Its derivation from underlying reversible dynamical systems is however under control only in special cases [1] [2], and it is not clear whether one may characterize a unique set of conditions leading to irreversible behaviour, or one should rather think in terms of different schemes and limit procedures, applying to different physical situations [3] [4] [5] [6].

Since the classical work of Boltzmann, it is in fact well known that the presence of a large (infinite) number of degrees of freedom plays a crucial rôle for the appearence of irreversible behaviour, and that other conditions are necessary, involving at least dynamical instabilities which grow suitably with the number of degrees of freedom, toghether with restrictions on the states appearing “at time zero” [7] [8] [9] [10] [11].

Moreover, a distinction between “macroscopic” and “microscopic” observables plays an important rôle in the general discussion, and statistical arguments enter in an essential way, either through a classification of the states in terms of macroscopic and microscopic variables [12][5], or in their very description [1][2][10][11].

The restriction to a suitable subset of observables is also at the basis of the discussion of irreversible behaviour for a fixed subsystem of an infinite system, and an irreversible time evolution has in fact been derived in general by Davies in the “weak coupling”, or “diffusive” limit, for finite subsystems of infinite quantum systems in pure states [13]. The resulting scheme is, however, only indirectly related to the one advocated in the Boltzmann–Grad limit, and, at least, the problem arises of identifying common structures behind different constructions and limit procedures.

Since in all cases irreversible behaviour is obtained as a limit description in the variation of some parameter, it is worthwhile to investigate whether this ingredient may also be sufficient, i.e. whether *irreversible time evolution may be the generic result of taking such limits.* The rôle of the other conditions would then be interpreted as related to the description of different classes of physical situations, in which different parameters and limits become relevant, in relation with specific structures.

In models, the above mentioned parameters appear in the description of the structure of the system (the number of particles in the Boltzmann–Grad case), of the states (the correlation functions at time zero) and of the dynamics (the rescaling of the particle radius); however, since dynamical instabilities seem to be an essential ingredient if irreversibility, it is reasonable to concentrate the attention on parameters which enter in the definition of the time evolution, leaving for the moment the system and its states invariant. Such a possibility is not far from the situation in the Boltzmann–Grad limit since, in the spatially homogeneous case, that limit can be formulated in terms of an infinite system of hard spheres, in a fixed time zero configuration, in terms of a suitable rescaling of the radius ($r \to \lambda r$) and of the time scale ($t \to \lambda^{-2} t$ in three space dimensions).

We are therefore led to investigate limits of sequences of reversible dynamical systems. More precisely, on the basis of the general interpretation of dynamical systems in terms of observables and states, the most general relevant limit is the limit for the result of measurements of fixed observables, given any state in some relevant class.

Mathematically, in the Hamiltonian case, observables may be identified with regular functions on the phase space, states with measures on the phase space, and the problem is to take limits of expectation values of observables over states evolved in time under groups of transformations defined by Hamiltonians depending on a parameter.

More generally, i.e. for quantum systems and infinite systems, observables may be identified (see e.g. [14]) with the hermitean elements of a C* algebra, states with positive linear functionals, time evolution with a group of automorphisms of the observable algebra, and the relevant limit is the weak limit of the time evolution automorphisms, taken with respect to a subspace of the state space, interpreted as the set of states of interest.

In this form, the problem is similar to the one that appears in the framework of the algebraic formulation of Quantum Field Theory and Statistical Mechanics [15], in relation with the construction of the time evolution of infinite quantum systems in presence of long range interactions. A general analysis of limits of automorphisms arising in that context has been developed has in fact been developed in refs. [16] [17] [18] [19].

The relatively strong assumptions which had to be required in that analysis in order that the limit still give rise to a group of automorphisms, i.e. to a reversible dynamical system, leave open the possibility that the result of such limits may be interpreted, in some generality, in terms of irreversible time evolution.

The purpose of the present paper is to show that this is the case, i.e. that maps having all the properties usually assumed to characterize irreversible time evolution appear quite in general as weak limits of reversible dynamical systems, on the basis of only two ingredients:

i) existence of a limit in a parameter, which describes *growing dynamical instability*

ii) statistical *regularity* of the states involved in the limit procedure.

The first requirement seems to be essential in order to obtain the loss of information which is implicit in irreversible evolution. The second one has a clear mathematical rôle in defining the kind of limit which is taken, and can be physically interpreted (see below) in terms of the statistical description of the states of finite subsystems arising, through a space ergodic mean, from a pure state of an infinite system.

It must be stressed that the present scheme for the origin of irreversibility is *not* incompatible with the usually advocated ones, and that our purpose is rather to focus the attention on an abstract mechanism which *by itself* gives rise to irreversibility *in a generic way*, and may therefore be a good candidate to explain the generality in which irreversible behaviour appears in the description of nature.

In particular, the fact that irreversible behaviour may appear also in finite systems does not contradict the idea that irreversibility comes from the loss of information "at infinity" in infinite systems. First, it is clear that finite systems *with a continuous phase space* have room enough, in the small, in order to let information become "inaccessible on a given scale", and this can be interpreted as a generalization of the Boltzmann mechanism. More important, the application of our approach to relevant infinite systems should yield irreversible behaviour exactly when the range of the interactions (or the change of the time scale) is made to grow fast enough in order to give rise to "loss of information at infinity"; this follows from the fact that the condition that the limit still give an automorphism of a ("quasi–local") observable algebra has in general the interpretation of allowing for a sufficiently local description of the dynamics, i.e. of excluding loss of information at infinity (see Sect. 2 for the relation between automorphism property and reversible interpretation); the rôle of such condition is particularly clear in quantum spin systems, where the phase space can be thought as locally discrete, and time evolution remains in fact reversible if the range of the interaction remains suitably bounded [16].

¿From a general mathematical point of view, the essential point of our results is that all the features that seem to characterize the description of irreversible time evolution (see Sect.2), and are sometimes even *assumed as an independent source of information* on the behaviour of physical systems [20], arise *generically*, from reversible dynamical systems, as a consequence of taking limits in some parameter which enters in the dynamics.

We restrict our attention in this paper to classical dynamical systems, under assumptions which are typically satisfied in Hamiltonian models with a finite number of degrees of freedom. In this sense, our results do not directly apply to infinite systems in general. However, if a finite system is interpreted as a (non-interacting) subsystem of an infinite system, the latter being obtained, e.g., by taking copies of the system, indexed by an integer, then any statistical state, i.e. any probability measure μ on its state space, defines a product measure ρ on the state space of the infinite system, which is ergodic under translations of the parameter (see e.g. [21]). The original state is therefore automatically interpreted in terms of the statistics defined by the ergodic mean with respect to the integer parameter indexing the subsystems, for ρ–almost all fixed configuration of the infinite system (see [22], [5]). Our results can therefore be interpreted as holding in a class of infinite systems, for which

i) the presence of an infinite number of subsystems is only used as the basis for a statistical description of the states of each of them;

ii) the origin of growing dynamical instability is separated from the increasing of the number of components, and is assumed to be obtained from the variation of a parameter in a finite dynamical system.

No restriction on the observables of the system is involved in our analysis, even if a relation with the notion of "macroscopic" observables may be regarded as implicit in the interpretation of the statistical description in terms of the subalgebra of observables generated by the ergodic means over a parameter indexing the subsystem of an extended system.

In Sect.2 we make our framework more precise and give the main results; Sect.3 contains the proofs; in Sect.4 a model is discussed, bearing some similarity to the situation arising in the Boltzmann–Grad limit, giving rise to an irreversible time evolution, and reproducing the general structures characterized in Sect.2.

2. Assumptions and results

We consider classical dynamical systems, defined as in [23] by groups of continuous transformations Γ_t, $t \in \mathbb{R}$ of a locally compact Hausdorff topological space X. We will discuss continuous transformations Γ_n^t depending on a parameter $n \in \mathbb{N}$, and their limits as $n \to \infty$.

The observables of the system are identified with the C^* algebra of the continuous functions on X vanishing at infinity, $C^0(X)$, i.e., the completion, in the sup norm, of $C_0^0(X)$, the space of continuous functions of compact support. The transformations Γ_n^t define as usual groups of automorphism α_n^t of $C^0(X)$, $\alpha_n^t f(x) = f(\Gamma_n^{-t}(x))$.

Conversely, by Gelfand's isomorphism [24], any group of automorphisms of a commutative C^* algebra defines a group of continuous transformation of its spectrum, a locally compact Hausdorff space (the set of multiplicative linear functionals being clearly left invariant by the transpose of an automorphism).

The states of the system are the positive normalized linear functional ω on $C^0(X)$, i.e., by the Riesz–Markov theorem, the regular Borel probability measures on X.

The only restrictions on α_n^t that we will assume are the existence of a common invariant measure, and a condition which keeps the time evolution "far from infinity" uniformly in n; more specifically, we will assume

M) a σ finite regular Borel measure μ is defined on X, and left invariant by all Γ_n^t, $n \in \mathbb{N}$.

F) for all real t and compact set $K \subset X$ there is a compact set $M(K,t) \subset X$ such that

$$\Gamma_n^t(K) \subset M \qquad\qquad \forall n \in \mathbb{N}$$

Given the invariant measure μ, a distinguished class of states is given by the set of probability measures ν which are continuous with respect to μ: $\nu(x) = h(x)\mu(x)$, $h(x) \in L^1(X, d\mu)$. The set $\mathcal{F}$ of such states, which will be called "regular", is the positive part of a closed subspace of the dual of $C^0(X)$ [19], and defines a "full folium" of states in the sense of [25]. Regular states will be identified in the following with functions in $L^1(X, d\mu)$.

The invariance of $d\mu$ implies that $\mathcal{F}$ is left invariant by the transpose $\alpha_n^{t\,*}$ of α_n^t, defined by $\alpha_n^{t\,*}\omega(f) = \omega(\alpha_n^t f)$. It also implies that time evolution can be formulated "a la Koopman" [26] in $L^2(X, d\mu)$: $\forall g \in L^1 \cap L^2$, $U_n^t g$ is defined by $\alpha_n^{t\,*} g$, and extends by continuity to a unitary group in $L^2(X, d\mu)$.

The above general structure covers in particular the case of Hamiltonian systems, for which X is the phase space, Γ^t the Hamiltonian time evolution, $d\mu$ the (infinite) Liouville measure and the regular states are the probability measures on the phase space with well defined (L^1) density.

We do not assume that $d\mu$ is a finite measure; in fact, even in the Hamiltonian case, a sequence of groups of transformations does not in general leave invariant a common compact subspace, and there is no common invariant finite measure. For the same reason, X is assumed to be only locally compact.

The advantage of the above formulation is to make explicit the rôle of the observables and of the states of the system, so that the discussion of the limit $n \to \infty$ can be done with a clear physical interpretation, once the preceding remarks on the origin of the statistical description of the states of a finite system are taken into account.

The existence of a *limit description* of the dynamical systems (X, Γ_n^t) as $n \to \infty$, given a time zero state ω, amounts in fact to convergence of all the mean values

$\omega(\alpha_n^t f)$, $f \in C^0(X)$. As we will see (Proposition 1), this is also equivalent to convergence of $\omega(\alpha_n^t \chi)$, for all characteristic functions χ of μ measurable subsets of X, i.e. convergence of the frequence of answers for all yes–no experiments based on μ measurable subsets of X, given the state ω.

Since we will admit as time zero states all regular states, we will investigate the limit of α_n^t in the weak topology $\tau_{\mathcal{F}}$ defined by $\mathcal{F}$ on $C^0(X)$. $\tau_{\mathcal{F}}$ can also be described as the ultraweak topology [27] associated to the representation of $C^0(X)$ as multiplication operators in $L^2(X, d\mu)$, and the $\tau_{\mathcal{F}}$ closure of $C^0(X)$ can be identified with $L^\infty(X, d\mu)$.

In the following, we will also use the strong and ultrastrong topologies defined on $C^0(X)$ and $L^\infty(X, d\mu)$ by the same representation; for bounded sequences f_n the strong and ultrastrong topology coincide, and f_n converge to f iff

$$\int |f_n(x) - f(x)|^2 \, h(x) d\mu(x) \to 0 \qquad \forall h(x) \in L^1(X, d\mu)$$

We will assume that the $\tau_{\mathcal{F}}$ weak limit of α_n^t exists for all t, and investigate under which condition it can be interpreted as an irreversible time evolution. More detailed results, which do not assume convergence for all times, are given in [28]. The following result holds for any $\tau_{\mathcal{F}}$ convergent sequence of automorphisms of $C^0(X)$:

Proposition 1

If a sequence of automorphisms α_n^t of $C^0(X)$, satisfying M, converges in the weak topology $\tau_{\mathcal{F}}$, for all $t \in \mathbb{R}$, then

i) the limit defines $\tau_{\mathcal{F}}$ continuous linear positive maps α^t of $C^0(X)$ into $L^\infty(X, d\mu)$, with norm at most one.

ii) the transpose $\alpha^{t\,*}$ of α^t leaves $\mathcal{F}$ invariant;

iii) α^t extends by continuity to a linear positive mapping of all $L^\infty(X, d\mu)$ into itself.

iv) $\alpha^{t\,*} = \alpha^{-t}$ on $L^1 \cap L^\infty$;

v) $\forall p \in [1, \infty)\ \forall h \in L^p \cap L^1 \quad ||\alpha^{t\,*} h||_p \leq ||h||_p$;

If the sequence α_n^t also satisfies property F, then

vi) $\alpha^{t\,*}$ preserve the L^1 norm on $(L^1 \cap L^\infty)^+$, and α^t leaves the identity invariant;

vii) the $\tau_{\mathcal{F}}$ continuous extension of α_n^t to $L^\infty(X, d\mu)$ also converges $\tau_{\mathcal{F}}$ for $n \to \infty$ to α^t on L^∞.

Positive maps are well known candidates for the description of irreversible evolution. Moreover, property *vi)* characterizes *bistochastic maps*, and it is generally assumed as fundamental [29] [20] for an axiomatic description of irreversible processes. In the present framework, *vi) automatically holds* for all maps which arise from weak limits of classical dynamical systems.

All the results of Proposition 1 hold, however, even for limits which still describe groups of automorphisms of $L^\infty(X, d\mu)$. In order to characterize limits describing irreversible evolution, one must ask whether the mapping α^t are invertible, and whether the group law $\alpha^t\,\alpha^s = \alpha^{t+s}$ is satisfied. A priori, as candidates for the description of irreversible time evolution arising in the above limit we have therefore:

i) Failure of the group law for α^t

ii) failure of invertibility of α^t.

It is also clear that

iii) failure of the morphism property of α^t

gives a good notion of irreversible behaviour. In fact, the transpose $\alpha^{t\,*}$ of a linear positive map α^t of any abelian C^* algebra, with invariant identity, sends pure states into pure states if and only if α^t is a morphism, as it is easily seen by using the Gelfand construction. Property iii) has therefore in general the interpretation of a loss of information on the system, corresponding to a transformation of pure into mixed states. More concretely, a transformation of $L^\infty(X, d\mu)$ is a morphism iff it maps characteristic functions of measurable sets into themselves, i.e. iff all yes–no questions are transformed into yes–no questions, equivalently, no information is lost in the time evolution.

Moreover, in the same generality, both *ii)* and *iii)* imply *i)*. Only *iii)* $\Rightarrow$ *i)* is in fact non–trivial, and follows from the fact that if the α^t satisfies the group law, then α^{-t} inverts α^t and is a positive map, so that $\alpha^{-t\,*}$ sends mixed states into mixed states, and therefore invertibility implies that $\alpha^{t\,*}$ sends pure states into pure states.

The following Proposition shows that in the present framework the notions i)–iii) coincide, so that *there is a unique notion of irreversibility for weak limits of classical dynamical systems*; it also provides necessary and sufficient conditions for the appearence of irreversibility.

Proposition 2

The following properties are equivalent for the weak limit α^t of a sequence of automorphisms α_n^t of $C^0(X)$, satisfying M:

i) $\forall t \in \mathbb{R}$, α^t is an automorphism of $L^\infty(X, d\mu)$;

ii) α^t, $t \in \mathbb{R}$, is a one parameter group of automorphisms of $L^\infty(X, d\mu)$;

iii) $\forall t \in \mathbb{R}$, α_n^t *converges in the ultrastrong topology associated to the representation of* $C^0(X)$ *as multiplication operators in* $L^2(X, d\mu)$, *and* $\alpha^t(1) = 1$;

iv) For some $p \in (1, \infty)$ $\alpha^t : C_0^0(X) \mapsto L^\infty(X, d\mu)$ *preserves the* L^p *norm, for all* t,

$$||\alpha^t f||_p = ||f||_p \quad \forall f \in C_0^0(X) \quad , \quad t \in \mathbb{R} \quad ;$$

v) For some $p \in (1, \infty)$ $\alpha^{t\,*}$ *preserves the* L^p *norm, for all* t:

$$||\alpha^{t\,*} h||_p = ||h||_p \quad \forall h \in L^1 \cap L^p \ , \quad t \in \mathbb{R} \quad ;$$

vi) For all $p \in [1, \infty]$, *for all* t, $\alpha^{t\,*}$ *preserves the* L^p *norm.*

Similar results hold for the convergence of the unitary group U_n^t which implement the time evolution in $L^2(X, d\mu)$:

Proposition 3

Assuming M, weak $\tau_{\mathcal{F}}$ *convergence of* α_n^t *is equivalent to convergence of* U_n^t *in the weak operator topology in* $L^2(X, d\mu)$. *Assuming convergence and denoting by* U^t *the weak limit of* U_n^t, *conditions i) - vi) of Proposition 2 are equivalent to each of the following:*

i) $\forall t \in \mathbb{R}$ *the operators* U_n^t *converge strongly in* $L^2(X, d\mu)$;

ii) $\forall t \in \mathbb{R}$ *the operators* U^t *are unitary;*

iii) The operators U^t *form a one–parameter unitary group.*

The following result shows that dynamical instabilities growing with n (at fixed time) are necessary for irreversible behaviour to take place in the limit $n \to \infty$:

Proposition 4

Assume that X *is a metric space, with distance* $d(x, y)$, *and that the maps* Γ_n^t, *satisfying M and F, are stable, at fixed time, uniformly in* n, *except possibly on "locally small subsets"; i.e. assume:* $\forall$ *compact* $K \subset X$, $\varepsilon > 0$, $t \in \mathbb{R}$, $\exists M(K, \varepsilon, t) \in \mathbb{R}$ *such that*

$$d(\Gamma_n^t(x), \Gamma_n^t(y)) < M\, d(x, y) \qquad \forall n \in \mathbb{N},\ \forall x, y \in K \setminus S$$

$S = S(K, \varepsilon, t)$ *an open subset of* K *of* μ *measure smaller than* ε.

Then, if the sequence α_n^t *converges in the weak* $\tau_{\mathcal{F}}$ *topology, it also converges strongly and* α^t *is a group of automorphisms of* $L^\infty(X, d\mu)$.

In general, the map α^t has an immediate probabilistic interpretation: given any state $h \in L^1(X, d\mu)$ and denoting by χ_B the characteristic function of a Borel set $B \subset X$, the mean value

$$\int h(x)\, \alpha^t \chi_B(x)\, d\mu(x)$$

gives by definition the probability that a point distributed with $h(x) d\mu(x)$ at time 0 ends in B at time t. $\alpha^t \chi_B(x)$ is therefore interpreted as a probability $P_t(x, B)$ that a point starting at x at time 0 falls in B at time t.

Since $\alpha^t \chi_B$ is a characteristic function of a measurable set for all B if and only if α^t is an automorphism of $L^\infty(X, d\mu)$, $P_t(x, B)$ takes values in $\{0, 1\}$ ($d\mu$ a.e.) if and only if there is no irreversibility.

The situation is therefore very close to that of a stochastic process, as it also follows by a closer examination of limits at different times. Consider in fact finite sequences of measurements at different times, $t_1 \ldots t_k$, of observables $f_1 \ldots f_k$, and assume that, given a regular state $\omega \in \mathcal{F}$, the mean values

$$\omega(\alpha_n^{t_1} f_1 \ldots \alpha_n^{t_k} f_k)$$

converge for $n \to \infty$.

The following Proposition shows that, if such a limit exists, it defines a stochastic process, i.e. a measure $d\rho_\omega(x(t))$ on trajectories in X, and gives necessary and sufficient condition for the process to be Markov.

Proposition 5

Let α_n^t satisfy M and F and converge weakly for $n \to \infty$; assume that for a fixed $\omega \in \mathcal{F}$, and for all finite sequences $f_1 \ldots f_k \in C^0(X)$ and $0 \le t_1 < t_2 < \ldots < t_k \in \mathbb{R}$

$$\omega(\alpha_n^{t_1} f_1 \ldots \alpha_n^{t_k} f_k)$$

converge for $n \to \infty$.

Then, there exists a unique regular Borel probability measure $d\rho_\omega$ on $\prod_{t \in \mathbb{R}^+} \dot{X}_t$, $\dot{X}$ the one point compactification of X, such that

$$\lim_n \omega(\alpha_n^{t_1} f_1 \ldots \alpha_n^{t_k} f_k) = \int d\rho_\omega(x(t))\, f_1(x(t_1)) \ldots f_k(x(t_k))$$

For $\omega(x)$ of compact support K, the measure $d\rho_\omega$ is supported on a product of compact sets K_t, $t > 0$

The process defined by $d\rho_\omega$ is Markov, with time independent transition function $P_t(x,B) = \alpha^t \chi_B(x)$, if

$$\lim_n \omega(\alpha_n^{t_1} f_1 \ldots \alpha_n^{t_k} f_k) =$$

$$= \lim_{n_k} \ldots \lim_{n_2} \lim_{n_1} \omega(\alpha_{n_1}^{t_1}(f_1 \alpha_{n_2}^{t_2-t_1}(\ldots(f_{k-1}\alpha_{n_k}^{t_k-t_{k-1}} f_k)\ldots)))$$

This condition is also necessary for the time independent Markov property if $\omega(x)$ is a.e. different from zero.

In Sect.4 a model will be discussed leading to an irreversible time evolution, actually defining a time independent Markov process.

3. Proofs

As mentioned above, states in $\mathcal{F}$ will be identified with their representative function in $L^1(X, d\mu)$. If the state $\omega \in \mathcal{F}$ is represented by the function $h \in L^1(X, d\mu)$, we will write $\alpha_n^{t\,*} h$ for the representative function of $\alpha_n^{t\,*}\omega$.

It is immediate to see that in $\mathcal{F}$ the norm of the dual $C^0(X)^*$ of $C^0(X)$ coincides with the $L^1(X, d\mu)$ norm. The $\mathcal{F}$ notation will be preferred to that based on L^1 in arguments where the dual $C^0(X)^*$ of $C^0(X)$ is involved. Since $d\mu$ is sigma–finite, the continuous functions with compact support belong to $L^1(X, d\mu)$ and are dense there; $L^1 \cap C^0$ is therefore dense in $L^1 = \mathcal{F}$.

By weak and strong $\tau_{\mathcal{F}}$ topology we will denote respectively the weak topology defined on $C^0(X)$ by the linear functionals in $\mathcal{F}$ and the ultrastrong topology defined as above by the representation of $C^0(X)$ in $L^2(X, d\mu)$, identical with the strong topology for bounded sets. By $\tau_{\mathcal{F}}$ topology and $\tau_{\mathcal{F}}$ continuity we will refer to the weak one. The following Lemma is preliminary:

Lemma 6

i) $\forall t \in \mathbb{R}$, $\alpha_n^{t\,*}$ maps $\mathcal{F}$ into $\mathcal{F}$ and

$$\alpha_n^{t\,*} h(x) = h(\Gamma_n^t x) \tag{3.1}$$

ii) α_n^t is continuous in the weak (and in the strong) $\tau_{\mathcal{F}}$ topology, and extends therefore by continuity to an automorphism of the Von Neumann algebra $L^\infty(X, d\mu)$;

iii) $\forall f \in L^1 \cap L^\infty$ $\alpha_n^{t\,*} f = \alpha_n^{-t} f$;

iv) $\forall p \in [1,\infty], \quad \forall f \in L^1 \cap L^p \quad ||\alpha_n^{t\,*} f||_p = ||f||_p\,.$

Proof.

Eq. (3.1) follows immediately from the invariance of $d\mu$: $\forall h \in L^1(X,d\mu)$, $h(\Gamma_n^t x) \in L^1(X,d\mu)$ and

$$\int h(\Gamma_n^t x)\, f(x)\; d\mu = \int h(x)\, f(\Gamma_n^{-t} x)\; d\mu = \int h(x)\, \alpha_n^t f\; d\mu \qquad \forall f \in C^0(X) \tag{3.2}$$

Invariance of $d\mu$ implies conservation of all L^p norms. Invariance of $\mathcal{F}$ implies (see e.g. [19]) weak $\tau_{\mathcal{F}}$ continuity of α_n^t, which extends by continuity to the weak closure L^∞ of $C^0(X)$.

Proof of Proposition 1.

i): For all f in $C^0(X)$, the weak $\tau_{\mathcal{F}}$ limit of $\alpha_n^t f$ defines a continuous linear functional on $L^1(X,d\mu)$, with norm bounded by $\sup|f|$, i.e. an element of $L^\infty(X,d\mu)$; the limit α^t of α_n^t defines therefore a linear map of $C^0(X)$ into $L^\infty(X,d\mu)$, of norm at most one; its transpose $\alpha^{t\,*}$ a priori maps $\mathcal{F}$ into $C^0(X)^*$, and is continuous in the norm of $C^0(X)^*$, denoted by $||\cdot||_*$: $\forall h \in L^1$

$$||\alpha^{t\,*} h||_* = \sup_{f\in C^0(X)\ ||f||_\infty = 1} |\lim_n \int h\, \alpha_n^t f\; d\mu| \le ||h||_1$$

Weak $\tau_{\mathcal{F}}$ continuity of α^t is equivalent [17] to stability of $\mathcal{F}$ under $\alpha^{t\,*}$, i.e. point *ii)*; for positivity see *iii)* below.

ii): Consider first states represented by functions $h \in L^1 \cap C^0$; by Lemma 1, $\alpha_n^{t\,*} h = \alpha_n^{-t} h$, which $\tau_{\mathcal{F}}$ converges for $n \to \infty$ to $\alpha^{-t} h$. This means that $\forall f \in C_0^0(X)$

$$\int \alpha^{t\,*} h\, f\; d\mu = \lim_n \int \alpha_n^{t\,*} h\, f\; d\mu = \lim_n \int \alpha_n^{-t} h\, f\; d\mu = \int \alpha^{-t} h\, f\; d\mu \tag{3.3}$$

so that

$$\alpha^{t\,*} h = \alpha^{-t} h \qquad \forall h \in L^1 \cap C^0 \quad . \tag{3.4}$$

Moreover,

$$||\alpha^{-t} h||_1 = ||\alpha^{t\,*} h||_1 = ||\alpha^{t\,*} h||_* \quad . \tag{3.5}$$

Eq. (3.5), continuity in $C^0(X)^*$ norm of $\alpha^{t\,*}$ and density of $L^1 \cap C^0$ in $\mathcal{F}$ imply stability of $\mathcal{F}$ under $\alpha^{t\,*}$.

iii): α^t extends by $\tau_{\mathcal{F}}$ continuity to the Von Neumann algebra L^∞. Positivity of α^t is equivalent to

$$\int h(x)\,\alpha^t f(x)\,d\mu \geq 0 \qquad \forall f \geq 0\,,\quad f \in L^\infty\,,\quad h \geq 0\,,\quad h \in L^1 \tag{3.6}$$

For $f \in C^0(X)$, eq. (3.6) follows from the positivity of the same expression with α^t_n replacing α^t, for $f \in L^\infty$ by definition of continuous extension of α^t.

iv): Eq. (3.5) extends to all $h \in L^1 \cap L^\infty$, since, $\forall h \in L^1 \cap L^\infty$, $f \in L^1 \cap C^0$,

$$\int \alpha^{t\,*} h\, f\, d\mu = \int h\, \alpha^t f\, d\mu = \int h\, \alpha^{-t\,*} f\, d\mu = \int \alpha^{-t} h\, f\, d\mu \tag{3.7}$$

and $L^1 \cap C^0$ is dense in $C^0(X)$.

v) follows from Hölder inequality:

$$\int \alpha^{t\,*}_n h\, f\, d\mu \leq ||h||_p\, ||f||_q \qquad \forall h \in L^1 \cap L^p\,,\quad f \in C^0 \cap L^q\,,\quad n \in \mathbb{N}$$

which implies

$$\int \alpha^{t\,*} h\, f\, d\mu \leq ||h||_p\, ||f||_q$$

so that, by density of $C^0 \cap L^q$ in L^q

$$||\alpha^{t\,*} h||_p \leq ||h||_p \tag{3.8}$$

vi): In order to prove

$$||\alpha^{t\,*} h||_1 = ||h||_1 \qquad \forall h \in L^{1+} \tag{3.9}$$

consider first $h \in C^0_0(X)$. Denote by χ_K the characteristic function of a compact set $K \subset X$; using Lemma 1, condition F implies

$$\int \chi_K\, \alpha^{-t}_n h\, d\mu = \int \chi_K\, h\, d\mu \qquad \forall n \tag{3.10}$$

if K is sufficiently large, depending only on h and t. Given the support L of h, it is in fact enough to take $K \supset L$ such that $\Gamma^t_n(L) \subset K\ \forall n$. Using eqs.(3.4), (3.10) it follows

$$\int \alpha^{t\,*} h\, d\mu = \lim_K \int \chi_K\, \alpha^{-t} h\, d\mu =$$

$$= \lim_K \lim_n \int \chi_K\, \alpha^{-t}_n h\, d\mu = \lim_K \int \chi_K\, h\, d\mu = \int h\, d\mu$$

Eq.(3.9) then follows from density in L^1 of $C_0^0(X)$ and L^1 continuity of $\alpha^{t\,*}$. Eq.(3.9) also implies $\alpha^t 1 = 1$.

vii): Using Lemma 1 and eq.(3.4), $\forall h \in L^1 \cap L^\infty,\ f \in L^1 \cap C^0$

$$\int \alpha_n^t h\, f\; d\mu = \int h\, \alpha_n^{t\,*} f\; d\mu = \int h\, \alpha_n^{-t} f\; d\mu \to$$

$$\to \int h\, \alpha^{-t} f\; d\mu = \int h\, \alpha^{t\,*} f\; d\mu = \int \alpha^t h\, f\; d\mu \tag{3.11}$$

and $\tau_{\mathcal{F}}$ convergence of $\alpha_n^t h$ to $\alpha^t h$ follows from density of $L^1 \cap C^0$ in L^1 and uniform boundedness of $\alpha_n^t h$ in L^∞.
To obtain $\tau_{\mathcal{F}}$ convergence of $\alpha_n^t h$ to $\alpha^t h$ for all $h \in L^\infty$ consider a sequence $\{K_i\}$ of compact sets with χ_{K_i} converging $\tau_{\mathcal{F}}$ to 1 (it exists because μ is σ-finite); then, $\forall h \in L^{\infty +}$, $\forall l \in L^{1+}$

$$\int \alpha_n^t (h - h\chi_{K_i})\, l\; d\mu \leq \|h\|_\infty \int (1 - \alpha_n^t \chi_{K_i})\, l\; d\mu \tag{3.12}$$

but $\tau_{\mathcal{F}}$ convergence of α_n^t on $L^1 \cap L^\infty$, $\tau_{\mathcal{F}}$ continuity of α^t and $\alpha^t 1 = 1$ imply

$$\lim_i \lim_n \ \text{RHS of (3.12)} = 0 \quad ;$$

on the other hand, by $\tau_{\mathcal{F}}$ continuity of α^t,

$$\lim_i \lim_n \int \alpha_n^t (h\chi_{K_i})\, l\; d\mu = \int \alpha^t h\, l\; d\mu$$

and therefore

$$\lim_n \int \alpha_n^t h\, l\; d\mu = \int \alpha^t h\, l\; d\mu$$

which extends by linearity to $h \in L^\infty$ and $l \in L^1$.

Proof of Proposition 2.

vi) $\Rightarrow$ *v)* and *ii)* $\Rightarrow$ *i)* are obvious; *iv)* $\iff$ *v)* and *i)* $\Rightarrow$ *vi)* immediatly follow from Prop.1, (for the latter implication, see the proof of Lemma 6); it is therefore enough to prove *v)* $\Rightarrow$ *i)* and *vi)* $\Rightarrow$ *iii)* $\Rightarrow$ *ii)*.

v) $\Rightarrow$ *i)*: For any Borel set B of finite measure $d\mu$, $\alpha^{t\,*}\chi_B$ is real, non negative, and

$$||\alpha^{t\,*}\chi_B||_\infty \leq ||\chi_B||_\infty \leq 1 \quad ,$$

so that

$$(\alpha^{t\,*}\chi_B)^p \leq \alpha^{t\,*}\chi_B \quad . \tag{3.13}$$

$v)$ implies that for some $p \in (1,\infty)$

$$\int (\alpha^{t\,*}\chi_B)^p \, d\mu = \int \chi_B^p \, d\mu = \int \chi_B \, d\mu \geq \int \alpha^{t\,*}\chi_B \, d\mu \quad , \tag{3.14}$$

the last equality following from Prop.1, $v)$. Eqs. (3.13) and (3.14) imply

$$\alpha^{t\,*}\chi_B = (\alpha^{t\,*}\chi_B)^p$$

$d\mu$ a.e., so that $\alpha^{t\,*}\chi_B$ is a measurable function taking values in $\{0,1\}$, i.e. it is the characteristic function of a measurable set $B' \subset X$, and by eqs.(3.13), (3.14), $\mu(B') = \mu(B)$.
¿From linearity and the fact that L^∞ norms do not increase (Prop.1, i), it follows that disjoint sets B, C have disjoint images B', C'; α^t, coinciding with $\alpha^{-t\,*}$ on $L^1 \cap L^\infty$, (Prop.1, iv), is therefore a morphism of the algebra of simple functions based on sets of finite measure, and, by a density argument, of $L^\infty(X, d\mu)$. It also follows that $\alpha^{t\,*}$ preserves the L^1 norm on L^{1+}, which implies invertibility of α^t, actually $\alpha^{-t}\alpha^t = \alpha^t\alpha^{-t} = 1$; in fact, for all positive simple functions f, g,

$$\int f\alpha^{-t}\alpha^t g \, d\mu = \int \alpha^t f \alpha^t g \, d\mu = \int \alpha^t(fg) \, d\mu = \int fg \, d\mu \quad .$$

The same holds for $\alpha^t\alpha^{-t}$, and invertibility of α^t follows by a density argument.

$vi) \Rightarrow iii)$: convergence in the strong and ultrastrong topologies is equivalent for a bounded sequence [25], and amounts to

$$\int |\alpha_n^t f - \alpha^t f|^2 \, h \, d\mu \to 0 \qquad \forall h \in L^1(X, d\mu) \quad . \tag{3.15}$$

By a density argument, it is enough to obtain eq.(3.15) for all f in $C_0^0(X) \subset L^1 \cap L^\infty$; in this case, $\alpha_n^t f = \alpha_n^{-t\,*} f$ converges to $\alpha^t f = \alpha^{-t\,*} f$ strongly in $L^2(X, d\mu)$, as a consequence of weak L^2 convergence and of conservation of the L^2 norms in the limit. $\alpha^t 1 = 1$ immediately follows from conservation of the L^1 norm.

$iii) \Rightarrow ii)$: Strong convergence implies in general that α^t is a morphism. The group property follows since for all f, g in $C_0^0(X)$,

$$\int f\alpha^{t+s} g \, d\mu = \lim_n \int f\alpha_n^t \alpha_n^s g \, d\mu = \lim_n \int \alpha_n^{-t} f \alpha_n^s g \, d\mu \quad ; \tag{3.16}$$

strong convergence in L^2 of $\alpha_n^{-t} f$ and $\alpha_n^s g$ imply in fact

$$\lim_n \int \alpha_n^{-t} f \alpha_n^s g \, d\mu = \int \alpha^{-t} f \alpha^s g \, d\mu = \int f \alpha^t \alpha^s g \, d\mu \quad . \tag{3.17}$$

Proof of Proposition 3.

The algebra $C_0^0(X)$ of continuous functions of compact support is dense in $C^0(X)$ and in $L^1(X, d\mu)$ in the respective norms. Moreover, the sequence $\alpha_n^t h$ is uniformly bounded in L^1 and in C^0. $\tau_{\mathcal{F}}$ convergence of α_n^t forall $f \in C^0$ is therefore equivalent to convergence on $C_0^0(X)$.
In the same way, since $C_0^0(X)$ is dense in L^2 and U_n^t preserves the L^2 norm, $\tau_{\mathcal{F}}$ convergence is equivalent to

$$\forall h_1, h_2 \in L^2 \quad \exists \lim_n \int h_1^* \, U_n^t h_2 \, d\mu$$

i.e. to convergence of U_n^t in the weak topology w_{op} for operators in L^2
Denoting $U^t \equiv w_{op} - \lim_n U_n^t$, $\forall h_1, h_2 \in C_0^0(X)$,

$$\int h_1^* \, U^t h_2 \, d\mu = \lim_{n \to \infty} \int h_1^* \, \alpha_n^t h_2 \, d\mu = \int h_1^* \, \alpha^t h_2 \, d\mu \tag{3.18}$$

which implies $U^t h = \alpha^t h$ for $h \in C_0^0(X)$ and, by L^2 continuity of both U^t and α^t (Prop.1, v), for $h \in L^1 \cap L^\infty$. The rest of the proof of Prop.3 is then reduced to the following simple fact:

$\forall n \in \mathbb{N}$ let U_n^t be a one parameter group of unitary operators on a Hilbert space $\mathcal{H}$, converging weakly for $n \to \infty$, $\forall t \in \mathbb{R}$, $w_{op} - \lim_n U_n^t \equiv U^t$. Then the following are equivalent:

$a)$ $\forall t \in \mathbb{R}$, U_n^t converges strongly to U^t,

$b)$ $\forall t \in \mathbb{R} \, \forall h \in \mathcal{H} \;\; \|U^t h\| = \|h\|$,

$c)$ $\forall t \in \mathbb{R} \, U^t$ is unitary,

$d)$ U^t is a group of unitary operators.

In fact, $a) \Leftrightarrow b$ is well known, and $d) \Rightarrow c) \Rightarrow b)$ are trivial.

To prove $a) \Rightarrow d)$, we have, from Schwarz inequality

$$|(h, (U_n^t U_n^s - U^t U^s) h)| = |(h, (U_n^t U_n^s - U_n^t U^s + U_n^t U^s - U^t U^s) h)| \le$$

$$\le |(h, U_n^t (U_n^s - U^s) h)| + |(h, (U_n^t - U^t) U^s h)| \le$$

$$\leq \|h\|\ \|(U_n^s - U^s)h\| + |(h, (U_n^t - U^t)U^s h)| \quad ,$$

which implies the group property. In particular U^t is invertible, and unitarity follows since $a) \Rightarrow b)$.

Proof of Proposition 4.

For fixed compact $K \subset X$, $\varepsilon > 0$, $t \in \mathbb{R}$, $f \in C_0^0(X)$ with support in K, by property F all the functions $\alpha_n^t f(x) = f(\Gamma_n^{-t}(x))$ have support contained in a compact set K'. By assumption, there exists an open set S, of measure smaller than ε, such that

$$d(\Gamma_n^{-t}(x), \Gamma_n^{-t}(y)) < M(K', \varepsilon, -t)\, d(x,y)$$

for $x, y \in K' \setminus S$. Since f is of compact support and therefore uniformly continuous, it follows that the functions $\alpha_n^t f(x)$ are equicontinuous on $K' \setminus S$.

Since $K' \setminus S$ is closed and therefore compact, by the Ascoli–Arzelà theorem there exists a subsequence $\alpha_{n_k}^t f$ which converges there uniformly; its limit coincides, on $K' \setminus S$ with $\alpha^t f$, as a consequence of weak convergence of $\alpha_n^t f(x)$ to $\alpha^t f$. On compact sets, uniform convergence implies convergence in L^2; moreover, the functions $\alpha_n^t f(x)$ are bounded by $\sup|f|$, and therefore their L^2 norm coincides with the norm of their restriction to $K' \setminus S$ up to an error of order ε, and the same is true for $\alpha^t f(x)$; this implies

$$|\,||\alpha^t f||_{L^2}^2 - ||f||_{L^2}^2\,| \ \leq\ 2 \sup|f|^2 \varepsilon \qquad \forall \varepsilon > 0$$

It follows that α^t preserves the L^2 norm on $C_0^0(X)$, for all t, and therefore, by Proposition 2, *iv)*, it is a group of automorphisms of $L^\infty(X, d\mu)$.

Proof of Proposition 5.

We will use the shorthand $\{t_j\}_{1,k}$ for $0 \leq t_1 < t_2 < \ldots < t_k$, and $\{t_j\}_{1,]m[,k}$ for the same sequence without t_m; $X^{\{t_j\}_{1,k}}$ will denote the product of k copies of X, X_{t_j} the copy at time t_j; the state ω will be represented by the measure $\omega(x)\, d\mu$, $\omega(x) \in L^1(X, d\mu)$.

For fixed $\{t_j\}_{1,k}$, consider the algebra C^{sp} of functions $f \in C^0(X^{\{t_j\}_{1,k}})$ which are finite sums of products, i.e. $f(x_{1,k}) = \sum_i c_i f_1^i(x_1) \ldots f_k^i(x_k)$ with $f_j^i(x_j) \in C^0(X_{t_j})$. For fixed n the linear map of C^{sp} into itself

$$f(x_{1,k}) \to \sum_i c_i \prod_{j=1}^{k} \alpha_n^{t_j} f_j^i\,(x_j)$$

is $\| \|_\infty$ preserving and extends to an automorphism of the completion $\overline{C^{sp}}$ of C^{sp} in the sup norm; in particular, it is positive.

It follows that $\rho_\omega^{\{t_j\}_{1,k}}$, defined by

$$\rho_\omega^{\{t_j\}_{1,k}}(f) \equiv \lim_n \int_X \sum_i c_i \prod_{j=1}^k \alpha_n^{t_j} f_j^i(x)\, \omega(x)\, d\mu(x) \tag{3.19}$$

is a bounded positive linear functional over C^{sp}, with $\|\rho_\omega\| \leq \|\omega\|$. Since by the Stone Weierstrass theorem $\overline{C^{sp}} = C^0(X^{\{t_j\}_{1,k}})$, ρ_ω has a unique extension to C^0, with the same bound.

The functional so obtained defines, by the Riesz Markov theorem, a finite regular Borel measure, which we indicate with $\rho_\omega^{\{t_j\}_{1,k}}$, over $X^{\{t_j\}_{1,k}}$. If X is not compact it is technically convenient to consider $\rho_\omega^{\{t_j\}_{1,k}}$ over $\dot{X}^{\{t_j\}_{1,k}}$, the product of the 1-point compactified spaces $\dot{X}_{t_j}$. However, if ω has compact support K, the construction also lives in $\prod_{t\geq 0} K_t$ with K_t the compact sets given by condition F, $\alpha_n^t \chi_K \leq \chi_{K_t}$ $\forall n$.

The measures $\rho_\omega^{\{t_j\}_{1,k}}$, for all k-tuples $0 \leq t_1 < t_2 < \ldots < t_k$, define a (unique) measure over $\prod_{t\geq 0} \dot{X}_t$ iff they are compatible, i.e.

$$\rho_\omega^{\{t_j\}_{1,k}}(f_1(x_1)\ldots f_{m-1}(x_{m-1})\, f_{m+1}(x_{m+1})\ldots f_k(x_k)) =$$

$$= \rho_\omega^{\{t_j\}_{1,]m[,k}}(f_1(x_1)\ldots f_{m-1}(x_{m-1})\, f_{m+1}(x_{m+1})\ldots f_k(x_k)) \quad ..$$

Compatibility obviously holds for X compact, since then $1 \in C^0(X)$, and $\alpha_n^t 1 = 1$. For non compact X, compatibility follows from $\tau_{\mathcal{F}}$ continuity of α^t together with $\alpha^t 1 = 1$; take in fact an increasing sequence of positive functions $g_\nu \in C_0^0(X_{t_m})$ converging to 1 $d\mu$ almost everywhere. Clearly, for any positive $f_j \in C^0(X_{t_j})$, $j = 1, \ldots, k$, $j \neq m$,

$$\lim_\nu \lim_n \omega\,((1 - \alpha_n^{t_m} g_\nu) \prod_j \alpha_n^{t_j} f_j\,) \leq \lim_\nu \lim_n \omega\,(1 - \alpha_n^{t_m} g_\nu) \prod_j \|f_j\|_\infty \quad . \tag{3.20}$$

By the monotone convergence theorem, $g_\nu \prod_j f_j$ converge in $\| \|_{L^1(X^{\{t_j\}_{1,k}}, \rho_\omega^{\{t_j\}_{1,k}})}$ to $\prod_j f_j$, and therefore the l.h.s. of eq.(3.20) converges to

$$\rho_\omega^{\{t_j\}_{1,]m[,k}}(\prod_j f_j) - \rho_\omega^{\{t_j\}_{1,k}}(\prod_j f_j) \quad ;$$

since g_ν converge in $\| \|_{L^1(X, \alpha^{t_m *}\omega)}$ to 1, the r.h.s. converges to 0.

This implies the compatibility condition, since linear combinations of products of positive functions are dense in $C^0(X^{\{t_j\}_{1,]m[,k}})$.

¿From compatibility it also follows $\rho_\omega^{\{t_j\}_{1,k}}(1) = \omega(1) = 1$.
As regards the Markov property, let us indicate by

$$p(t_1, x_1; \ldots; t_{k-1}, x_{k-1} | t_k, f_k) \in L^\infty(\dot{X}^{\{t_j\}_{1,k-1}}, \rho_\omega^{\{t_j\}_{1,k-1}})$$

the conditional expectations of the process, characterized by

$$\forall k \geq 2,\ \forall j = 1, \ldots, k,\ \forall f_j \in L^\infty(\dot{X}_{t_j}, \rho_\omega^{t_j}) \qquad \int f_1(x_1) \ldots f_k(x_k)\, d\rho_\omega^{\{t_j\}_{1,k}} =$$

$$= \int f_1(x_1) \ldots f_{k-1}(x_{k-1})\, p(t_1, x_1; \ldots; t_{k-1}, x_{k-1} | t_k, f_k)\, d\rho_\omega^{\{t_j\}_{1,k-1}} \quad .$$

The process is Markov iff

$$p(t_1, x_1, \ldots, t_{k-1}, x_{k-1} | t_k, f_k) = p(t_{k-1}, x_{k-1} | t_k, f_k)$$

and it is time independent Markov iff it is Markov and

$$p(t_1, x_1 | t_2, f_2) = p(0, x_1 | (t_2 - t_1), f_2) \quad .$$

Hence, for a time independent Markov process, by iteration we obtain

$$\int f_1(x_1) \ldots f_k(x_k)\, d\rho_\omega^{\{t_j\}_{1,k}} =$$

$$= \int p(0, x_0 | t_1, f_1\, p(0, \cdot\, | t_2 - t_1, f_2 \ldots f_{k-1}\, p(0, \cdot\, | t_k - t_{k-1}, f_k) \ldots))\, \omega(x_0)\, d\mu \tag{3.21}$$

Notice now that, since the α_n^t are automorphisms, it is

$$\lim_{n_k} \ldots \lim_{n_2} \lim_{n_1} \omega(\alpha_{n_1}^{t_1} f_1 \ldots \alpha_{n_k}^{t_k} f_k) = \tag{3.22}$$

$$= \lim_{n_k} \ldots \lim_{n_2} \lim_{n_1} \omega(\alpha_{n_1}^{t_1} (f_1\, \alpha_{n_2}^{t_2 - t_1} (\ldots (f_{k-1}\, \alpha_{n_k}^{t_k - t_{k-1}} f_k) \ldots)))$$

which, by definition and continuity of α^t, converges to

$$\omega(\alpha^{t_1} (f_1\, \alpha^{t_2 - t_1} (\ldots (f_{k-1}\, \alpha^{t_k - t_{k-1}} f_k) \ldots))) \quad . \tag{3.23}$$

So, if the limit (3.19) coincides with (3.22), then the stochastic process ρ is Markov and time independent, with $p(0, x | t, f) = \alpha^t f\,(x)$.
The converse follows if $\omega(x)$ only vanishes on a set of zero measure; in fact, for $\omega(x) \neq 0$, $p(0, x | t, f) = \alpha^t f\,(x)$, and therefore, if the process is time independent Markov, i.e. if (3.21) holds, then it is also given by the limit (3.22), (3.23).

4. A model

We consider a sequence of dynamical systems defined by a particle moving horizontally in $\mathbb{R}^2$ with speed 1, and interacting with fixed vertical "rods", of equal height; the only effect of the interaction is assumed to be the change of the "spin" of the particle, taking values ± 1.

The space X of the configurations of the system is thus $\mathbb{R}^2 \times \{-1, 1\}$, on which we take the measure $\mu = d^2\mathbf{x} \otimes d\sigma$, with $\sigma(\{1\}) = \sigma(\{-1\}) = 1$. As space of states we will take $L^1(\mathbb{R}^2 \times \{-1, 1\}, \mu)$, i.e. the set of states defined by measures on the particle configuration space which are absolutely continuous with respect to the Lebesgue measure.

The dynamics will depend on a parameter n through the change of the positions and height of the rods, which will be given by a suitable scaling of the positions and of the height of the rods *in one given and fixed configuration* in a set to be specified in the following.

In general, a configuration ξ of rods is determined by their height h and by the set of their centers, $\xi \equiv \bigcup_{i \in \mathbb{N}} \{(x_i, y_i)\}$; we will always consider only configurations with a finite number of rods in any compact subset of $\mathbb{R}^2$, and denote by Ξ the set of all such configurations. For *fixed* $\xi \in \Xi$, and $h > 0$, we will study the limit $n \to \infty$ of the dynamics α_n^t defined by the interaction with rods of height $n^{-2}h$ in the scaled configuration $n^{-1}\xi \equiv \bigcup_{i \in \mathbb{N}} \{(n^{-1}x_i, n^{-1}y_i)\}$. The limit will be taken in the weak topology defined by the state space $\mathcal{F} \equiv L^1(\mathbb{R}^2 \times \{-1, 1\}, \mu)$. We will prove the following:

Proposition 7

For almost any choice of the configuration ξ with respect to the measure ρ defined below, the dynamical systems defined as above by the fixed configuration ξ converge weakly for $n \to \infty$, with respect to the set $\mathcal{F}$ of states, to a positive map, and give rise, as in Proposition 5, to a time independent Markov process.

We stress that for each finite n the dynamics of the model is *deterministic and reversible*, and only in the limit $n \to \infty$ the result for any given observable, in any given state in $\mathcal{F}$, is described by a Markov process.

The rôle of the measure ρ is to select the configurations of rods which appear "when their centers are given without correlations and with density one"; the construction of such a measure is necessarely a little absctract, but it can be done in a relatively straightforward way, following the procedure outlined in ref.[13], Sect.7.1.2.

In fact, (full details are given in [28]) the family of measures on Λ^j

$$d\rho_\Lambda^j \equiv \frac{e^{-|\Lambda|}}{j!}\, d\mathbf{x}_1 \dots d\mathbf{x}_j$$

with $j \in \mathbb{N}$ and Λ bounded Borel$\subset \mathbb{R}^2$, defines a bounded positive linear functional

$$\rho(f) \equiv \sum_{j=0}^{\infty} \frac{e^{-|\Lambda|}}{j!} \int_{\Lambda^j} d\mathbf{x}_1 \dots d\mathbf{x}_j\, f(\xi) \quad , \tag{4.1}$$

where $f : \Xi \to \mathbb{C}$ depends only on the points, exactly $\mathbf{x}_1 \dots \mathbf{x}_j$, of ξ inside Λ, over a suitable algebra $\mathcal{B}$ of functions on Ξ. The spectrum $\mathcal{E}$ of the $\| \, \|_\infty$ closure of $\mathcal{B}$ turns out be the $*$-w closure of Ξ (the $*$ weak topology being the weakest one which makes the functions in $\mathcal{B}$ continuous).

By the Riesz–Markov theorem, the positive functional ρ defines a Borel regular measure on $\mathcal{E}$, denoted for simplicity again by ρ, and it is not difficult to see that ρ is in fact concentrated on Ξ, i.e., $\rho(\Xi) = \rho(\mathcal{E}) = 1$ [28]. For the following, the essential point is that, for functions depending only on a finite number of points, integration with the measure ρ is simply given by eq. (4.1).

Proof of Proposition 7.

The time evolution of $\mathbf{x} \equiv (x, y)$ is a translation, denoted by $\mathbf{x} + t \equiv (x + t, y)$. For the spin σ, the effect of collisions is given by

$$\alpha_n^t \sigma = \sigma\, (-1)^{|n^{-1}\xi \,\cap\, B_{\mathbf{x},t,n^{-2}h}|} = \sigma\, (-1)^{|(\xi \,\cap\, B_{n\mathbf{x},nt,n^{-1}h}|} \quad , \tag{4.2}$$

$|I|$ being the number of points in the finite set I, and

$$B_{\mathbf{x},t,h} \equiv \{(\alpha, \beta) \in \mathbb{R}^2 \,|\, \beta \in [y - h/2, y + h/2),$$

$$\alpha \in [x, x + t) \text{ for } t > 0,\ \ \alpha \in [x + t, x) \text{ for } t < 0\}$$

For the states we have therefore:

$$\forall l \in L^1(X, d\mu) \quad \alpha_n^{t\,*} l(x, y, \sigma) = l(x + t, y, \sigma \cdot (-1)^{|\xi \,\cap\, B_{n\mathbf{x},nt,n^{-1}h}|})$$

and for the observables

$$\forall f \in C^0(X) \quad \alpha_n^t f(x, y, \sigma) = f(x - t, y, \sigma \cdot (-1)^{|\xi \,\cap\, B_{n(\mathbf{x}-t),nt,n^{-1}h}|}) \quad .$$

Clearly, μ is left invariant by α_n^t, for all t and n. The choice of a discontinuous interaction makes α_n^t not to map C^0 into itself. However, the α_n^t map C^0 into the $\| \, \|_\infty$ closure of $L^1 \cap L^\infty$, and define automorphisms of this space, which is enough for the proof of the results of Sect.2.

First of all we want to express the request of existence of a limit dynamics, obtained by scaling the obstacles, in terms of observables of the system which are directly related to the configuration ξ of rods. Denoting by $g_{\pm 1,\mathbf{x},t,h}(\xi)$ the characteristic functions of the sets

$$\{\mathbf{x} \in \mathbb{R}^2 \, : \, |\xi \cap B_{\mathbf{x},t,h}| \ \text{ is } \begin{matrix}\text{even}\\ \text{odd}\end{matrix}\} \ ,$$

from the fact that $\forall f \in C^0(X)$, $\forall l \in L^1(X, d\mu)$

$$\int_X \alpha_n^t f \, l \, d\mu = \sum_{\tau=\pm 1} \sum_{\sigma=\pm 1} \int g_{\tau, n(\mathbf{x}-t), nt, n^{-1}h}(\xi) f(\mathbf{x}-t, \tau\cdot\sigma) \, l(\mathbf{x},\sigma) \, d^2\mathbf{x}$$

and since the characteristic functions of bounded Borel subsets of $\mathbb{R}^2$ are dense in $L^1(\mathbb{R}^2, d^2\mathbf{x})$, it follows that the condition for the existence of a limit dynamics, i.e.

$$\forall f \in C^0(X), \quad \forall l \in L^1(X, d\mu) \qquad \exists \lim_n \int \alpha_n^t f \, l \, d\mu \equiv \int \alpha^t f \, l \, d\mu$$

is equivalent to the existence of the limit

$$\exists \lim_n \int \chi_B(\mathbf{x}) \, g_{\sigma, n(\mathbf{x}-t), nt, n^{-1}h}(\xi) \, d^2\mathbf{x} \ . \tag{4.3}$$

$\forall B$ bounded Borel subset of $\mathbb{R}^2$, and $\sigma = \pm 1$.

Moreover, calling $G_{\sigma,\mathbf{x},t,h}(\xi)$ the $\tau_{\mathcal{F}}$ limit of $g_{\sigma,n\mathbf{x},nt,n^{-1}h}(\xi)$, for the limit dynamics α^t we may write

$$\sum_{\sigma=\pm 1} \int \alpha^t f(\mathbf{x},\sigma) \, l(\mathbf{x},\sigma) \, d^2\mathbf{x} =$$

$$= \lim_n \sum_{\tau=\pm 1} \sum_{\sigma=\pm 1} \int g_{\sigma_1, n(\mathbf{x}-t), nt, n^{-1}h}(\xi) \, f(\mathbf{x}-t, \tau\cdot\sigma) \, l(\mathbf{x},\sigma) \, d^2\mathbf{x} =$$

$$= \sum_{\tau=\pm 1} \sum_{\sigma=\pm 1} \int G_{\sigma_1, \mathbf{x}-t, t, h}(\xi) \, f(\mathbf{x}-t, \tau\cdot\sigma) \, l(\mathbf{x},\sigma) \, d^2\mathbf{x} \ .$$

In the same way it turns out that the existence of a limit for the measurement of k observables at k successive times $0 \le t_1 < t_2 < \ldots < t_k$ is equivalent to $\tau_{\mathcal{F}}$

convergence of the products $\prod_{i=1}^k g_{\sigma_i,n(\mathbf{x}-t_i),nt_i,n^{-1}h}(\xi)$. Moreover, denoting $t_0 \equiv 0$ and $\rho_i \equiv \prod_{i\le j}\sigma_j$, it follows

$$\prod_{i=1}^k g_{\sigma_i,n(\mathbf{x}-t_i),n(t_i-t_{i-1}),n^{-1}h}(\xi) = \prod_{i=1}^k g_{\rho_i,n(\mathbf{x}-t_i),nt_i,n^{-1}h}(\xi) \quad ,$$

and we may write the request of convergence in a form that will be more suitable to demonstrate the Markov time independent property for the limit stochastic process:

$$\forall B \text{ bounded Borel set } \subset \mathbb{R}^2, \forall k = 1,2,\ldots\ , \ \forall\sigma_i = \pm 1 \quad , \quad i = 1,\ldots,k$$

$$\exists \lim_n \int \chi_B(\mathbf{x}) \prod_{i=1}^k g_{\sigma,n(\mathbf{x}-t_i),n(t_i-t_{i-1}),n^{-1}h}(\xi)\, d^2\mathbf{x} \quad . \tag{4.4}$$

Now, from the construction of ρ it follows that the characteristic functions of the sets $\{\mathbf{x}\in\mathbb{R}^2 \,:\, |\xi\cap B_{\mathbf{x},t,h}| = n\}$, denoted by $\tilde g_{n,\mathbf{x},t,h}(\xi)$, are in $L^1(\Xi,\rho)$, and their integral is given by

$$\int_\Xi \tilde g_{n,\mathbf{x},t,h}(\xi)\, d\rho(\xi) = \sum_{j=0}^\infty \frac{e^{-|\Lambda|}}{j!}\int_{\Lambda^j} d\mathbf{x}_1\ldots d\mathbf{x}_j\, \tilde g_{n,\mathbf{x},t,h}(\xi) = e^{-th}\frac{|th|^n}{n!} \quad .$$

Moreover, if $\forall i\neq j\ \ B_{\mathbf{x}_i,t_i,h_i}\cap B_{\mathbf{x}_j,t_j,h_j} = \emptyset$, then

$$\int_\Xi \prod_{i=1}^k \tilde g_{n_i,\mathbf{x}_i,t_i,h_i}(\xi)\, d\rho(\xi) = \prod_{i=1}^k \int_\Xi \tilde g_{n_i,\mathbf{x}_i,t_i,h_i}(\xi)\, d\rho(\xi)$$

(which expresse the fact that the positions of the rods of ρ-almost each $\xi\in\Xi$ are uncorrelated).

Expressing the functions $g_{\sigma,\mathbf{x},t,h}$ in terms of the $\tilde g_{n,\mathbf{x},t,h}$, we obtain

$$\int_\Xi g_{\pm 1,\mathbf{x},t,h}(\xi)\, d\rho(\xi) = \frac{1\pm e^{-2|t|h}}{2}$$

and for $B_{\mathbf{x}_i,t_i,h_i}\cap B_{\mathbf{x}_j,t_j,h_j} = \emptyset$

$$\int_\Xi \prod_{i=1}^k g_{\sigma_i,\mathbf{x}_i,t_i,h_i}(\xi)\, d\rho(\xi) = \prod_{i=1}^k \int_\Xi g_{\sigma_i,\mathbf{x}_i,t_i,h_i}(\xi)\, d\rho(\xi) = \prod_{i=1}^k \frac{1+\sigma_i\cdot e^{-2|t_i|h_i}}{2} \quad .$$

The aim is now to show that the conditions in eqs. (4.3) and (4.4) are fulfilled for ρ-almost all $\xi\in\Xi$. We will show that those limits exist in the $L^1(\Xi,\rho)$ norm, and hence pointwise ρ-almost everywhere on Ξ for a subsequence.

Consider for simplicity the single time case: $\forall B$ bounded Borel $\subset \mathbb{R}^2$:

$$\int_\Xi d\rho(\xi) \left| \int_B d^2\mathbf{x}\, g_{\sigma,n(\mathbf{x}-t),nt,n^{-1}h}(\xi) \;-\; |B|\, \frac{1+\sigma\cdot e^{-2|t|h}}{2} \right|^2 \;=$$

$$= \int_\Xi d\rho(\xi) \int_B d^2\mathbf{x}\, g_{\sigma,n(\mathbf{x}-t),nt,n^{-1}h}(\xi) \int_B d^2\mathbf{x}'\, g_{\sigma,n(\mathbf{x}'-t),nt,n^{-1}h}(\xi) +$$

$$- \int_\Xi d\rho(\xi)\, (|B|\, \frac{1+\sigma\cdot e^{-2|t|h}}{2})^2$$

where we applied Fubini's theorem to g, which is measurable over $(\mathbb{R}^2 \times \Xi, d^2\mathbf{x} \otimes d\rho)$, since it is pointwise approximated by a sequence of functions of the form

$$\sum_{\text{finite}} \phi_i(\mathbf{x}) \cdot \chi_i(\xi)$$

with $\phi_i(\mathbf{x})$ and $\chi_i(\xi)$ measurable.

Since the points of almost all ξ are uncorrelated, in the above integration no contribution comes from the set

$$Q \equiv \{(\mathbf{x},\mathbf{x}') \in B \times B \,|\, n|y-y'| \geq n^{-1}h\} \;\; .$$

As $n \to \infty$, Q becomes all of $B \times B$: more precisely, taken a rectangle in $\mathbb{R}^2$ of base Δ_1 and height Δ_2 which contains B, it is

$$|(B \times B) \setminus Q| \leq \int_B d^2\mathbf{x}\, \Delta_1 \int_{y-n^{-2}h}^{y+n^{-2}h} dy' \;\leq\; n^{-2}\, 2h\, \Delta_1^2\, \Delta_2$$

and therefore, since $\|g\|_\infty = 1$, we conclude

$$\int_\Xi d\rho(\xi) \left| \int_B d^2\mathbf{x}\, g_{\sigma,n(\mathbf{x}-t),nt,n^{-1}h}(\xi) \;-\; |B|\, \frac{1+\sigma\cdot e^{-2|t|h}}{2} \right|^2 \;\leq\; n^{-2}\, 2h\, \Delta_1^2\, \Delta_2 \quad . \tag{4.5}$$

The same reasoning and the same inequality apply for functions of the form

$$\prod_{i=1}^{k} g_{\sigma_i,n(\mathbf{x}-t_i),n(t_i-t_{i-1}),n^{-1}h}(\xi) \quad , \text{ with } \; t_0 = 0 < t_1 < \ldots < t_k \quad ,$$

which describe all measurements at k successive times. Moreover, since the g's in the above product depend on ξ only through points in disjoint sets, and since the points of almost all the ξ's are uncorrelated, it follows that

$$\int_B d^2\mathbf{x} \prod_{i=1}^{k} g_{\sigma_i, n(\mathbf{x}-t_i), n(t_i-t_{i-1}), n^{-1}h}(\xi)$$

converges, for the moment in $\|\ \|_2$ over (Ξ, ρ), to the product of the limits, i.e. to

$$|B| \prod_{i=1}^{k} \frac{1+\sigma_i \cdot e^{-2|t_i|h}}{2} \quad .$$

Since $\rho(\Xi) = 1$, convergence also takes palce in the L^1 norm. Now, if a sequence $\{f_j\}$ converges in $\|\ \|_1$ to f and $\sum_{j=1}^{\infty} \|f_j - f_{j-1}\|_1 \leq \infty$, then f_j converge pointwise a.e. to f ([26], vol.1, pag.18), and in our case it is sufficient to take a subsequence with $n_j = 2^j$ to assure ρ–a.e. pointwise convergence.

We can thus state that for ρ-almost all $\xi \in \Xi$, for any denumerable family $\mathcal{R}$ of bounded Borel sets $B \subset \mathbb{R}^2$, $\forall k = 1, 2, \ldots$, $\forall \rho_i = \pm 1$, $i = 1, \ldots, k$, $\forall t_0 = 0 < t_1 < \ldots < t_k$ rational it is

$$\lim_n \int_B d^2\mathbf{x} \prod_{i=1}^{k} g_{\sigma_i, 2^n(\mathbf{x}-t_i), 2^n(t_i-t_{i-1}), 2^{-n}h}(\xi) = |B| \prod_{i=1}^{k} \frac{1+\sigma_i \cdot e^{-2|t_i-t_{i-1}|h}}{2} \quad .$$

By taking $\mathcal{R}$ such that the linear combinations of characteristic functions χ_B are dense in $L^1(\mathbb{R}^2, d^2\mathbf{x})$, and using the $\|\ \|_\infty$ boundedness of the integrand, uniform in n, we extend convergence to any B bounded Borel $\subset \mathbb{R}^2$, which is the condition of existence of a limit time evolution (the case $k = 1$), and in general of a limit stochastic process, with the restriction of the times t_i to a denumerable subset of $\mathbb{R}$.

In particular, the limit dynamics is given by

$$\alpha^t f(\mathbf{x}, \sigma) = \sum_{\rho=\pm 1} f(\mathbf{x} - t, \rho \cdot \sigma) \frac{1 + \rho\, e^{-2|t|h}}{2} \tag{4.6}$$

for almost all configurations of rods and a denumerable set of times t.

The convergence of $\prod_{i=1}^{k} g$ to $\prod_{i=1}^{k} G$ implies that measurements at k successive times define a Markov and time independent process, with transition function

$$P_t((\mathbf{x}, \sigma), B) = \alpha^t \chi_B(\mathbf{x}, \sigma) \qquad ;$$

in fact,

$$\alpha_n^t f(\mathbf{x}, \sigma) = \sum_{\rho=\pm 1} f(\mathbf{x} - t, \rho \cdot \sigma) g_{\rho, n(\mathbf{x}-t), nt, n^{-1}h}(\xi)$$

so that

$$\alpha_{2^n}^{t_1}(f_1\alpha_{2^n}^{t_2-t_1}(f_2\alpha_{2^n}^{t_3-t_2}\ldots(f_{k-1}\alpha_{2^n}^{t_k-t_{k-1}}f_k)\ldots))(\mathbf{x},\sigma)=$$

$$=\prod_{i=1}^{k}\sum_{\rho_i=\pm 1} f_i(\mathbf{x}-t_i,\sigma\cdot\prod_{j=1}^{i}\rho_j)\ g_{\rho_i,2^n(\mathbf{x}-t_i),2^n(t_i-t_{i-1}),2^{-n}h}(\xi)$$

which $\tau_{\mathcal{F}}$ converges as $n\to\infty$ to

$$\prod_{i=1}^{k}\sum_{\rho_i=\pm 1} f_i(\mathbf{x}-t_i,\sigma\cdot\prod_{j=1}^{i}\rho_j)\ \frac{1+\rho_i e^{-2(t_i-t_{i-1})h}}{2}=$$

$$=\alpha^{t_1}(f_1\alpha^{t_2-t_1}(f_2\alpha^{t_3-t_2}\ldots(f_{k-1}\alpha^{t_k-t_{k-1}}f_k)\ldots))(\mathbf{x},\sigma)\quad , \tag{4.7}$$

which is the condition for the stochastic process to be Markov and time independent. The process is defined by the l.h.s. of eq.(4.7) for all positive times, and the restriction to a denumerable set of times is only involved in the construction of a fixed set of configurations of rods (actually of ρ–measure one) for which weak convergence holds.

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