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TOPICS IN NONHAMILTONIAN INTERACTION OF HAMILTONIAN DYNAMIC SYSTEMS

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I. Introduction

The evolution of hamiltonian systems (defined by Poisson brackets and hamiltonians) attracts a lot of attention (see f.e. [1]). Many of such systems are associated with Lie algebras, in this case Poisson brackets (the so–called Lie–Berezin brackets) have a linear form and are restored from the commutator in the Lie algebra [2] (it should be marked that nonlinear Poisson brackets are also of interest [3]). The introducing of external fields (which interact with the system noncanonically, in general) transforms Lie algebras into isocommutator algebras (Lie isoalgebras) [4]. If the external field possesses a symmetry governed by the Lie algebra $\mathfrak{g}$, then it is rather convenient to describe the systems by Lie $\mathfrak{g}$–bunches [5]. By use of them one may construct various hamiltonian systems noncanonically controlled by external fields, which dynamics maybe also hamiltonian, in particular. Such picture is rather realistic if one supposes that a counteraction of the system on the external fields maybe neglected. Nevertheless, it is not so in many cases. Therefore, it is interesting to consider a dynamics of two hamiltonian systems, which interact with each other noncanonically. A convenient algebraic structure to describe such dynamics seems to be of the following definition.

Definition 1. The pair (V_1, V_2) of linear spaces is called *an isotopic pair* iff there are defined two mappings $m_1 : V_2 \otimes \bigwedge^2 V_1 \mapsto V_1$ and $m_2 : V_1 \otimes \bigwedge^2 V_2 \mapsto V_2$ such that the mappings $(X,Y) \mapsto [X,Y]_A = m_1(A,X,Y)$ $(X,Y \in V_1,\ A \in V_2)$ and $(A,B) \mapsto [A,B]_X = m_2(X,A,B)$ $(A,B \in V_2,\ X \in V_1)$ obey the Jacobi identity for all values of a subscript parameter (such operations will be called *isocommutators* and the subscript parameters will be called *isotopies*) and are compatible to each other, i.e. the identities

$$\begin{aligned}[X,Y]_{[A,B]_Z} =& \tfrac{1}{2}([[X,Z]_A,Y]_B + [[X,Y]_A,Z]_B + [[Z,Y]_A,X]_B - \\ & [[X,Z]_B,Y]_A - [[X,Y]_B,Z]_A - [[Z,Y]_B,X]_A)\end{aligned}$$

and

$$\begin{aligned}[A,B]_{[X,Y]_C} =& \tfrac{1}{2}([[A,C]_X,B]_Y + [[A,B]_X,C]_Y + [[C,B]_X,A]_Y - \\ & [[A,C]_Y,B]_X - [[A,B]_Y,C]_X - [[C,B]_Y,A]_X)\end{aligned}$$

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$(X,Y,Z\in V_1,\ A,B,C\in V_2)$ hold.

This definition is a result of an axiomatization of the following construction. Let's consider an associative algebra $\mathcal{A}$ (f.e. any matrix algebra) and two linear subspaces V_1 and V_2 in it such that V_1 is closed under the isocommutators $(X,Y)\mapsto[X,Y]_A=XAY-YAX$ with isotopies A from V_2, whereas V_2 is closed under the isocommutators $(A,B)\mapsto[A,B]_X=AXB-BXA$ with isotopies X from V_1. If a family (linear space) V_1 of operators forms an isocommutator algebra with a family (linear space) V_2 of admissible isotopies then (V_1,V_2) is an isotopic pair via the so–called "isotopic duality" [5][1].

It should be mentioned that the isocommutators define families of Poisson brackets $\{\cdot,\cdot\}_A$ and $\{\cdot,\cdot\}_X$ $(A\in V_2,\ X\in V_1)$ in the spaces $S^{\cdot}(V_1)$ and $S^{\cdot}(V_2)$, respectively.

Definition 2. Let's consider two elements $\mathcal{H}_1$ and $\mathcal{H}_2$ ("hamiltonians") in $S^{\cdot}(V_1)$ and $S^{\cdot}(V_2)$, respectively. The equations

$$\dot X_t=\{\mathcal{H}_1,X_t\}_{A_t},\quad \dot A_t=\{\mathcal{H}_2,A_t\}_{X_t},$$

where $X_t\in V_1$ and $A_t\in V_2$ will be called *the (nonlinear) dynamical equations associated with the isotopic pair (V_1,V_2) and "hamiltonians" $\mathcal{H}_1$ and $\mathcal{H}_2$* (it should be marked that "hamiltonians" are not even integrals of motion in a general situation).

Let's consider several simple but crucial examples now. We shall treat the subject purely mathematically and shall not specify its concrete physical implications of a possible interest.

II. Noncanonically coupled rotators and Euler–Arnold tops

2.1. Nonlinear integrable dynamics of noncanonically coupled rotators. Let's consider the Lie $\mathrm{sl}(2,\mathbb{C})$–bunch in π_1 (the adjoint representation of $\mathrm{sl}(2,\mathbb{C})$) [5]. It is defined by the next formulas

$$\begin{array}{lll} [m_{-1},m_0]_{l_0}=m_{-1} & [m_{-1},m_0]_{l_{-1}}=0 & [m_{-1},m_0]_{l_1}=2m_0\\ [m_1,m_{-1}]_{l_0}=0 & [m_1,m_{-1}]_{l_{-1}}=2m_{-1} & [m_1,m_{-1}]_{l_1}=-2m_1\\ [m_1,m_0]_{l_0}=m_1 & [m_1,m_0]_{l_{-1}}=2m_0 & [m_1,m_0]_{l_1}=0 \end{array}\tag{1}$$

where l_i $(i=-1,0,1)$ are generators of $\mathrm{sl}(2,\mathbb{C})$ $([l_i,l_j]=(i-j)l_{i+j})$, m_i $(i=-1,0,1)$ form a basis in π_1 $(l_i(m_j)=(i-j)m_{i+j})$.

As it was marked in [5] the operators l_i form an isocommutator Lie algebra with respect to isotopies from π_1 via the "isotopic duality", namely

$$\begin{array}{lll} [l_{-1},l_0]_{m_0}=-l_{-1} & [l_{-1},l_0]_{m_{-1}}=0 & [l_{-1},l_0]_{m_1}=-2l_0\\ [l_1,l_{-1}]_{m_0}=0 & [l_1,l_{-1}]_{m_{-1}}=-2l_{-1} & [l_1,l_{-1}]_{m_1}=2l_1\\ [l_1,l_0]_{m_0}=-l_1 & [l_1,l_0]_{m_{-1}}=-2l_0 & [l_1,l_0]_{m_1}=0 \end{array}\tag{2}$$

It can be easily verified that isocommutators (1) and (2) define a structure of an isotopic pair in $\pi_1\oplus\pi_1$. Let's denote the first summand by π_1^+ and the second

[1] The "isotopic duality" maybe speculatively regarded as a certain "mathematical manifestation" of the Third Law of Classical Dynamics.

summand by π_1^-. Let's also consider two fixed elements $\Omega^\pm$ in $\pi_1^\pm$, respectively ($\Omega^+=\Omega^+_{-1}m_{-1}+\Omega^+_0m_0+\Omega^+_1m_1$, $\Omega^-=\Omega^-_{-1}l_{-1}+\Omega^-_0l_0+\Omega^-_1l_1$), and two variables A and B from π_1^+ and π_1^- ($A=A_{-1}m_{-1}+A_0m_0+A_1m_1$ and $B=B_{-1}l_{-1}+B_0l_0+B_1l_1$). The dynamical equations (with linear "hamiltonians" defined by $\Omega^\pm$) will have the form

$$\begin{cases} \dot{A}_{-1}=\Omega^+_{-1}(A_0B_0-2A_1B_{-1})-\Omega^+_0A_{-1}B_0+2\Omega^+_1A_{-1}B_{-1} \\ \dot{A}_0=2\Omega^+_{-1}A_0B_1+2\Omega^+_0(A_1B_{-1}-A_{-1}B_1)+2\Omega^+_1A_0B_{-1} \\ \dot{A}_1=2\Omega^+_{-1}A_1B_1-\Omega^+_0A_1B_0+\Omega^+_1(A_0B_0-2A_{-1}B_1) \end{cases}$$

$$\begin{cases} \dot{B}_{-1}=-\Omega^-_{-1}(B_0A_0-2B_1A_{-1})+\Omega^-_0B_{-1}A_0-2\Omega^-_1B_{-1}A_{-1} \\ \dot{B}_0=-2\Omega^-_{-1}B_0A_1-2\Omega^-_0(B_1A_{-1}-B_{-1}A_1)-2\Omega^+_1B_0A_{-1} \\ \dot{B}_1=-2\Omega^-_{-1}B_1A_1+\Omega^-_0B_1A_0-\Omega^-_1(B_0A_0-2B_{-1}A_1) \end{cases}$$

It is rather convenient to consider the compact real form of the isotopic pair (π_1^+,π_1^-). Let's denote

$$l_x=\tfrac{i}{2}(l_1-l_{-1}),\quad l_y=\tfrac{1}{2}(l_1+l_{-1}),\quad l_z=il_0$$
$$m_x=\tfrac{i}{2}(m_1-m_{-1}),\quad m_y=\tfrac{1}{2}(m_1+m_{-1}),\quad m_z=im_0.$$

The dynamical equation maybe rewritten as

$$\begin{cases} \dot{A}_x=-\Omega^+_x(A_yB_y+A_zB_z)+\Omega^+_yA_xB_y+\Omega^+_zA_xB_z \\ \dot{A}_y=-\Omega^+_y(A_xB_x+A_zB_z)+\Omega^+_xA_yB_x+\Omega^+_zA_yB_z \\ \dot{A}_z=-\Omega^+_z(A_xB_x+A_yB_y)+\Omega^+_xA_zB_x+\Omega^+_yA_zB_y \end{cases}$$

$$\begin{cases} \dot{B}_x=-\Omega^-_x(B_yA_y+B_zA_z)+\Omega^-_yB_xA_y+\Omega^-_zB_xA_z \\ \dot{B}_y=-\Omega^-_y(B_xA_x+B_zA_z)+\Omega^-_xB_yA_x+\Omega^-_zB_yA_z \\ \dot{B}_z=-\Omega^-_z(B_xA_x+B_yA_y)+\Omega^-_xB_zA_x+\Omega^-_yB_zA_y \end{cases}$$

after the change $\Omega^-\to-\Omega^-$. They maybe also written in a more compact form

$$\begin{cases} \dot{A}=-\langle A,B\rangle\,\Omega^++\big\langle\Omega^+,B\big\rangle\,A \\ \dot{B}=-\langle A,B\rangle\,\Omega^-+\big\langle\Omega^-,A\big\rangle\,B \end{cases}$$

here brackets $\langle\cdot,\cdot\rangle$ denote su(2)–invariant inner products in $\pi_1^\pm$. Such nonlinear dynamics maybe interpreted as one for the noncanonically coupled (linear) rotators. It possesses two trivial integrals of motion $K=\langle A,B\rangle$ and $L=\langle A,\Omega^-\rangle-\langle B,\Omega^+\rangle$.

Let's denote $\langle\Omega^+,B\rangle+\frac{1}{2}L=\langle\Omega^-,A\rangle-\frac{1}{2}L$ by y. Then

$$\begin{cases} \dot{A}=-K\Omega^++(y-\frac{1}{2}L)A \\ \dot{B}=-K\Omega^-+(y+\frac{1}{2}L)B \end{cases}$$

whereas

$$\dot{y}=y^2-(\tfrac{1}{4}L^2+K\beta),$$

$\beta=\langle\Omega^+,\Omega^-\rangle$, so the dynamical equations are integrable in elementary functions.

2.2. Nonlinear integrable dynamics of noncanonically coupled Euler–Arnold tops. Noncanonically coupled Euler–Arnold tops are realized by means of the same isotopic pair as noncanonically coupled rotators but with quadratic "hamiltonians" $\mathcal{H}_1 = \langle A, TA\rangle$ and $\mathcal{H}_2 = \langle B, TB\rangle$ ($T = T^*$). The corresponding dynamical equations maybe written as

$$\begin{cases} \dot{A} = -\langle A, B\rangle TA + \langle TA, B\rangle A \\ \dot{B} = -\langle A, B\rangle TB + \langle TB, A\rangle B \end{cases}$$

Such system admits an integral of motion $K = \langle A, B\rangle$; the equations maybe rewritten as

$$\begin{cases} \dot{A} = (\rho - KT)A \\ \dot{B} = (\rho - KT)B \end{cases}$$

where $\rho = \langle TA, B\rangle = \langle TB, A\rangle$ obeys the ordinary nonlinear differential equation of the second degree:

$$\ddot{\rho} + (4K\alpha - 6\rho)\dot{\rho} + 4\rho^3 - 4K\alpha\rho^2 - 4K^2\beta\rho - 4K^2\gamma = 0,$$

$$\dot{\rho}|_{t=0} = -2K\langle TA_0, TB_0\rangle + \rho^2_{|_{t=0}},$$

where $\alpha = \mathrm{tr}(T)$, $\beta = \frac{1}{2}(\mathrm{tr}(T^2) - \mathrm{tr}^2(T))$ is a sum of main minors of the second order of symmetric matrix T, $\gamma = \det(T)$.

It is convenient to make a change of variables $y = \rho - \frac{K\alpha}{3}$:

$$\ddot{y} - 6y\dot{y} + 4y^3 - \Gamma y - \Delta = 0,$$

where $\Gamma = 4K^2(\beta + \frac{1}{3}\alpha^2)$, $\Delta = \frac{4K^3}{27}(8\alpha^3 + 9\alpha\beta + 27\gamma)$.

One may look for solutions of this differential equation by its splitting into two ones $\dot{y} = f(y; C)$ and $ff'_y - 6yf + 4y^3 - \Gamma y - \Delta = 0$. The first differential equation is solved in the inversed quadratures $\int \frac{dy}{f(y;C)} = t + \tilde{C}$. The second one admits a trivial partial solution $f_0(y) = 2y^2 + by + c$ ($b^2 - 2c = \Gamma$, $bc = \Delta$) and in this case $y = y_0(t)$ is expressed in elementary functions; a general solution f of the second equation has the form $f = f_0 + g$, where $gg'_y - (2y - b)g + (2y^2 + by + c)g'_y = 0$.

III. Noncanonically coupled oscillators and their chains

3.1. Nonlinear integrable dynamics of noncanonically coupled oscillators. Let's consider a standard representation of the Heisenberg algebra with three generators p, q and r ($[p, q] = r$, $[p, r] = [q, r] = 0$) by 3×3 matrices:

$$p = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad r = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The linear space of admissible isotopies is generated by three elements a, b and c, which are represented by matrices

$$a = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The isocommutators have the form

$$\begin{array}{lll} [p,q]_a = 0 & [p,q]_b = 0 & [p,q]_c = r \\ [p,r]_a = r & [p,r]_b = 0 & [p,r]_c = 0 \\ [q,r]_a = 0 & [q,r]_b = -r & [q,r]_c = 0 \end{array}$$

The elements a, b and c are closed themselves under isocommutators with isotopies p, q and r via the "isotopic duality":

$$\begin{array}{lll} [a,b]_p = 0 & [a,b]_q = 0 & [a,b]_r = c \\ [a,c]_p = c & [a,c]_q = 0 & [b,c]_r = 0 \\ [b,c]_p = 0 & [b,c]_q = -c & [a,c]_r = 0 \end{array}$$

The linear spaces generated by p, q, r and a, b, c form an isotopic pair. One may associate noncanonically coupled oscillators with it; namely, let's consider p, q, r and a, b, c as linear functionals (we shall denote them by capitals) on dual spaces; "hamiltonians" $\mathcal{H}_1$ and $\mathcal{H}_2$ will be of the form $\mathcal{H}_1 = P^2 + Q^2$ and $\mathcal{H}_2 = A^2 + B^2$, the dynamical equations will be written as

$$\begin{cases} \dot{Q} = 2RPC \\ \dot{P} = -2RQC \\ \dot{R} = 2PRA - 2QRB \end{cases} \qquad \begin{cases} \dot{A} = -2CBR \\ \dot{B} = 2CAR \\ \dot{C} = 2ACP - 2BCQ \end{cases}$$

It should be marked that "hamiltonians" are integrals of motion here, also three integrals maybe written: $\mathcal{M} = AQ - BP$, $\mathcal{N} = BQ + AP$ and $\mathcal{L} = CR - BP - AQ = CR - 2BP - \mathcal{M} = CR - 2AQ + \mathcal{M}$. The presence of five integrals essentially simplifies a picture, so an integration of the system of two noncanonically coupled oscillators becomes an easy but interesting exercise. Let's perform it. Put $\mathcal{H}_1 = h_1^2$, $\mathcal{H}_2^2 = h_2^2$, $P = h_1 \cos\varphi$, $Q = h_1 \sin\varphi$, $A = h_2 \cos\psi$, $B = h_2 \sin\psi$, then the condition $\dot{\mathcal{M}} = \dot{\mathcal{N}} = 0$ gives $\cos(\varphi - \psi) = -\frac{\mathcal{N}}{h_1 h_2}$, $\sin(\varphi - \psi) = \frac{\mathcal{M}}{h_1 h_2}$, so $\vartheta := \psi - \varphi = \arctan(\frac{\mathcal{M}}{\mathcal{N}})$. Also $\dot{\varphi} = \dot{\psi} = 2CR$, $\dot{R} = 2Rh_1h_2\cos(\varphi + \psi)$, $\dot{C} = 2Ch_1h_2\cos(\varphi + \psi)$, hence $C = \varkappa R$ and $\dot{\varphi} = 2\varkappa R^2$, $\dot{R} = 2h_1h_2R\cos(2\varphi + \vartheta)$. Let's consider R as a function of φ then $RR'_\varphi = h_1h_2\cos(2\varphi + \vartheta)$ and $R = \sqrt{\frac{\mathcal{L}}{\varkappa} + \frac{h_1h_2}{\varkappa}\sin(2\varphi + \vartheta)}$ (to receive this fact one may also use a conservation law $\dot{\mathcal{L}} = 0$). It should be marked that the plane curve, which is defined by the equation $R = R(\varphi)$ in polar coordinates, is the Booth lemniscate. Substituting the resulted expression for $R = R(\varphi)$ into the formula for $\dot{\varphi}$ one obtains that $\dot{\varphi} = 2\mathcal{L} + 2h_1h_2\sin(2\varphi + \vartheta)$.

3.2. Dynamics of periodic and nonperiodic chains of noncanonically coupled oscillators. The dynamical equations for two noncanonically coupled oscillators are immediately generalized on (periodic or infinite nonperiodic) chains of them, namely

$$\begin{cases} \dot{Q}_i = R_iP_i(R_{i-1} + R_{i+1}) \\ \dot{P}_i = -R_iQ_i(R_{i-1} + R_{i+1}) \\ \dot{R}_i = R_iP_i(P_{i-1} + P_{i+1}) - R_iQ_i(Q_{i-1} + Q_{i+1}) \end{cases}$$

The "hamiltonians" $\mathcal{H}_i = P_i^2+Q_i^2$ are certainly integrals of motion. Put $\mathcal{H}_i = h_i^2$, $P_i = h_i \cos\varphi_i$, $Q_i = h_i \sin\varphi_i$, then $\dot{\varphi}_i = R_i(R_{i-1}+R_{i+1})$, $\dot{R}_i = R_i h_i(h_{i-1}\cos(\varphi_i + \varphi_{i-1}) + h_{i+1}\cos(\varphi_i+\varphi_{i+1}))$.

Let's introduce new "coupled" variables $\psi_i = \varphi_i + \varphi_{i+1}$ and $S_i = R_i R_{i+1}$ as well as $H_i = h_i h_{i+1}$. The dynamical equations are rewritten as follows

$$\begin{cases} \dot{\psi}_i = S_{i-1} + 2S_i + S_{i+1} \\ \dot{S}_i = S_i(H_{i-1}\cos\psi_{i-1} + 2H_i\cos\psi_i + H_{i+1}\cos\psi_{i+1}) \end{cases}$$

Let's put $T_i = H_i e^{\sqrt{-1}\psi_i}$ now, then

$$\begin{cases} \dot{S}_i = \mathrm{Re}(T_{i-1} + 2T_i + T_{i+1})S_i \\ \dot{T}_i = \sqrt{-1}(S_{i-1} + 2S_i + S_{i+1})T_i \end{cases}$$

3.3. A continuum limit of the noncanonically coupled oscillator chain. One may consider noncanonically coupled oscillator chain with changed signs, i.e.

$$\begin{cases} \dot{Q}_i = R_i P_i(R_{i+1} - R_{i-1}) \\ \dot{P}_i = -R_i Q_i(R_{i+1} - R_{i-1}) \\ \dot{R}_i = R_i P_i(P_{i+1} - P_{i-1}) - R_i Q_i(Q_{i+1} - Q_{i-1}) \end{cases}$$

Such chain admits a natural continuum (field) limit

$$\begin{cases} \dot{q} = prr'_x \\ \dot{p} = -qrr'_x \\ \dot{r} = rpp'_x - rqq'_x \end{cases}$$

$q=q(t,x)$, $p=p(t,x)$, $r=r(t,x)$. The function $h^2=p^2+q^2$ is an integral of motion, so it is convenient to put $p = h\cos\varphi$, $q=h\sin\varphi$. Then

$$\begin{cases} \dot{\phi} = rr'_x \\ \dot{r} = -rh^2\sin 2\varphi\varphi'_x \end{cases}$$

Let's put $s=r^2$, $t=h^2e^{2\sqrt{-1}\varphi}$, then

$$\begin{cases} \dot{s} = -\,\mathrm{Re}(t'_x)s \\ \dot{t} = \sqrt{-1}s'_x t \end{cases}$$

IV. Other examples

4.1. Dynamic system connected with the isotopic pair of 3×3 symmetric and skew–symmetric matrices. One of the most interesting examples of isotopic pairs is related to $n\times n$ symmetric and skew–symmetric matrices. Let's consider the simplest case $n = 3$. It is rather convenient to introduce the following basises:

$$l_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad l_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad l_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

and

$$m_{xy} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m_{yz} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad m_{xz} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$m_{xx} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m_{yy} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m_{zz} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Elements l_a form a basis in the space of skew–symmetric matrices, whereas w_{ab} form a basis in the space of symmetric matrices. The isocommutators have the form

$$\begin{aligned} [m_{ab}, m_{cd}]_{l_e} &= \epsilon_{ace} m_{bd} + \epsilon_{ade} m_{bc} + \epsilon_{bce} m_{ad} + \epsilon_{bde} m_{ac}, \\ [l_a, l_b]_{m_{cd}} &= \epsilon_{abc} l_d + \epsilon_{abd} l_c, \end{aligned}$$

here ϵ_{abc} is a totally antisymmetric tensor.

Let's fix a skew–symmetric matrix Ω^- and a symmetric matrix Ω^+; they maybe regarded as linear "hamiltonians" for the considered isotopic pair. If the linear functionals defined by l_a and m_{ab} in the dual spaces are denoted by capitals L_a and M_{ab}, respectively, then the dynamical equation will have the form

$$\begin{cases} \dot{L}_a = \epsilon_{bcd} \Omega^-_b L_c M_{ad} \\ \dot{M}_{ab} = \epsilon_{cde} \Omega^+_{ac} M_{bd} L_e + \epsilon_{cde} \Omega^+_{bc} M_{ad} L_e \end{cases}$$

Unfortunately, I do not know whether this system is integrable.

4.2. The hybrid coupling: "elastoplastic" spring. It is rather interesting to consider the case then two hamiltoinian systems are coupled by interaction terms in a hamiltonian and simultaneously noncanonically. Below we shall consider an "elastoplastic" string, which is a hybrid of the ordinary "elastic" spring with noncanonically coupled oscillators.

The isotopic pair is the same as for noncanonically coupled oscillators but the "hamiltonian" is of the form $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 - 2QB - 2PA$; the dynamical equations have the form

$$\begin{cases} \dot{Q} = 2RC(P - A) \\ \dot{P} = -2RC(Q - B) \\ \dot{R} = -2(P - A)^2 R + 2(Q - B)^2 R \end{cases} \qquad \begin{cases} \dot{A} = 2RC(Q - B) \\ \dot{B} = -2RC(P - A) \\ \dot{C} = -2(P - A)^2 C + 2(Q - B)^2 C \end{cases}$$

Let's denote $D = P - A$, $G = Q - B$, then $\mathcal{J}^2 = D^2 + G^2$ is an integral of motion so it is convenient to put $D = \mathcal{J}\sin\varphi$, $G = \mathcal{J}\cos\varphi$. Moreover, $C = \lambda R$ and $\dot{R} = 2\mathcal{J}^2R^2\cos 2\varphi$, whereas $\dot{\varphi} = -4\lambda R^2$. Hence, $-2\lambda RR'_\varphi = \mathcal{J}^2\cos 2\varphi$ and $R = \sqrt{\frac{\mathcal{L}}{\lambda} - \frac{1}{2\lambda}\mathcal{J}^2\sin 2\varphi}$ (and the corresponding plane curve is the Booth lemniscate again), whereas $\dot{\varphi} = -4\mathcal{L} + 2\mathcal{J}^2\sin 2\varphi$.

4.3. "Elastoplastic" spring with general interaction potential. The described picture is straightforwardly generalized on arbitrary interaction potentials. Namely, let's consider the "hamiltonian" of the form $\mathcal{H} = (P - A)^2 + V(Q - B)$, where V is an interaction potential. The dynamical equations are written in the form

$$\begin{cases} \dot{Q} = 2RC(P - A) \\ \dot{P} = -RCV'(Q - B) \\ \dot{R} = -2(P - A)^2R + (Q - B)V'(Q - B)R \end{cases}$$

$$\begin{cases} \dot{A} = RCV'(Q - B) \\ \dot{B} = -2RC(P - A) \\ \dot{C} = -2(P - A)^2C + (Q - B)V'(Q - B)C \end{cases}$$

Let's denote $D = P - A$, $G = Q - B$; $\mathcal{J}^2 = D^2 + V(G)$ is an integral of motion so it is convenient to put $D = \mathcal{J}\sin\varphi$, $G = V^{-1}(\mathcal{J}^2\cos^2\varphi)$. Moreover $C = \lambda R$ and $\dot{R} = \left[-2\mathcal{J}^2\sin^2\varphi + V^{-1}(\mathcal{J}^2\cos^2\varphi)V'(V^{-1}(\mathcal{J}^2\cos^2\varphi))\right] R$, whereas $\dot{\varphi} = -\frac{2\lambda R^2}{\mathcal{J}\cos\varphi} \times V'(V^{-1}(\mathcal{J}^2\cos^2\varphi))$. Hence $-2\lambda RR'_\varphi V'(V^{-1}(\mathcal{J}^2\cos^2\varphi)) = \mathcal{J}\cos\varphi\left[-2\mathcal{J}^2\sin^2\varphi + V^{-1}(\mathcal{J}^2\cos^2\varphi)V'(V^{-1}(\mathcal{J}^2\cos^2\varphi))\right]$ and

$$R = \sqrt{\frac{1}{\lambda}\int \mathcal{J}\cos\varphi\frac{2\mathcal{J}^2\sin^2\varphi - V^{-1}(\mathcal{J}^2\cos^2\varphi)V'(V^{-1}(\mathcal{J}^2\cos^2\varphi))}{V'(V^{-1}(\mathcal{J}^2\cos^2\varphi))}d\varphi},$$

whereas

$$\dot{\varphi} = \frac{2}{\cos\varphi}\int \cos\varphi\left[V^{-1}(\mathcal{J}^2\cos^2\varphi) - \frac{2\mathcal{J}^2\sin^2\varphi}{V'(V^{-1}(\mathcal{J}^2\cos^2\varphi))}\right]d\varphi.$$

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