DECAY OF CORRELATIONS FOR PIECEWISE EXPANDING MAPS

CARLANGELO LIVERANI

University of Rome *Tor Vergata*

ABSTRACT. This paper investigates the decay of correlations in a large class of non-Markov one-dimensional expanding maps. The method employed is a special version of a general approach recently proposed by the author. Explicit bounds on the rate of decay of correlations are obtained.

CONTENT

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§0 INTRODUCTION.

In this paper we apply a new technique–introduced in [L] to study the decay of correlations in hyperbolic systems–to expanding one-dimensional non-Markov maps (see section 2 for a precise definition). A main feature of such a method is the possibility of obtaining explicit bounds on the rate of decay.

The standard approach to this problem[1] is to study the Perron–Frobenius (P–F) operator via Ionescu-Tulcea and Marinescu type spectral theorems [ITM]. Unfortunately, this suffices to prove exponential decay of correlations but does not provide any constructive bound on the rate of decay. One can hope to obtain some bounds using the theory of Ruelle zeta functions [Ru], but it is not clear in which generality this can be accomplished.

Here, we will study the P-F operator as well, but from a different point of view: we will see that there exists a convex cone of functions that is mapped "strictly" inside itself by the P-F operator. We will then be able to take advantage of an idea by Garrett Birkhoff: he showed that one can associate an Hilbert metric to the above mentioned cone and that such a metric is contracted by the P-F operator. Once we obtain a contraction all the wanted consequences are the result of straightforward arguments.

The possibility to obtain bounds on the rate of decay was recently addressed also by P. Collet [Co]. He used techniques developed for the study of markov chains [DS]. Here we show that the approach proposed in [L] is more direct, produces substantially better bounds for the class of maps studied in [Co], and we apply it to a larger class of maps.

If T is a map of the type under consideration (see §2) and μ is the associated invariant probability measure, absolutely continuous with respect to the Lebesgue measure (i.e., there exists $\phi \in L^1([0,1])$ such that $d\mu = \phi dx$), then I prove the following:

Theorem 0.1. *If $\inf_{x\in[0,1]} \phi(x) \geq \gamma > 0$ and the system (T, μ) is mixing, then there exists b, $K > 0$, $\Lambda \in (0,1)$ such that, for each function $f \in L^1([0,1])$ and $g \in BV$ (the space of function of bounded variation), $\int_0^1 g = 1$,*

$$\left| \int_0^1 g f \circ T^n - \int_0^1 f d\mu \right| \leq K\Lambda^{-n} \|f\|_1 \left(1 + b \bigvee_0^1 g\right).$$

In addition, it is possible to state a concrete condition that insures $\inf \phi \geq \gamma > 0$ (see Lemma 4.2), and, most of all, I will obtain explicit estimates for the constants K, b, Λ (although they may not be optimal).

The paper is organized as follows: in section 1 I provide a brief review of some basic facts on which the subsequent arguments rest. Section 2 contains a detailed description of the class of maps under consideration, and it presents the main ingredient in the proof of Theorem 0.1: a cone of functions invariant under the action of

[1] See [LY] where it is proven the existence of an invariant measure absolutely continuous with respect to Lebesgue and [HK] where the decay of correlations is investigated.

the Perron-Frobenius operator. In section 3 the bound on the decay of correlations is proven, but under apparently stronger hypotheses than the ones in the statement of Theorem 0.1. In section 4 I show that the hypotheses of Theorem 0.1 imply the assumptions used in section 3, whereby concluding the argument. As a byproduct, I also answer to some questions, concerning general properties of expanding maps, recently raised by P. Collet [Co]. Finally, the paper includes an appendix where I give numerical bounds, for all the constants involved, in a concrete example; whereby emphasizing the constructive nature of the present approach.

§1 OPERATORS AND INVARIANT CONVEX CONES.

This section illustrates some results in lattice theory originally due to Garrett Birkhoff [B2]. More details and the proofs of the following results can be found in [L].

Consider a topological vector space $\mathbb{V}$ with a partial ordering "$\preceq$," that is a vector lattice.[2] We require the partial order to be "continuous," i.e. given $\{f_n\} \in \mathbb{V}$ $\lim_{n\to\infty} f_n = f$, if $f_n \succeq g$ for each n, then $f \succeq g$. We call such vector lattices "integrally closed."[3]

We define the closed convex cone[4] $\mathcal{C} = \{f \in \mathbb{V} \mid f \neq 0,\ f \succeq 0\}$ (hereafter, the term "closed cone" $\mathcal{C}$ will mean that $\mathcal{C} \cup \{0\}$ is closed), and the equivalence relation "$\sim$": $f \sim g$ iff there exists $\lambda \in \mathbb{R}^+\backslash\{0\}$ such that $f = \lambda g$. If we call $\widetilde{\mathcal{C}}$ the quotient of $\mathcal{C}$ with respect to $\sim$, then $\widetilde{\mathcal{C}}$ is a closed convex set. Conversely, given a closed convex cone $\mathcal{C} \subset \mathbb{V}$, enjoying the property $\mathcal{C} \cap -\mathcal{C} = \emptyset$, we can define an order relation by

$$f \preceq g \iff g - f \in \mathcal{C} \cup \{0\}.$$

Henceforth, each time that we specify a convex cone we will assume the corresponding order relation and vice versa.

It is then possible to define a projective metric Θ (Hilbert metric),[5] in $\mathcal{C}$, by the construction:

$$\begin{aligned}
\alpha(f,\, g) &= \sup\{\lambda \in \mathbb{R}^+ \mid \lambda f \preceq g\} \\
\beta(f,\, g) &= \inf\{\mu \in \mathbb{R}^+ \mid g \preceq \mu f\} \\
\Theta(f,\, g) &= \log\left[\frac{\beta(f,\, g)}{\alpha(f,\, g)}\right]
\end{aligned}$$

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[2] We are assuming the partial order to be well behaved with respect to the algebraic structure: for each $f, g \in \mathbb{V}$ $f \succeq g \iff f - g \succeq 0$; for each $f \in \mathbb{V}$, $\lambda \in \mathbb{R}^+\backslash\{0\}$ $f \succeq 0 \Longrightarrow \lambda f \succeq 0$; for each $f \in \mathbb{V}$ $f \succeq 0$ and $f \preceq 0$ imply $f = 0$ (antisymmetry of the order relation).

[3] To be precise, in the literature "integrally closed" is used in a weaker sense. First, $\mathbb{V}$ does not need a topology. Second, it suffices that for $\{\alpha_n\} \in \mathbb{R}$, $\alpha_n \to \alpha$; $f, g \in \mathbb{V}$, if $\alpha_n f \succeq g$, then $\alpha f \succeq g$. Here we will ignore these and other subtleties: our task is limited to a brief account of the results relevant to the present context.

[4] Here, by "cone," we mean any set such that, if f belongs to the set, then λf belongs to it as well, for each $\lambda > 0$.

[5] In fact, we define a semi–metric, since $f \sim g \Rightarrow \Theta(f, g) = 0$. The metric that we describe corresponds to the conventional Hilbert metric on $\widetilde{\mathcal{C}}$.

where we take $\alpha = 0$ and $\beta = \infty$ if the corresponding sets are empty.

The importance of the previous constructions is due, in our context, to the following theorem.

Theorem 1.1. *Let $\mathbb{V}_1$, and $\mathbb{V}_2$ be two integrally closed vector lattices; $T : \mathbb{V}_1 \to \mathbb{V}_2$ a linear map such that $T(\mathcal{C}_1) \subset \mathcal{C}_2$, for the two corresponding closed convex cones $\mathcal{C}_1 \subset \mathbb{V}_1$ and $\mathcal{C}_2 \subset \mathbb{V}_2$. Let Θ_i be the Hilbert metric corresponding to the cone $\mathcal{C}_i$. Setting $\Delta = \sup_{f,\, g \in T(\mathcal{C}_1)} \Theta_2(f,\, g)$ we have*

$$\Theta_2(Tf,\, Tg) \leq \tanh\left(\frac{\Delta}{4}\right) \Theta_1(f,\, g) \qquad \forall f,\, g \in \mathcal{C}_1$$

$(\tanh(\infty) = 1)$.

Remark 1.2. *If $T(\mathcal{C}_1) \subset \mathcal{C}_2$ then it follows that $\Theta_2(Tf,\, Tg) \leq \Theta_1(f,\, g)$. However, a uniform rate of contraction depends on the diameter of the image being finite.*

In particular, if an operator maps a convex cone strictly inside itself (in the sense that the diameter of the image is finite), then it is a contraction in the Hilbert metric. This implies the existence of a “positive” eigenfunction (provided the cone is complete with respect to the Hilbert metric), and, with some additional work, the existence of a gap in the spectrum of T (see [B1] for details). The relevance of this theorem for the study of invariant measures and their ergodic properties is obvious.

It is natural to wonder about the strength of the Hilbert metric compared to other, more usual, metrics. While, in general, the answer depends on the cone, it is nevertheless possible to state an interesting result.

Lemma 1.3. *Let $\|\cdot\|$ be a norm on the vector lattice $\mathbb{V}$, and suppose that, for each $f,\, g \in \mathbb{V}$,*

$$-f \preceq g \preceq f \Longrightarrow \|f\| \geq \|g\|.$$

Then, given $f,\, g \in \mathcal{C} \subset \mathbb{V}$ for which $\|f\| = \|g\|$,

$$\|f - g\| \leq \left(e^{\Theta(f,\, g)} - 1\right) \|f\|.$$

§2 THE MAP AND AN INVARIANT CONE.

We consider a map T from the interval $[0,\, 1]$ into itself.

We assume that there exists a partition (mod–0, with respect to the Lebesgue measure) $\mathcal{A}_0$ of $[0,\, 1]$ into open intervals such that, for each interval $I \in \mathcal{A}_0$, the map T, restricted to I, can be extended to a C^2 map on an open interval containing the closure of I. In addition, we assume that $|DT| \geq \lambda > 1$ (expansivity).

The simplest examples, in this class of maps, occur when, for each $I \in \mathcal{A}_0$, TI is equal, mod-0, to the union of elements of $\mathcal{A}_0$. This case is called “Markov case,” and it is well understood in the literature; in particular, explicit bounds on the rate of decay of correlations are available [F], [H], [L]. If the above mentioned property fails, the map is called "non–Markov," this is the case addressed here.

Thanks to the work of Lasota and Yorke [LY] it is known that any piecewise smooth expanding map has at least one invariant measure absolutely continuous with respect to Lebesgue.[6] We will call ϕ the density, with respect to the Lebesgue measure, of such an invariant measure μ, and assume that the dynamical systems $(T,\, \mu)$ is mixing.

The results of [HK] and [BK] imply that the class of maps under consideration exhibits exponential decay of correlations. But, no explicit bound on the rate of decay is available in such a generality.

A first technical obstacle is that the set $\{x \in [0,\, 1] \mid \phi(x) = 0\}$ may have positive Lebesgue measure. This is a concrete possibility and it is very easy to construct examples with this property. The simplest case of this behavior occurs when there exists an attracting set; clearly such a set supports all the invariant measures in the class under consideration (i.e., measures that are absolutely continuous with respect to the Lebesgue measure). We will assume that this is not the case. In principle, such an assumption does not imply any loss of generality: we can alway study the restriction of the map to an invariant set.[7] Yet, some more complex situations may be possible, we are therefore forced to further reduce the class of maps under consideration.

On the one hand, we assume that $|DT| \geq \lambda > 2$ (here no loss of generality is implied: this condition can always be satisfied replacing T by an appropriate power). On the other hand we ask that there exists $\gamma > 0$ such that $\inf_{x\in[0,\,1]} \phi(x) \geq \gamma$.

To study the decay of correlations it is useful to introduce the Perron-Frobenius (P-F) operator $\widetilde{T}$. Such operator is defined by the relation[8]

$$\int gf \circ T = \int f\widetilde{T}g$$

for each $f \in L^1([0,\, 1])$ and $g \in L^\infty([0,\, 1])$. A direct computation shows that

$$\widetilde{T}g(x) = \sum_{y\in T^{-1}(x)} g(y)|D_yT|^{-1}.$$

We will show that the P-F operator leaves invariant a cone of functions; the decay of the correlations will then follow from the theory discussed in §1.

Let BV be the space of functions of bounded variation on $[0,\, 1]$. The cones that we will use in this paper are:

$$\mathcal{C}_a = \left\{ g \in BV \;\middle|\; g(x) \geq 0 \;\forall x \in [0,\, 1];\; \bigvee_0^1 g \leq a \int_0^1 g \right\},$$

[6] See [BK] for a generalization of such a result.

[7] In fact, there is nothing sacred about the interval $[0,\, 1]$.

[8] If not otherwise stated, all the integrals are between 0 and 1, and taken with respect to the Lebesgue measure.

for $a > 0$.[9]

The main ingredient for studying the maps under consideration is the following inequality (due to Lasota and Yorke) [LY]: for each $g \in BV$

$$\bigvee_0^1 \widetilde{T} g \leq 2\lambda^{-1} \bigvee_0^1 g + A \int_0^1 |g| \tag{2.1}$$

where $A = \lambda^{-2} \|D^2 T\|_\infty + 2b_0^{-1}\lambda^{-1}$; $b_0 = \inf_{I \in \mathcal{A}_0} |I|$.[10] The relevance of the cones $\mathcal{C}_a$ and the inequality (2.1) are exemplified by the following result.

Lemma 2.1. *For each $a > \frac{A}{1-2\lambda^{-1}}$, there exists $\sigma < 1$ such that*

$$\widetilde{T}\mathcal{C}_a \subset \mathcal{C}_{\sigma a}.$$

Proof. For each $g \in \mathcal{C}_a$, inequality (2.1) yields

$$\bigvee_0^1 \widetilde{T} g \leq 2\lambda^{-1} \bigvee_0^1 g + A \int_0^1 g \leq (2\lambda^{-1} a + A) \int_0^1 g.$$

The result follows by choosing $a = A(\sigma - 2\lambda^{-1})^{-1}$, and noticing that $\int g = \int \widetilde{T} g$.

§3 DECAY OF CORRELATIONS.

In the previous section we have found a cone of functions that is left invariant by the P-F operator. This it is not quite enough to obtain a contraction in the corresponding Hilber metric: the diameter of the image must also be investigated. The nature of the problem is elucidated by the following Lemma.

Lemma 3.1. *Calling Θ_a the Hilbert metric associated to the cone $\mathcal{C}_a$, for each $\nu < 1$, and $g \in \mathcal{C}_{\nu a}$*

$$\Theta_a(g, 1) \leq \ln \left[\frac{\max\left\{ (1+\nu)\int_0^1 g;\ \sup_{x\in[0,1]} g(x) \right\}}{\min\left\{ (1-\nu)\int_0^1 g;\ \inf_{x\in[0,1]} g(x) \right\}} \right].$$

[9] By $\bigvee_0^1 g$ we mean the variation of the function g in the interval $[0, 1]$.

[10] The alert reader has certainly noticed that we have defined $\widetilde{T}$ only on $L^\infty([0, 1])$, to define it on BV it is necessary to specify the value of $\widetilde{T} g$ at all points, that is also at the boundaries of the intervals belonging to the partition $\mathcal{A}_0$. In fact, we want to define all the powers of $\widetilde{T}$ as well, so we may be in trouble at countably many points (all the preimages of the boundaries of the partition). We will not worry about such points since they can be consistently ignored (loosely speaking, the value of $\widetilde{T} g$ at such points can be defined by taking left and/or right limits–see [HK] or [BK] for details).

Proof. We have to find the set of λ and μ such that $\lambda 1 \preceq g \preceq \mu 1$. Let us start with the first inequality; it is satisfied iff $g - \lambda \in \mathcal{C}_a$, i.e.

$$\lambda \leq g(x) \qquad \forall x \in [0,\, 1]$$

and

$$\lambda \leq \int_0^1 g - a^{-1} \bigvee_0^1 g.$$

Consequently,

$$\alpha = \sup \lambda = \min \left\{ \inf_{x \in [0,\, 1]} g(x);\; \int_0^1 g - a^{-1} \bigvee_0^1 g \right\},$$

see §1 for a definition of α and β in the present context. Since $g \in \mathcal{C}_{\nu a}$, it follows

$$\int_0^1 g - a^{-1} \bigvee_0^1 g \geq (1 - \nu) \int_0^1 g,$$

that is $\alpha \geq \min\{(1-\nu) \int g,\, \inf g\}$. Analogously, one can compute $\beta \leq \max\{(1 + \nu) \int g,\, \sup g\}$.

Note that, up to now, we do not have any control on the inf of a function belonging to our cones, therefore the above Lemma shows not only that more work is needed, but also in which direction to concentrate our efforts.

The first step is to notice that, if a function belongs to the cone $\mathcal{C}_a$, then it cannot be small too often. This is made precise by the following.

Lemma 3.2. *Given a partition, mod-0, $\mathcal{P}$ of $[0,\, 1]$, if each $p \in \mathcal{P}$ is a connected interval with Lebesgue measure less than $\frac{1}{2a}$, then, for each $g \in \mathcal{C}_a$, there exists $p_0 \in \mathcal{P}$ such that*

$$g(x) \geq \frac{1}{2} \int_0^1 g \quad \forall x \in p_0.$$

Proof. Consider the set

$$\mathcal{P}_- = \left\{ p \in \mathcal{P} \;\middle|\; \exists x_p \in p \,:\, g(x_p) < \frac{1}{2} \int_0^1 g \right\}.$$

Clearly the Lemma is proven if we show that $\mathcal{P}_- \neq \mathcal{P}$. Let us suppose the contrary. For

$$\int_p g \leq m(p) \left(g(x_p) + \bigvee_p g \right) < \frac{m(p)}{2} \int_0^1 g + \frac{1}{2a} \bigvee_p g,$$

and remembering that $g \in \mathcal{C}_a$, we have

$$\int_0^1 g < \frac{1}{2} \int_0^1 g + \frac{1}{2a} \bigvee_0^1 g \leq \int_0^1 g$$

which is a contradiction.

To continue, we define a particular class of partitions:

Definition 3.3. *For each $n \in \mathbb{N}$,*

$$\mathcal{A}_n = \bigvee_{j=0}^{n} T^{-j}\mathcal{A}_0.$$

It is immediately clear that T^{n+1} is monotone and regular on each element of the partition $\mathcal{A}_n$ (in fact, this could be used as an alternative definition of $\mathcal{A}_n$); moreover, $\mathcal{A}_n$ consists of intervals with Lebesgue measure smaller than λ^{-n}.

We can then choose n_0 such that all the elements of $\mathcal{A}_{n_0}$ have measure less than $\frac{1}{2a}$ (for example $n_0 = \left[\frac{\ln 2a}{\ln\lambda}\right] + 1$ would do).

Definition 3.4. *We call a map "covering" if, for each $n \in \mathbb{N}$ there exists $N(n)$ such that, for each $I \in \mathcal{A}_n$,*[11]

$$T^{N(n)}I = [0,\, 1].$$

The above property corresponds to condition $(H2)$ in [Co]. The importance of the notion of "covering" is emphasized by the following Lemma.

Lemma 3.5. *If the map T is covering, then, for each $a > \frac{A}{1-2\lambda^{-1}}$,*

$$diam(\widetilde{T}^{N(n_0)}\mathcal{C}_a) \le \Delta < \infty.$$

Proof. Let $g \in \mathcal{C}_a$, then, according to Lemma 3.2, there exists $I_0 \in \mathcal{A}_{n_0}$ such that $g(x) \ge \frac{1}{2}\int_0^1 g$ for each $x \in I_0$. By the covering property, for each $x \in [0,\, 1]$ (apart, at most, finitely many points) there exists $y \in I_0$ such that $T^{N(n_0)}y = x$; hence

$$\left(\widetilde{T}^{N(n_0)}g\right)(x) = \sum_{y \in T^{-N(n_0)}x} g(y)|D_yT^{N(n_0)}|^{-1} \ge \frac{\int_0^1 g}{2\|DT\|_\infty^{N(n_0)}}.$$

According to Lemma 2.1, $\widetilde{T}^{N(n_0)}\mathcal{C}_a \subset \mathcal{C}_{\sigma_1 a}$, with[12]

$$\sigma_1 = (2\lambda^{-1})^{N(n_0)} + \frac{1-(2\lambda^{-1})^{N(n_0)}}{1-2\lambda^{-1}}Aa^{-1}.$$

Let $\delta(g) = \frac{\inf \widetilde{T}^{N(n_0)}g}{\int g}$, up to now we have seen that $\delta(g) \ge (2\|DT\|^{N(n_0)})^{-1}$ for all $g \in \mathcal{C}_a$. Using Lemma 3.1, we can then estimate

$$\begin{aligned}
\operatorname{diam}\left(\widetilde{T}^{N(n_0)}\mathcal{C}_a\right) &\le 2\sup_{g\in\mathcal{C}_a} \ln\left[\frac{\max\left\{(1+\sigma_1)\int g;\ \inf \widetilde{T}^{N(n_0)}g + \bigvee_0^1 \widetilde{T}^{N(n_0)}g\right\}}{\min\left\{(1-\sigma_1)\int g;\ \inf\widetilde{T}^{N(n_0)}g\right\}}\right] \\
&\le 2\sup_{g\in\mathcal{C}_a}\ln\left[\frac{\max\{(1+\sigma_1);\ \delta(g)+a\sigma_1\}}{\min\{(1-\sigma_1);\ \delta(g)\}}\right] \\
&\le 2\ln\left[\frac{\max\{(1+\sigma_1);\ 1+a\sigma_1\}}{\min\left\{(1-\sigma_1);\ (2\|DT\|^{N(n_0)})^{-1}\right\}}\right].
\end{aligned}$$

[11]We have already remarked that what happens at the boundaries of the partitions is immaterial. In the same vein, each time that we write an equality between sets we always mean it apart a finite number of points.

[12]To obtain the following formula it is enough to notice that, iterating (2.1), for each $k > 0$, holds $\bigvee_0^1 \widetilde{T}^k g \le (2\lambda^{-1})^k \bigvee_0^1 g + \frac{1-(2\lambda^{-1})^k}{1-2\lambda^{-1}}A\int_0^1 g$; apply then Lemma 2.1 directly to $\widetilde{T}^{N(n_0)}$.

The above Lemma, together with the results of §1, implies exponential decay of the correlations for covering maps.

Theorem 3.6. *If T is covering, then, for each $f \in L^1([0,\,1])$ and $g \in BV$, $\int_0^1 g = 1$,*

$$\left|\int_0^1 gf\circ T - \int_0^1 f d\mu\right| \le K_n \Lambda^n \|f\|_1 \left(1 + b\bigvee_0^1 g\right),$$

with $\Lambda = \tanh(\frac{\Delta}{4})^{\frac{1}{N(n_0)}}$, $K_n = e^{\Lambda^{n-N(n_0)}\Delta}\Lambda^{-N(n_0)}\Delta\|\phi\|_\infty$ and $b = (a - B)^{-1}$ ($B = \frac{A}{1-2\lambda^{-1}}$).

Proof. Consider $g \in \mathcal{C}_a$, $a > \frac{A}{1-2\lambda^{-1}}$, normalized so that $\int_0^1 g = 1$ (i.e., g can be thought as the density of a measure). Then,

$$\left|\int_0^1 f(\widetilde{T}^n g - \phi)dx\right| \le \|f\|_1 \left\|\frac{\widetilde{T}^n g}{\phi} - 1\right\|_\infty \|\phi\|_\infty.$$

Lemma 3.7. *If $\mathcal{C}_+ = \{g \in BV \mid g(x) \ge 0, \quad \forall x \in [0,\,1]\}$, and Θ_+ is the corresponding Hilbert metric, then, for each $g_1,\, g_2 \in \mathcal{C}_a$,*

$$\Theta_+(g_1,\, g_2) \le \Theta(g_1,\, g_2)$$

Proof. Since $\mathcal{C}_+ \supset \mathcal{C}_a$, the identity is a map from BV to itself that maps $\mathcal{C}_a$ into $\mathcal{C}_+$. The result follows then from Theorem 1.1.

A simple computation yields

$$\Theta_+(g_1,\, g_2) = \ln \sup_{x,\, y \in [0,\,1]} \frac{g_1(x)g_2(y)}{g_1(y)g_2(x)}.$$

Using the previous facts, and the trivial equality

$$\frac{(\widetilde{T}^n g)(x)}{\phi(x)} = \frac{\widetilde{T}^n g(x)\phi(y)}{\widetilde{T}^n g(y)\phi(x)}\frac{\widetilde{T}^n g(y)}{\phi(y)},$$

we have

$$e^{-\Theta_+(\widetilde{T}^n g,\,\phi)}\frac{\widetilde{T}^n g(y)}{\phi(y)} \le \frac{\widetilde{T}^n g(x)}{\phi(x)} \le e^{\Theta_+(\widetilde{T}^n g,\,\phi)}\frac{\widetilde{T}^n g(y)}{\phi(y)}$$

for each $x,\, y \in [0,\,1]$. Because $\int_0^1(\widetilde{T}^n g - \phi) = 0$, there must exist $y_n^+,\, y_n^- \in [0,\,1]$ such that $\widetilde{T}^n g(y_n^-) \le \phi(y_n^-)$ and $\widetilde{T}^n g(y_n^+) \ge \phi(y_n^+)$. Using the previous inequalities, with $y = y_n^-$ and $y = y_n^+$, respectively, we obtain, for each $x \in [0,\,1]$,

$$e^{-\Theta_+(\widetilde{T}^n g,\,\phi)} \le \frac{\widetilde{T}^n g(x)}{\phi(x)} \le e^{\Theta_+(\widetilde{T}^n g,\,\phi)}$$

and

$$\left\|\frac{\widetilde{T}^n g}{\phi} - 1\right\|_\infty \le e^{\Theta_+(\widetilde{T}^n g,\,\phi)} - 1 \le e^{\Theta(\widetilde{T}^n g,\,\phi)} - 1.$$

According to Theorem 1.1 and Lemma 3.5

$$\begin{aligned}\Theta(\widetilde{T}^n g,\, \phi) &\leq \Theta([\widetilde{T}^{N(n_0)}]^{[\frac{n}{N(n_0)}]} g,\, [\widetilde{T}^{N(n_0)}]^{[\frac{n}{N(n_0)}]}\phi) \\ &\leq \tanh(\frac{\Delta}{4})^{[\frac{n}{N(n_0)}]-1}\Theta(\widetilde{T}^{N(n_0)} g,\, \phi).\end{aligned}$$

Hence,

$$\left\| \frac{\widetilde{T}^n g}{\phi} - 1 \right\|_\infty \leq e^{\Lambda^{n-N(n_0)}\Theta(\widetilde{T}^{N(n_0)}g,\,\phi)} - 1 \leq e^{\Lambda^{n-N(n_0)}\Delta}\Delta\Lambda^{n-N(n_0)}.$$

This estimate shows that for each $f \in L^1([0,\, 1])$, $g \in \mathcal{C}_a$

$$\begin{aligned}\left| \int_0^1 f\circ T^n g dx - \int_0^1 f\phi dx \right| &\leq K_n \|f\|_1 \Lambda^n \\ K_n = e^{\Lambda^{n-N(n_0)}\Delta}\Delta\Lambda^{-N(n_0)}\|\phi\|_\infty.\end{aligned}$$

Let us now consider $g \in BV$, $g \geq 0$, and $\int_0^1 g = 1$. If $\bigvee_0^1 g \leq a$, we have the above estimate, otherwise we define $g_\rho = (g + \rho\phi)(1+\rho)^{-1}$, then $\int_0^1 g_\rho = 1$ and we have

$$\bigvee_0^1 g_\rho = \left[\bigvee_0^1 g + \rho \bigvee_0^1 \phi\right](1+\rho)^{-1}.$$

Iterating (2.1) one obtains $\bigvee_0^1 \phi \leq \frac{A}{1-2\lambda^{-1}} = B$, then

$$\bigvee_0^1 g_\rho \leq \left[\bigvee_0^1 g + \rho B\right](1+\rho)^{-1}$$

Choosing,

$$\rho = \frac{\bigvee_0^1 g - a}{a - B}$$

we have $g_\rho \in \mathcal{C}_a$; hence, the decay of correlations for g_ρ implies the decay of correlations for g. The result for arbitrary $g \in BV$ follows, since any function can be written as the difference of two positive functions.

In the next section we will see that, provided $\phi(x) > 0$, all the mixing maps are covering, whereby completing the proof of Theorem 0.1.

§4 GENERAL PROPERTIES OF EXPANDING MAPS.

In this section we address some questions concerning piecewise expanding maps posed by P. Collet in [Co]. We will see that the property that some image of any interval covers all [0, 1], is a quite general feature of mixing maps. This shows that the results obtained in the paper apply to a wide class of maps.

We start by giving a checkable criterion for the hypotheses of Theorem 0.1.

Definition 4.1. *We call a map "weakly–covering" if there exists $N_0 \in \mathbb{N}$ such that, for each $I \in \mathcal{A}_0$,*

$$\bigcup_{j=0}^{N_0} T^j I = [0,\, 1].$$

This is a weaker version of $(H1)$ in [Co]. The next Lemma shows that weak–covering is all is needed to insure that the first of the hypotheses of Theorem 0.1 holds.

Lemma 4.2. *If a map is weakly-covering, then there exists $\gamma > 0$ such that* $\inf \phi \geq \gamma$.

Proof. A consequence of weak-covering and expansivity is that the property of being weakly-covering does not depend substantially on the partition.

Sub–Lemma 4.3. *If a map is weakly-covering, then, for each $n \in \mathbb{N}$, there exists $N_0(n) \in \mathbb{N}$ such that, for each $I \in \mathcal{A}_n$,*

$$\bigcup_{j=0}^{N_0(n)} T^j I = [0,\, 1].$$

Proof. Let $I \in \mathcal{A}_n$, then T^n is smooth on I, accordingly $|T^n I| \geq \lambda^n |I|$. If $T^n I$ covers an element of $\mathcal{A}_0$ then the Lemma is proven; if not, since $T^n I$ is connected, it can intersect at most two elements of $\mathcal{A}_0$, so it is naturally broken in, at most, two pieces, let I_1 be the larger of the two, clearly $|I_1| \geq \lambda^n |I|/2$. Let $b_n = \inf\limits_{J \in \mathcal{A}_n} |J|$; then $|I_1| \geq \lambda^n \frac{b_n}{2}$.

We can then carry out a recursive argument: consider TI_1; by construction it is connected, intersect it with the elements of $\mathcal{A}_0$, either it will cover one element or it will be divided in, at most, two sub-intervals, call I_2 the larger one; consider TI_2, and so on. It follows that $|I_{k+1}| \geq \left(\frac{\lambda}{2}\right)^k |I_1|$, which implies that, eventually, I_k will cover an element of $\mathcal{A}_0$, this is all is needed to prove the Lemma.

By definition $\widetilde{T}\phi = \phi$, in addition, by Lemma 3.2 there exists $I_0 \in \mathcal{A}_{n_0}$ such that $\phi(x) \geq \frac{1}{2}$ for each $x \in I_0$. By Sub-Lemma 4.3, for each $x \in [0,\, 1]$ there exists $j \leq N_0(n_0)$ and $y_* \in I_0$ such that $T^j y_* = x$. Hence,

$$\phi(x) = \widetilde{T}^j \phi(x) = \sum_{y \in T^{-j}x} \phi(y) |D_y T^j|^{-1} \geq \phi(y_*) \|DT\|_\infty^{-j} \geq \frac{1}{2} \|DT\|_\infty^{-N_0(n_0)}.$$

The main result of this section is contained in the following Theorem.

Theorem 4.4. *If an expanding map is mixing and* $\inf \phi \geq \gamma > 0$, *then it is covering.*

Proof. For each interval $I \subset [0,\, 1]$, define

$$\chi_I(x) = \begin{cases} \dfrac{1}{|I|} & \text{for } x \in I \\ 0 & \text{for } x \notin I \end{cases}$$

then, $\bigvee_0^1 \chi_I \leq \frac{2}{|I|}$ and $\int_0^1 \chi_I = 1$. Given any two intervals I, I', the mixing property implies

$$\lim_{n\to\infty} \int_0^1 \chi_{I'} \chi_I \circ T^n = \int_0^1 \chi_I \phi \geq \gamma.$$

Consider some $n_1 \in \mathbb{N}$ (to be chosen later), there exists $N_* \in \mathbb{N}$ such that, for each I, $I' \in \mathcal{A}_{n_1}$

$$\int_0^1 \chi_I \widetilde{T}^n \chi_{I'} \geq \frac{\gamma}{2} \qquad \forall n \geq N_*.$$

Choose $I_0 \in \mathcal{A}_{n_1}$, consider $\widetilde{T}^n \chi_{I_0}$, from (2.1) follows

$$\bigvee_0^1 \widetilde{T}^n \chi_{I_0} \leq (2\lambda^{-1})^n \frac{2}{|I_0|} + A \sum_{i=0}^{n-1} (2\lambda^{-1})^i \int_0^1 \chi_{I_0} \leq (2\lambda^{-1})^n \frac{2}{|I_0|} + \frac{A}{1-2\lambda^{-1}}.$$

Let $b_n = \inf_{I\in\mathcal{A}_n} |I|$, choose $N_1 \geq N_*$ such that

$$\frac{2(2\lambda^{-1})^{N_1}}{b_{n_1}} \leq \frac{A}{1-2\lambda^{-1}}.$$

Then,

$$\bigvee_0^1 \widetilde{T}^n \chi_{I_0} \leq \frac{2A}{1-2\lambda^{-1}} \quad \forall n \geq N_1.$$

Consider the set $\mathcal{B}_- = \{I \in \mathcal{A}_{n_1} \mid \text{exists } x \in I : \widetilde{T}^{N_1} \chi_{I_0}(x) < \frac{\gamma}{4}\}$, and let $L = \#\mathcal{B}_-$.[13]

First of all, notice that for each $I \in \mathcal{A}_{n_1}$ there exists $y \in I$ such that $\widetilde{T}^{N_1}\chi_{I_0}(y) \geq \frac{\gamma}{2}$, if not

$$\int_0^1 \chi_I \widetilde{T}^{N_1} \chi_{I_0} < \frac{\gamma}{2} \int_0^1 \chi_I = \frac{\gamma}{2}$$

contrary to our assumptions on N_1.

Consequently, for each $I \in \mathcal{B}_-$,

$$\bigvee_I \widetilde{T}^{N_1} \chi_{I_0} \geq \frac{\gamma}{4},$$

which implies

$$L \leq \frac{8A}{(1-2\lambda^{-1})\gamma} \equiv L_0.$$

But, if $\#\{T^{-n}x\} \leq L_0$ we have

$$\gamma \leq \phi(x) = \sum_{y\in T^{-n}x} \phi(y) |D_y T^n|^{-1} \leq L_0 \|\phi\|_\infty \lambda^{-n}$$

[13] By # we mean the cardinality of a set.

which shows that, for $N_2 = \left[\frac{\ln L_0 \|\phi\|_\infty \gamma^{-1}}{\ln \lambda}\right] + 1$, $\#\{T^{-N_2}x\} > L_0$.

Choose $n_1 = N_2$; since $T^{-n_1}x$ has at most one point in each element of $\mathcal{A}_{n_1}$ and $\#\{T^{-n_1}x\} > L$, it follows that there exists $y \in T^{-n_1}x$ such that $y \in I \notin \mathcal{B}_-$; hence $T^{n_1+N_1}I_0 = [0, 1]$.

The statement is then proven by using the same reasoning employed in Sub-Lemma 4.3.

Summarizing, if a map is weakly-covering then, in force of Lemma 4.2, we have $\gamma > 0$ and, by Theorem 4.4, the map is covering. Hence, we can prove the exponential decay of correlations thanks to Theorem 3.6. We have proved Theorem 0.1.

APPENDIX (Constants).

This appendix contains explicit formulas for all the constants involved, therefore substantiating my claim that the present approach provides explicit bounds for the rate of decay of correlations.

Note that all the estimates, and the best choice of a, depend on the function $N(n)$. In an application $N(n)$ can be computed for the values one is interested in. Unfortunately, it is not clear how to derive a general bound for $N(n)$ since it contains global informations on the dynamics of the map.

To be concrete we will study an example.

Example

Let us consider the map

$$T(x) = \begin{cases} \frac{9}{2}\left(\frac{1}{9} - x\right) & x \in \left(0, \frac{1}{9}\right) \\ \frac{9}{2}\left(x - \frac{1}{9}\right) & x \in \left(\frac{1}{9}, \frac{3}{9}\right) \\ \frac{9}{2}\left(\frac{5}{9} - x\right) & x \in \left(\frac{3}{9}, \frac{5}{9}\right) \\ \frac{9}{2}\left(x - \frac{5}{9}\right) & x \in \left(\frac{5}{9}, \frac{7}{9}\right) \\ \frac{9}{2}(1 - x) & x \in \left(\frac{7}{9}, 1\right) \end{cases}$$

The partition on which T is defined is:

$$\mathcal{A}_0 = \left\{\left(0, \frac{1}{9}\right); \left(\frac{1}{9}, \frac{1}{3}\right); \left(\frac{1}{3}, \frac{5}{9}\right); \left(\frac{5}{9}, \frac{7}{9}\right); \left(\frac{7}{9}, 1\right)\right\}$$

The map satisfies our assumptions since $|DT| = \frac{9}{2} > 2$ and it is easy to check that it is covering (remember that, by Lemma 4.2 and Theorem 4.4, it suffice to check that T is weakly-covering).

The smaller element of the partition $\mathcal{A}_0$ has size $\frac{1}{9}$, accordingly $A = 4$ (see (2.1)). The choice of a is subject to the constraint $a > \frac{A}{1-2\lambda^{-1}} = \frac{36}{5}$ (see Lemma 2.1), while n_0 must satisfy $\sup_{I \in \mathcal{A}_{n_0}} |I| \leq \frac{1}{2a}$, it is then immediately clear that we must choose $n_0 = 1$.

The partition $\mathcal{A}_1$ is

$$\mathcal{A}_1 = \left\{ \left(0, \frac{1}{27}\right); \left(\frac{1}{27}, \frac{7}{81}\right); \left(\frac{7}{81}, \frac{1}{9}\right); \left(\frac{1}{9}, \frac{11}{81}\right); \left(\frac{11}{81}, \frac{15}{81}\right); \dots \right\}$$

Clearly $b_1 = \frac{2}{81}$.

All the elements of $\mathcal{A}_1$ have as image an element of $\mathcal{A}_0$ that it is mapped on all $[0, 1]$ apart for the ones that are mapped on $(0, \frac{1}{9})$ and for $(0, \frac{1}{27})$. A direct computation shows that $T^3(0, \frac{1}{27}) = [0, 1]$ and $T^2(0, \frac{1}{9}) = [0, 1]$, so $N(1) = 3$.

If we choose $a = 7.25$, then $\sigma_1 \leq .994$ (see Lemma 3.5 for a definition of σ_1). Since $a\sigma_1 \geq 1 + \sigma_1$ and $1 - \sigma_1 \geq \frac{1}{2}\left(\frac{2}{9}\right)^3$, the formula in the proof of Lemma 3.5 yields

$$\frac{\Delta}{2} \leq \ln\left[1 + 2a\sigma_1 \|DT\|^{N(1)}\right] \leq \ln 1314$$

$$\Lambda = \tanh(\frac{\Delta}{4})^{\frac{1}{3}} \approx 1 - \frac{2}{3}e^{-\frac{\Delta}{2}} \leq 1 - \frac{1}{1971}$$

moreover, $\lim_{n\to\infty} K_n \leq 140$ and $b \leq .3$.

The above estimates are probably off by a couple of orders of magnitude, but they should not be considered too unsatisfactory: using the only other known rigorous estimates, [Co], one would have been lead to consider at least $N(20)$ obtaining an estimate of $(1-\Lambda)^{-1}$ wrong by at least 20 orders of magnitude. In addition, the present estimate can certainly be improved: first of all it is easy to check that for each $x \in [0, 1]$ $\#\{T^{-3}x\} \geq 2$, this allows immediately to divide by two the number inside the log (cf. the proof of Lemma 3.5); second, some advantages can be obtained by using a different partition (instead of the dynamical one) and a different number (instead than $\frac{1}{2}$) in Lemma 3.2.[14] Moreover, the result can conceivably be improved by choosing a cone of functions better adapted to the particular example at hand.

However, the problem of finding optimal bounds it is a business all in itself and not the one we have been concerned with in this paper.

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[14] Using these two ideas (i.e., the partition $\{(0, \frac{1}{9}); (\frac{1}{9}, \frac{2}{9}); (\frac{2}{9}, \frac{3}{9}); \dots\}$ and $\frac{7}{36}$ instead than $\frac{1}{2}$, which allows to set $n_0 = 0$) it is already possible to obtain the improved estimate $\Lambda \leq 1 - \frac{1}{377}$.

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Liverani Carlangelo, Mathematics Department, University of Rome II, Tor Vergata, 00133 Rome, Italy.

E-mail address: liverani@mat.utovrm.it