Large field versus small field expansions and Sobolev inequalities

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Abstract

We study a model for a two dimensional random interface $\phi(x)$, $x \in {}^2$ described by a massless gaussian measure perturbed by a weak potential $V(\phi) = \frac{\varepsilon^2}{2}(\mathrm{e}^{-\alpha\phi} - 1)^2$. Such a model occurs for instance in a phenomenological description of the wetting transition. We prove that, provided α is small enough, the two-point function decreases exponentially with a rate of order $m \equiv \varepsilon\alpha$ which is just the mean field value. The large field region problem due to the fact that $V(\phi)$ remains bounded when $\phi \to +\infty$ is treated by means of a large field versus small field expansion combined with elementary Sobolev inequalities. The paper is intended to be accessible to non-experts.

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1 Introduction

A fundamental difficulty in statistical mechanics and constructive field theory is to perform the thermodynamic limit in a mathematicaly rigorous way. In statistical mechanics the existence of such a limit simply corresponds to the extensivity of thermodynamic potentials like the pressure, the free energy and so on. In field theory, it proves in the nontrivial cases the existence of non-gaussian translation invariant measures on some distribution spaces. One of the methods to control this limit is the cluster expansion. It is basically perturbative but allows for very general types of interactions and gives detailed information about the limiting theory in contrast with non-perturbative methods, like correlation inequalities. Let us briefly recall the setting in field theory when no renormalisation is needed. One is given a gaussian measure $\mathrm{d}\mu_C(\phi)$ with covariance C on a space of distributions, e.g. $\mathcal{S}'({}^n)$, and a functional $F_\Lambda[\phi]$ depending only on the field in the volume $\Lambda \subset {}^n$. F_Λ is usually assumed to have some factorisation properties, for example $F_\Lambda = \prod_{\Delta \subset \Lambda} F_\Delta$, where the Δ's are squares covering Λ. The problem then is to control the $|\Lambda| \to \infty$ limit of the free energy $f_\Lambda \equiv |\Lambda|^{-1} \log Z(\Lambda)$ where $Z(\Lambda) \equiv \int \mathrm{d}\mu_C(\phi) F_\Lambda[\phi]$. The cluster expansion is an expansion in the volume for $Z(\Lambda)$. More precisely $Z(\Lambda)$ is written as a finite sum of contributions associated to each partition of the volume Λ into disjoint subsets X_j, each X_j being a collection of squares $\Delta \subset \Lambda$. The contribution to a given partition factorises over the subsets X_j. The sets X_j are called clusters or polymers. The factors $z(X_j)$ associated to a given polymer is its activity. So basically the cluster expansion is a rewriting of the partition function $Z(\Lambda)$ as a system of polymers X_j interacting through a hard core exclusion. The standard algebraic machinery of Mayer series [1], therefore allows to write a formal power series in the activities $z(X_j)$ for the free energy $f \equiv \lim_{\Lambda \to \infty} f_\Lambda$ in the thermodynamic limit. The hard work really begins with showing that $|z(X_j)|$ is small enough for this series to converge towards f. To be a little bit more specific consider the case where $F_\Delta = \exp(-\lambda V_\Delta[\phi])$, where $V_\Delta[\phi]$ is some potential. There are two ways in which $|z(X_j)|$ should be small. First $|z(X_j)|$ has to become exponentially small with the volume $|X_j|$ occupied by the polymer X_j. One necessary condition for this is that $|\lambda|$ be small enough. Second, $|z(X_j)|$ must become small whenever the squares Δ comprising X_j are far apart from each other. From the explicit formula for the activity one can show that a sufficient condition for this is that the propagator $C(x, y)$ decays sufficiently fast at infinity. Actually, integrability at infinity is enough. An example satisfying this condition is the free field of mass $m > 0$ for which $C(x, y) \leq \mathbf{C} \exp(-m|x-y|)$, $\mathbf{C} > 0$ as $|x-y| \to \infty$.

In this paper we want to study a model in which a two-dimensional massless gaussian measure is perturbed by a weak potential. We choose $V_\Lambda[\phi] \equiv \int_\Lambda V(\phi(x))\mathrm{d}x$ with $V(\phi) \equiv \frac{\varepsilon^2}{2}(\mathrm{e}^{-\alpha\phi} - 1)^2$. Notice that for $|\phi| \ll \alpha^{-1}$ we have the quadratic approximation $V(\phi) \simeq \frac{m^2\phi^2}{2}$ where $m \equiv \varepsilon\alpha$. The main point about $V(\phi)$ is that it remains bounded when $\phi \to +\infty$. Hence the interacting measure will be effectively massless in the regions where the field is large and positive. This is actually the major source of difficulty and interest for this model. Recall that the massless propagator is not integrable, therefore a naive cluster expansion will not work. Actually the kernel of the covariance is not even defined in two dimensions in the infinite volume limit. There are three reasons for choosing this particular V.

- The V above appears in a phenomenological description of the wetting transition [2, 3]. The field $\phi(x)$ corresponds for example to the height of a liquid/vapor interface above a wall. $V(\phi)$ is a weak potential which tends to localise the interface near the wall. If we put Dirichlet boundary conditions on $\partial\Lambda$, the massless gaussian measure is defined by the covariance $C_{0,\Lambda} \equiv (-\Delta_D)^{-1}$ and just penalizes high curvatures of the interface ϕ. A direct computation shows that $C_{0,\Lambda}(x, x) \sim \log|\Lambda|$, hence we expect that in the thermodynamic limit $\langle\phi(x)^2\rangle_V \to \infty$ as $\varepsilon \to 0$. This phenomenon is supposed to describe how a liquid/vapor interface delocalises as the temperature T approaches from below the critical wetting temperature T_w. One can identify ε with $T_w - T > 0$ see [4].

- The particular form of the V chosen above seems to be well suited for a multiscale analysis, which would allow to control the model outside the mean-field regime where the mass is equal to $\alpha\varepsilon$. We shall not deal with these questions here, see however section 2.

- At last $V(\phi)$ is a simple example for which a large field versus small field decomposition is needed. Many of the most interesting models in constructive field theory (non-Abelian gauge theories [5], two-dimensional Gross-Neuveu models [6]) or many body theory (BCS theory of supraconductivity) require a similar treatment.

Our aim is to prove that, at least in some regime of the parameters α and ε, the potential $V(\phi)$ generates a mass $m' = O(m)$, in other words we want to prove that the connected two-point functions verifies $|\langle\phi(x)\phi(y)\rangle_V - \langle\phi(x)\rangle_V\langle\phi(y)\rangle_V| \leq \mathbf{C}\mathrm{e}^{-m'|x-y|}$.

To show this we first have to find good definitions for what is meant by small and large field regions. Then we must show that big large-field regions, in which there is no exponential decay of the correlations, are rare. That this is true can be understood intuitively as follows. Take a square $\Delta \subset \Lambda$ (whose size will have to be chosen properly), and suppose ϕ is large in mean on Δ. There are just two possibilities: either ϕ is more or less constant on Δ in which case $\exp(-V_\Delta[\phi])$ is small, or this last factor is not small but in that case ϕ necessarily has a large mean curvature and thus $\exp[-\frac{1}{2}\int_\Delta(\nabla\phi)^2]\mathrm{d}x$ is small. This is where the Sobolev inequalities enter. They allow to treat this alternative in a natural and efficient way. These inequalities replaces a substantial part of the very heavy formalism of [7, 8]. As an input for the Sobolev inequalities we need to extract a factor $\exp[-\frac{1}{2}\int_\Delta(\nabla\phi)^2]\mathrm{d}x$ per square in the large field regions from a "locally massless" non-translation invariant gaussian measure. This is a technically non-trivial problem which could have applications in other contexts. It is solved by means of some operator inequalities.

The paper is organized as follows. In section 2 we describe mathematically the model we want to study and state our main result. Section 3 shows how to generate a cluster expansion with small versus large field decomposition. To generate this expansion we make an easy adaptation from Brydges formula in [9]. Its main advantage is its compactness and that it avoids the pain of going through many recursive formulas. A polymer X in this new cluster expansion will be a collection of squares $\{\Delta\}_{j=1}^{s(X)}$ *and* large field regions $\{\gamma\}_{j=1}^{l(X)}$. In section 4 we shall bound the activities $z(X)$ and prove the convergence of the cluster expansion. The last section contains the proof of two crucial propositions. The first uses Sobolev inequalities to show that in the large field region the product $\exp(-V_\Delta[\phi])\exp[-\frac{1}{2}\int_\Delta(\nabla\phi)^2\mathrm{d}x]$ is very small if α is small. The second shows how to extract the factor $\exp[-\frac{1}{2}\int_\Delta(\nabla\phi)^2\mathrm{d}x]$ from the locally massless gaussian measure which appears in the expression for the activities. Proofs are given in details, even when the arguments are rather standard. This makes the paper a little bit longer but hopefully more readable for people who are, like me, beginners in the cluster expansion business.

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Figure 1: The functions V and $\overline{V}$.

2 Definition of the model

In this section we define mathematically the model of random interface we want to study. For each $\Lambda \subset {}^2$ we define a probability measure $\mathrm{d}\mu_{\Lambda,V}(\phi)$ on the space of tempered distributions $\phi \in \mathcal{S}'({}^2)$. The measure $\mathrm{d}\mu_{\Lambda,V}(\phi)$ will be constructed as a perturbation of a gaussian measure $\mathrm{d}\mu_{\Lambda,0}$ which is massless inside Λ by some very flat potential $V(\phi)$. It turns out that $\mathrm{d}\mu_{\Lambda,V}(\phi)$ will actually be supported on C^∞ functions. For V we choose

$$V(\phi) \equiv \varepsilon^2 \left(-\mathrm{e}^{-\alpha\phi} + \frac{1}{2}\mathrm{e}^{-2\alpha\phi} + \frac{1}{2} \right) = \frac{\varepsilon^2}{2}(\mathrm{e}^{-\alpha\phi} - 1)^2 = \frac{m^2\phi^2}{2} + \overline{V}(\phi), \tag{2.1}$$

where $m \equiv \varepsilon\alpha$. We shall always assume $\alpha < 1$. For any $\Lambda \subset {}^2$ we also define the two functionals

$$V_\Lambda[\phi] \equiv \int_\Lambda V(\phi(x))\mathrm{d}x, \tag{2.2}$$

$$\overline{V}_\Lambda[\phi] \equiv \int_\Lambda \overline{V}(\phi(x))\mathrm{d}x. \tag{2.3}$$

Using Taylors remainder formula we have

$$\overline{V}(\phi) \equiv \frac{1}{2}\varepsilon^2\alpha^3\phi^3 \int_0^1 (1-t)^2(\mathrm{e}^{-\alpha t\phi} - 4\mathrm{e}^{-2\alpha t\phi})\mathrm{d}t. \tag{2.4}$$

The functions V and $\overline{V}$ are plotted in figure 1.

Let us now define the free measure $\mathrm{d}\mu_{\Lambda,0}$. It is constructed as follows. Take the massive translation invariant measure $\mathrm{d}\mu_C(\phi)$ with the expected mass m whose covariance is given by

$$C(x,y) \equiv \frac{1}{(2\pi)^2}\int_{{}^2} \mathrm{e}^{ip\cdot(x-y)}\widehat{C}(p)\mathrm{d}p, \quad \text{with} \quad \widehat{C}(p) \equiv \frac{\mathrm{e}^{-p^2/\kappa}}{p^2+m^2}, \tag{2.5}$$

then define

$$\mathrm{d}\mu_{\Lambda,0}(\phi) \equiv \exp\left[\frac{1}{2}\int_\Lambda m^2\phi^2(x)\mathrm{d}x\right]\mathrm{d}\mu_C(\phi). \tag{2.6}$$

The exponential factor just kills the mass inside Λ and leaves a massive field outside Λ. The reason for this somewhat strange boundary conditions is that they nicely fit the cluster expansion formalism below. The parameter κ is an ultraviolet cutoff. In the context of a wetting model, the field would only be defined on the points of a ^{2} lattice and the free measure would look like

$$\exp\left(-\sum_{\substack{x,y\in\Lambda\cap{}^2 \\ |x-y|=1}} [\phi(x)-\phi(y)]^2 \right) \prod_{x\in\Lambda\cap{}^2} \mathrm{d}\phi(x). \tag{2.7}$$

In order that $\mathrm{d}\mu_{\Lambda,0}$ be a good approximation to this lattice measure we should take the parameter κ equal to 1. However as was mentioned in the introduction, this model seems to be well suited for a multiscale analysis which would allow to control the model outside the mean field regime. We hope to be able to report on progress in this direction soon. For the moment let us just mention that such an analysis starts with a decomposition of the propagator C into contributions from different "frequency slices" [10]. Following a renormalisation group strategy, one would integrate out degrees of freedom of higher frequencies until the ultraviolet cutoff, initially equal to 1 is brought down to a value close to the infrared cutoff given by the expected mass m^2. Having this in mind we shall choose $\kappa = m^2 \ll 1$. The rest of this paper can therefore be considered as the first step in a multiscale analysis namely the study of the effective model obtained after all higher frequency modes have been integrated out.

The partition function and n-point functions is defined in the usual way by

$$Z(\Lambda) \equiv \int \mathrm{e}^{-V_\Lambda[\phi]} \mathrm{d}\mu_{\Lambda,0}(\phi) = \int \mathrm{e}^{-\overline{V}_\Lambda[\phi]} \mathrm{d}\mu_C(\phi), \tag{2.8}$$

and

$$\begin{aligned} \langle \phi(x_1)...\phi(x_n)\rangle_{\Lambda,V} &\equiv Z(\Lambda)^{-1} \int \phi(x_1)...\phi(x_n) \mathrm{e}^{-V_\Lambda[\phi]} \mathrm{d}\mu_{\Lambda,0}(\phi) \\ &= Z(\Lambda)^{-1} \int \phi(x_1)...\phi(x_n) \mathrm{e}^{-\overline{V}_\Lambda[\phi]} \mathrm{d}\mu_C(\phi). \end{aligned} \tag{2.9}$$

In the sequel $\mathbf{C}$ will always denote some positive numerical constant, independant of α and ε. Our aim is to prove the following

Theorem 1 *For $\kappa = m^2$ and for α small enough, the connected two-point function verifies*

$$|\langle \phi(x)\phi(y)\rangle_V^{\mathrm{c}}| \equiv \lim_{|\Lambda|\to\infty} \left| \langle \phi(x)\phi(y)\rangle_{\Lambda,V} - \langle \phi(x)\rangle_{\Lambda,V} \langle \phi(y)\rangle_{\Lambda,V} \right| \leq \mathbf{C}\mathrm{e}^{-m'|x-y|},$$

where $m' = O(m)$.

3 Cluster expansion with small versus large field decomposition

In this section we show how to write our model as a gaz of polymers interacting only through a hard core exclusion. We first define technically what we mean by a large field configuration γ, and give an associated partition of the identity. Next we shall insert this decomposition in the expressions (2.8) and (2.9) for the partition function and for the correlation function. Then for *each* fixed large field configuration γ we perform a usual cluster expansion where the massive $\mathrm{d}\mu_C$ in (2.5) will be the perturbed measure see the remark below (3.38). At last we shall sum over all possible γ's in Λ. This will yield the polymer system.

3.1 A partition of the identity

As a first step we choose a function $\chi(t)$ which smoothly interpolates between 1 and 0 on the interval $[0, 1]$. We take for example

$$\chi(t) \equiv \begin{cases} 1 & \text{if} \quad 0 \leq t \leq \frac{1}{2}, \\ \mathrm{e}^2 \left(1 - \mathrm{e}^{-\frac{1}{t-\frac{1}{2}}}\right) \mathrm{e}^{-\frac{1}{1-t}} & \text{if} \quad \frac{1}{2} \leq t \leq 1, \\ 0 & \text{if} \quad t \geq 1. \end{cases} \tag{3.10}$$

An explicit computation shows that

$$\left|\frac{\mathrm{d}^n\chi(t)}{\mathrm{d}t^n}\right| \leq \mathbf{C}(n!)^2. \tag{3.11}$$

Note also that $\chi^{(n)}(t) = 0$ if $t \notin [\frac{1}{2}, 1]$ and $n \geq 1$. Next let $\Delta \subset {}^2$ be a square and $g(\phi)$ a function which goes to ∞ when $|\phi| \to \infty$. The size of the square Δ will be chosen at the end of the section and the precise form of g in section 4. Define the functional

$$\chi_\Delta[\phi] \equiv \chi\left(\frac{1}{|\Delta|}\int_\Delta g(\phi(x))\mathrm{d}x\right). \tag{3.12}$$

It vanishes when the field ϕ is large in Δ in some complicated sense. Consider that Λ is covered by a pavement of squares Δ, and let γ denote a collection of such squares. We call γ a large field configuration. The partition of the identity is defined as follows

$$\begin{aligned} 1 &= \prod_{\Delta\subset\Lambda} 1 = \prod_{\Delta\subset\Lambda} \left(\chi_\Delta[\phi] + (1-\chi_\Delta[\phi])\right) \\ &= \sum_{\gamma\subset\Lambda}\prod_{\Delta\in\gamma}(1-\chi_\Delta[\phi]) \prod_{\Delta\in\Lambda\setminus\gamma}\chi_\Delta[\phi] \equiv \sum_{\gamma\subset\Lambda}\chi_\gamma[\phi]. \end{aligned} \tag{3.13}$$

We say that the field is large inside γ whenever $\chi_\gamma[\phi] \neq 0$. Now we insert this partition in the expression (2.9) for the n-point function

$$\langle\phi(x_1)...\phi(x_n)\rangle_{\Lambda,V} = \frac{\displaystyle\sum_{\gamma\subset\Lambda}\int \chi_\gamma[\phi]\,\phi(x_1)...\phi(x_n)\,\mathrm{e}^{-V_\gamma[\phi]}\mathrm{e}^{-\overline{V}_{\Lambda\setminus\gamma}[\phi]}\mathrm{e}^{\frac{1}{2}\int_\gamma m^2\phi^2\mathrm{d}x}\mathrm{d}\mu_C(\phi)}{\displaystyle\sum_{\gamma\subset\Lambda}\int \chi_\gamma[\phi]\,\mathrm{e}^{-V_\gamma[\phi]}\mathrm{e}^{-\overline{V}_{\Lambda\setminus\gamma}[\phi]}\mathrm{e}^{\frac{1}{2}\int_\gamma m^2\phi^2\mathrm{d}x}\mathrm{d}\mu_C(\phi)}. \tag{3.14}$$

The exponential $\mathrm{e}^{-\overline{V}_\Lambda[\phi]}$ has been splitted into 3 factors to suggest that in the small field region $\Lambda\setminus\gamma$, $\overline{V}_{\Lambda\setminus\gamma}$ has to be thought of as the perturbation whereas in the large field region γ, the mass counterterm has to be combined with $\mathrm{d}\mu_C(\phi)$ to form a measure massless in γ, and e^{-V_γ} is a "small" factor. These remarks will hopefully become more clear in section 4.

3.2 Definition of the polymers

In the standard cluster expansion, the volume Λ is usualy divided into squares Δ. Here we shall proceed slightly differently. For each large field configuration γ we shall give a different partition D_γ of Λ, and then perform a cluster expansion for *each* term in the numerator and denominator in (3.14) with respect to the corresponding partition D_γ. The precise definition of a polymer as we shall see is a little bit complicated. The reason for this is technical and will become clear only in section 5. For the moment let us just say that in order to extract factors $\exp[-\frac{1}{2}\int_\Delta(\nabla\phi)^2]\mathrm{d}x$ per square Δ from the locally massless measure, the polymers have to be sufficiently "fat" to avoid strong boundary effects. This is the reason for all this corridor business below.

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Figure 2: A polymer $X = \{\Delta_1, ..., \Delta_6, \gamma_1, \gamma_2\}$, and a tree T connecting its elements.

To each large field configuration γ we associate a first corridor $B(\gamma)$ made of all squares $\Delta \notin \gamma$ which touch γ either through an edge or a corner. Define $\overline{\gamma} \equiv \gamma \cup B(\gamma)$. Next fix $L > 0$ and $\eta > 0$ and define

$$\begin{array}{lcl} r_1 \equiv \frac{L}{\eta} m^{-1} & & r_2 - r_1 = L m^{-1} \\ & \text{so that} & \\ r_2 \equiv \left(1 + \frac{1}{\eta}\right) L m^{-1} & & \frac{r_2}{r_1} = 1 + \eta. \end{array} \tag{3.15}$$

Define a second boundary $\widetilde{B}(\gamma) \equiv \{x \in \Lambda \backslash \overline{\gamma} \ |\text{dist}(x, \overline{\gamma}) \leq r_2\}$ and $\tilde{\gamma} \equiv \overline{\gamma} \cup \widetilde{B}(\gamma)$. At last define $\Gamma \equiv \tilde{\gamma} \cup \{\Delta | \Delta \cap \tilde{\gamma} \neq \}$. Partition Γ into maximal connected subsets Γ_j: $\Gamma = \cup_j^p \Gamma_j$, and define $\gamma_j \equiv \gamma \cap \Gamma_j$. Notice that a particular γ_j might be disconnected and that $|\Gamma_j| \leq \mathbf{C} |\gamma_j|$, where $|.|$ means the volume in ^{2}. Note also that $\gamma \neq \gamma'$ may both lead to the same Γ_j's. The partition D_γ of Λ is just the collection of all the Γ_j, $j = 1, ..., q$ and all the squares Δ_l, $l = 1, ..., p$ of $\Lambda \backslash \Gamma$:

$$D_\gamma \equiv \{\Delta_1, ..., \Delta_p, \Gamma_1, ..., \Gamma_q\} \equiv \{v_1, ..., v_{p+q}\}.$$

An element v_j of D_γ is called a vertex. A family of large field regions $\{\gamma_j\}_{j=1}^n$ is said to be compatible if there exist a configuration γ such that $\gamma_j = \Gamma_j \cap \gamma$, $\forall j$, in particular $\gamma_i \cap \gamma_j =$ if $i \neq j$. A polymer X is a collection of squares $\{\Delta\}_{j=1}^{s(X)}$ and compatible large field regions $\{\gamma_j\}_{j=1}^{l(X)}$ all mutually disjoint. We also define $\underline{X}$:

$$\begin{array}{rcl} X & = & \{\Delta_1, ..., \Delta_{s(X)}, \gamma_1, ..., \gamma_{l(X)}\}, \\ \underline{X} & = & \{\Delta_1, ..., \Delta_{s(X)}, \Gamma_1, ..., \Gamma_{l(X)}\}. \end{array}$$

Figure 2 and 4 illustrate some of these definitions.

3.3 The cluster expansion

We first recall the notion of connected function and what the tree-graph expansion is [9, 11]. Except for the large field problem most of the material in this subsection is standard, we introduce it neverthless for completeness and to introduce a clear set of notations. Let E be a countable set and $\mathcal{P}(E)$ the set of its finite subsets X. Let $\psi : \mathcal{P}(E) \rightarrow$ a function on the subsets of E. Within this subsection $|X|$ denotes the cardinality of a subset $X \subset E$. The associated connected function ψ_c on $\mathcal{P}(E)$ is then defined recursively in the following way

$$\begin{array}{rcll} \psi_c() & \equiv & 0, & \\ \psi_c(X) & \equiv & \psi(X) & \text{if} \quad |X| = 1, \\ \psi(X) & \equiv & \displaystyle\sum_{k=1}^{|X|} \sum_{\substack{\{X_1 ... X_k\} \\ X_i \cap X_j = \\ \cup_j X_j = X}} \prod_{j=1}^{k} \psi_c(X_j) & \text{if} \quad |X| \geq 2. \end{array} \tag{3.16}$$

The inner sum is over all partitions of the subset X into k disjoint subsets X_j. The connected function ψ_{c} appears for instance in an identity between a formal power series and its logarithm. A setting which is sufficiently general for all our purposes is the following. Take $E =$ the natural numbers. Let Ω be a set of objects $\xi_j, j = 1, 2, ...$ (later called polymers) to which we associate variables $z(\xi_j)$ (later called activities). Suppose we have symmetric functions ψ from Ω^k to for all $k \geq 1$. We denote in general $X = \{i_1, ..., i_k\} \subset$ and in particular $N \equiv \{1, ..., N\}$. We define $\psi(X)$'s in which the ξ_j's enter as parameters:

$$\begin{aligned} \psi() &= \psi_0 \in , & (3.17) \\ \psi(X) &= \psi(X;\xi) \equiv \psi(\xi_{i_1}, ..., \xi_{i_k}) \quad \text{if} \quad X = \{i_1, ..., i_k\}, & (3.18) \end{aligned}$$

and similar formulas for ψ_{c}. With these definition we have the following identity between formal power series, assuming $\psi_0 = 1$:

$$\begin{aligned} \text{if} \quad \langle z, \psi\rangle &\equiv 1 + \sum_{N\geq 1} \frac{1}{N!} \sum_{(\xi_1,\ldots,\xi_N)\in\Omega^N} \prod_{j=1}^{N} z(\xi_j)\psi(N;\xi), & (3.19) \\ \text{then } \log\langle z, \psi\rangle &= \sum_{N\geq 1} \frac{1}{N!} \sum_{(\xi_1,\ldots,\xi_N)\in\Omega^N} \prod_{j=1}^{N} z(\xi_j)\psi_{\mathrm{c}}(N;\xi) = \langle z, \psi_{\mathrm{c}}\rangle. & (3.20) \end{aligned}$$

One easy way to see this is to define a $*$-product between two ψ's see e.g. [1]:

$$(\psi_1 * \psi_2)(X) \equiv \sum_{Y\subset X} \psi_1(Y)\psi_2(X\backslash Y). \tag{3.21}$$

Correspondingly when $\psi() = 0$ one defines $*$-powers and $*$-exponentials:

$$(*\exp\psi)(X) \equiv \sum_{n\geq 1} \frac{1}{n!}\psi^{*n}(X) \text{ for } |X| \geq 1, \tag{3.22}$$

and by definition $(*\exp\psi)() \equiv 1$. Formula (3.20) immediately follows by checking that $\langle z, \psi_1 * \psi_2\rangle = \langle z, \psi_1\rangle\langle z, \psi_2\rangle$ and noticing that $\psi = *\exp\psi_{\mathrm{c}}$.

In general there is no simple formula for the ψ_{c}'s in term of the ψ's. However in the particular case where $\psi(X)$ is formally a Boltzman weight of a two-body interaction there is such a formula, namely the tree-graph identity [9]. Indeed suppose

$$\psi(X) = \mathrm{e}^{U(X)} \quad \text{where} \quad U(X) = \frac{1}{2}\sum_{i,j\in X} u(ij). \tag{3.23}$$

The connected function has then an explicit representation. Namely for $|X| \geq 2$, see e.g. [9]:

$$\begin{aligned} \psi_{\mathrm{c}}(X) = \left(\mathrm{e}^{U}\right)_{\mathrm{c}}(X) &= \sum_{T \text{ on } X} \prod_{(ij)\in T} u(ij) \int \mathrm{d}^T\!s\; \mathrm{e}^{U(X,s)} & (3.24) \\ \text{where} \quad U(X,s) &\equiv \frac{1}{2}\sum_{i,j\in X} [1 - s_{\max}(ij)]\, u(ij). \end{aligned}$$

The sum is over all tree graphs having the points of X as vertices, and (ij) denotes the link of the tree T connecting i and j. To each link (ij) in T there is a parameter $s_{(ij)} \in [0, 1]$

associated, $s \equiv \{s_{(ij)}\}_{(ij)\in T}$ and $\int \mathrm{d}^T s$ denotes an integral over all those variables. At last $s_{\max}(ij)$ denotes the largest of the parameters s along the unique path on T connecting i and j when $i \neq j$ and is zero for $i = j$.

Warning:

The above tree-graph expansion for a two-body interaction will be used *twice* in the following:

1. The first time we use (3.24) in (3.33) to get an explicit formula for the activities $K(X)$ in the polymer system representing $Z(\Lambda)$.
2. The polymer system being itself a particular case of a two-body interaction, we shall use (3.20) and (3.24) in (3.44) to get a criteria for the convergence of the series (3.43) representing $\log Z(\Lambda)$.

Beginners should probably meditate this again after having read this subsection.

Let us come back to the gaussian integrals we have to compute in (3.14). Denote by $F[\phi;\gamma]$ the integrand in the denominator for a given large field configuration γ. The above tree-graph expansion will serve as an algebraic device to reorganize in an efficient way the perturbation series generated by Wick's theorem. Efficient here means that we shall be able to prove convergence of the formal power series for $\log Z(\Lambda)$ or $\langle\phi(x_1)...\phi(x_n)\rangle_V^{\mathrm{c}}$. For this, fix a large field configuration γ and for each pair of vertices $v_i, v_j \in D_\gamma$ and $X \subseteq D_\gamma$ define the following commuting differential operators

$$\begin{aligned} u(ij) &\equiv \int_{v_i} \mathrm{d}x \int_{v_j} \mathrm{d}y \; C(x,y)\,\frac{\delta}{\delta\phi(x)}\,\frac{\delta}{\delta\phi(y)}, \\ U(X) &\equiv \frac{1}{2}\sum_{i,j\in X} u(ij), \end{aligned} \tag{3.25}$$

with the help of which Wick's theorem can be rewritten as (see e.g. [11])

$$\int \mathrm{d}\mu_C(\phi) F[\phi;\gamma] = \exp\left(\frac{1}{2}\sum_{i,j\in D_\gamma} u(ij)\right)\Bigg|_{\phi=0} F[\phi;\gamma] = \mathrm{e}^{U(D_\gamma)}\Big|_{\phi=0} F[\phi;\gamma]. \tag{3.26}$$

The exponential is defined by its expansion. Thus by continuity of both sides of the above equation, the formula makes sense when $F[\phi;\gamma]$ leads to an absolutely convergent series, which is what we shall show in section 4. Now take $E = D_\gamma$ and $X = E = D_\gamma$ in (3.16) and (3.23). Then using (3.16) and the fact that all $u(ij)$ commute we write the partition function as

$$\begin{aligned} Z(\Lambda) &= \sum_{\gamma\subset\Lambda}\left(\mathrm{e}^{U(D_\gamma)}\Big|_{\phi=0} F[\phi;\gamma]\right) \\ &\overset{(3.16)}{=} \sum_{\gamma\subset\Lambda}\sum_{k=1}^{|D_\gamma|}\sum_{\substack{\{X_1...X_k\}\\ X_i\cap X_j= \\ \cup_j X_j=D_\gamma}}\prod_{j=1}^{k}\left(\mathrm{e}^{U}\right)_{\mathrm{c}}(X_j)\Big|_{\phi=0} F[\phi;\gamma] \end{aligned} \tag{3.27}$$

$$= \sum_{\gamma\subset\Lambda} \sum_{k=1}^{|D_\gamma|} \sum_{\substack{\{X_1 ... X_k\} \\ X_i \cap X_j = \\ \cup_j X_j = D_\gamma}} \prod_{j=1}^{k} K(X_j) \tag{3.28}$$

$$= \sum_{k\geq 1} \sum_{\substack{\{X_1 ... X_k\} \\ X_i \cap X_j = \\ \cup_j \underline{X}_j = \Lambda}} \prod_{j=1}^{k} K(X_j), \tag{3.29}$$

where the X_j's in (3.29) are collections of elements of some D_γ, they are polymers in the sence of the previous subsection. Equation (3.28) uses the obvious fact that the functional $F[\phi;\gamma]$ factorizes over the different polymers X_j of a given partition $\{X_1, ..., X_k\}$ of D_γ:

$$\begin{aligned} F[\phi;\gamma] &= \prod_{j=1}^{k} F_{X_j}[\phi] \quad \text{where} \\ F_X[\phi] &\equiv \prod_{\Delta\in X} F_\Delta[\phi] \prod_{\gamma\in X} F_\gamma[\phi]. \end{aligned} \tag{3.30}$$

The two factors in (3.30) correspond respectively to the contribution of the small field squares Δ and the large field regions γ in the polymer X, they are given explicitely by

$$F_\Delta[\phi] \equiv \chi_\Delta[\phi] \mathrm{e}^{-\overline{V}_\Delta[\phi]} \tag{3.31}$$

$$F_\gamma[\phi] \equiv \left[\prod_{\Delta\in\gamma} (1 - \chi_\Delta[\phi]) \mathrm{e}^{-\overline{V}_\Delta[\phi]} \right] \left[\prod_{\Delta\in\Gamma\backslash\gamma} \chi_\Delta[\phi] \mathrm{e}^{-\overline{V}_\Delta[\phi]} \right]. \tag{3.32}$$

Γ is the fat large field region associated to γ. The second factor in (3.32) is the contribution of the boundary $\Gamma\backslash\gamma$ of γ. Equation (3.29) just sais that summing over all possible large field configurations γ removes the constraint that $X_j \subset D_\gamma$. To express $K(X)$ as a functional integral we first use the tree-graph expansion (3.24) to write for $|X| \geq 2$

$$K(X) \overset{(3.28)}{=} \left(\mathrm{e}^U\right)_{\mathrm{c}} (X) \Big|_{\phi=0} F_X[\phi] \overset{(3.24)}{=} \sum_{T \,\mathrm{on}\, X} \int \mathrm{d}^T_s \left(\mathrm{e}^{U(X,s)} \prod_{(ij)\in T} u(ij) \right) \Bigg|_{\phi=0} F_X[\phi], \tag{3.33}$$

where according to (3.24)

$$\begin{aligned} U(X,s) &\equiv \frac{1}{2} \sum_{i,j\in X} [1 - s^T_{\max}(ij)] \int_{v_i} \mathrm{d}x \int_{v_j} \mathrm{d}y C(x,y) \frac{\delta}{\delta\phi(x)} \frac{\delta}{\delta\phi(y)} \\ &= \frac{1}{2} \int_X \mathrm{d}x \int_X \mathrm{d}y \; C^T_s(x,y) \frac{\delta}{\delta\phi(x)} \frac{\delta}{\delta\phi(y)}, \end{aligned} \tag{3.34}$$

and the covariance with parameters $C^T_s(x,y)$ is defined by

$$C^T_s(x,y) \equiv \begin{cases} [1 - s^T_{\max}(ij)] C(x,y) & \text{if} \quad x \in v_i \text{ and } y \in v_j, i \neq j, \\ C(x,y) & \text{if} \quad x \text{ and } y \in v_i. \end{cases} \tag{3.35}$$

That $C_s^T(x,y)$ defines a positive operator C_s^T on $L^2({}^2)$ can be seen from the following formula (see [9]) which exhibits C_s^T as a convex sum of positive operators:

$$C_s^T(x,y) = \sum_{\substack{\text{partitions}\\ \pi=\{X_1,\dots,X_r\}\\ \text{of } X}} a_s^T(\pi) \sum_{\ell=1}^{r} \chi_{X_\ell}(x) C(x,y) \chi_{X_\ell}(y) \tag{3.36}$$

$$\text{where} \quad a_s^T(\pi) > 0 \quad \text{and} \quad \sum_\pi a_s^T(\pi) = 1, \tag{3.37}$$

where $\chi_Y(x)$ denotes the caracteristic function of the set $Y \subset {}^2$. Then using (3.26) backwards to go from a differential operator to a functional integral we get at last the expression for $K(X)$ in its full glory

$$K(X) = \tag{3.38}$$
$$= \begin{cases} \displaystyle\int \mathrm{d}\mu_C(\phi) F_X[\phi] & \text{if } |X| = 1, \\ \displaystyle\sum_{T \text{ on } X} \int \mathrm{d}^{T}s \int \mathrm{d}\mu_s^T(\phi) \prod_{(ij)\in T} \left(\int_{v_i} \mathrm{d}x \int_{v_j} \mathrm{d}y \, C(x,y) \frac{\delta}{\delta\phi(x)} \frac{\delta}{\delta\phi(y)} \right) F_X[\phi] & \text{if } |X| \geq 2, \end{cases}$$

where $\mathrm{d}\mu_s^T(\phi)$ is the gaussian measure whose covariance is C_s^T. The correlation lenght of $C(x,y)$ being equal to m^{-1}, see (4.50) below, we choose squares with volume $|\Delta| = m^{-2}$ which are hence approximately decorrelated by the measure $\mathrm{d}\mu(\phi)$. Figure 2 shows an example of a polymer and a tree on it.

Remark:

From the above formula for $K(X)$ we can understand why we chose to take $\mathrm{d}\mu_C(\phi)$ as the perturbed measure rather than the seemingly more natural $\mathrm{e}^{\frac{1}{2}\int_\gamma m^2\phi^2 \mathrm{d}x} \mathrm{d}\mu_C(\phi)$. Suppose indeed we choose the latter whose covariance C_γ will explicitely depend on γ. The new activities $K_\gamma(X)$ will now explicitely depend on the whole large field configuration γ. Also for each *fixed* γ the interaction is still factorized over the polymers X_j in D_γ it will not be factorized any more after summation over the γ's. In other words the polymers would feel a complicated many-body interaction rather than just the hard-core exclusion.

To get a real polymer system we still have to remove the constraint $\cup_j X_j = \Lambda$ (from now on we shall omit the underlining). For this we compute the following ratio:

$$\widetilde{Z}(\Lambda) \equiv \frac{Z(\Lambda)}{K(\Delta)^{N(\Lambda)}} = 1 + \sum_{k\geq 1} \frac{1}{k!} \sum_{\substack{X_1 \dots X_k \subset \Lambda \\ X_i \cap X_j = \\ N(X_j) \geq 2}} \prod_{j=1}^{k} z(X_j) \tag{3.39}$$

$$\text{with} \quad z(X) \equiv \frac{K(X)}{K(\Delta)^{N(X)}}, \tag{3.40}$$

where $N(.)$ is the number of squares Δ. The polymers containing just one square are thus ommited from the sum. We consider now that a polymer $X \subset D_\gamma$ is a point in the set Ω of all polymers. Thus the ξ_j's from the begining of this subsection are in our context the

X_j's. Introducing for $k \geq 2$

$$\psi(X_1, ..., X_k) \equiv \begin{cases} 1 \ \text{ if } \ X_i \cap X_j = & \forall i \neq j, \\ 0 \ \text{ if } \ X_i \cap X_j \neq & \text{for some } i \text{ and } j, \end{cases} \tag{3.41}$$

and $\psi(X_1) = 1$ we have using (3.20)

$$\begin{aligned} \widetilde{Z}(\Lambda) &= \sum_{k \geq 0} \frac{1}{k!} \sum_{X_1 ... X_k \subset \Lambda} \prod_{j=1}^{k} z(X_j) \psi(X_1, ..., X_k), && (3.42) \\ \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \log \widetilde{Z}(\Lambda) &= \sum_{k \geq 1} \frac{1}{k!} \sum_{\substack{X_1 ... X_k \\ \text{connected} \\ \cup_j X_j \ni O}} \prod_{j=1}^{k} z(X_j) \psi_{\mathrm{c}}(X_1, ..., X_k), && (3.43) \end{aligned}$$

where ψ_{c} is the connected function associated to ψ defined in (3.16) and (3.17), the word connected is justified in (3.44) below. The above ψ being of the type (3.23) we can use the tree-graph expansion (3.24) to get the ψ_{c}'s or a bound on them. Just take $u(ij) \equiv u(X_i, X_j) = -\beta$ if $X_i \cap X_j \neq$ and 0 when $X_i \cap X_j =$, then let $\beta \to +\infty$. Let us denote by $G(X_1, ..., X_k)$ the graph whose vertices are the X_j's and where a link connects X_i and X_j whenever they intersect. From (3.24) one has for $k \geq 2$:

$$\begin{aligned} |\psi_{\mathrm{c}}(X_1, ..., X_k)| &= \left| \lim_{\beta \to \infty} \sum_{T \text{ on } \{X_1, ..., X_k\}} \int \mathrm{d}^T s \prod_{(ij) \in T} u(X_i, X_j) \mathrm{e}^{U(X_1, ..., X_k; s)} \right| \\ &\leq \lim_{\beta \to \infty} \sum_{T \text{ on } \{X_1, ..., X_k\}} \int \mathrm{d}^T s \left| \prod_{(ij) \in T} u(X_i, X_j) \mathrm{e}^{(1 - s_{(ij)}) u(X_i, X_j)} \right| \\ &= \lim_{\beta \to \infty} \sum_{T \text{ on } \{X_1, ..., X_k\}} \prod_{(ij) \in T} \left(1 - \mathrm{e}^{u(X_i, X_j)}\right) \\ &= \sum_{T \text{ on } G(X_1, ..., X_k)} 1 \\ &= \text{number of tree-graphs on } G(X_1, ..., X_k), \end{aligned} \tag{3.44}$$

which is the famous tree-graph bound [9]. In particular $\psi_{\mathrm{c}}(X_1, ..., X_k)$ vanishes when $G(X_1, ..., X_k)$ is not connected. We warn the reader that the trees in (3.44) above have the polymer X_j as vertices whereas in (3.24) the trees where built on the polymer themselves. A simple application of Cayley's formula for the number of trees with given incidence numbers shows that the series (3.43) will converge provided that

$$\sum_{X \ni O} z(X) \mathrm{e}^{N(X)} < 1. \tag{3.45}$$

At last we come to the n-point function (2.9). We already have an expansion for the denominator divided by $K(\Delta)^{N(\Lambda)}$. Replacing $F[\phi; \gamma]$ by $\phi(x_1)...\phi(x_n) F[\phi; \gamma]$ generates a similar expansion for the numerator. We just need change the definition of the activities slightly. Fix a polymer X and a set of k points $\{x_1, ..., x_k\} \equiv \omega \subset \underline{X}$. Define

$$z(X; \omega) \equiv \begin{cases} z(X) & \text{if } \quad \omega = \ , \\ \begin{array}{l} \text{replace} \quad F_X[\phi] \quad \text{by} \quad \prod_{x_j \in \omega} \phi(x_j) F_X[\phi] \\ \text{in the definition of} \quad z(X) \end{array} & \text{if } \quad \omega \neq \ . \end{cases} \tag{3.46}$$

Denoting $\omega_j \equiv \{x_1, ..., x_n\} \cap \underline{X}_j$ we have

$$
\begin{aligned}
&\frac{1}{K(\Delta)^{N(\Lambda)}} \int \phi(x_1)...\phi(x_n) \mathrm{e}^{-\overline{V}_\Lambda[\phi]} \mathrm{d}\mu_C(\phi) \\
&= \sum_{k\geq 1} \sum_{\substack{X_1...X_k \subset \Lambda \\ N(X_j)\geq 2 \text{ or } \omega_j \neq \\ \cup_{j=1}^k \omega_j = \{x_1...x_n\}}} \frac{1}{k!} \prod_{j=1}^k z(X_j;\omega_j)\psi(X_1,...,X_k) \\
&\overset{(3.16)}{=} \sum_{\substack{\text{partitions} \\ \{\pi_1...\pi_m\} \\ \text{of } \{x_1...x_n\}}} \prod_{l=1}^m \left(\sum_{k\geq 1} \sum_{\substack{X_1...X_k \\ N(X_j)\geq 2 \text{ or } \omega_j \neq \\ \cup_{j=1}^k \omega_j = \pi_l}} \frac{1}{k!} \prod_{j=1}^k z(X_j;\omega_j)\psi_{\mathrm{c}}(X_1,...,X_k) \right) \widetilde{Z}(\Lambda) \quad (3.47)
\end{aligned}
$$

from which we readily identify the connected n point functions:

$$
\langle \phi(x_1)...\phi(x_n) \rangle_V^{\mathrm{c}} = \sum_{k\geq 1} \sum_{\substack{X_1...X_k \\ N(X_j)\geq 2 \text{ or } \omega_j \neq \\ \cup_{j=1}^k \omega_j = \{x_1...x_n\}}} \frac{1}{k!} \prod_{j=1}^k z(X_j;\omega_j)\psi_{\mathrm{c}}(X_1,...,X_k). \quad (3.48)
$$

The analog of (3.45) then implies convergence of (3.48). In next section we shall prove the following which implies (3.45) when α is small enough:

Theorem 2 *There is a function $\lambda(\alpha) \to 0$ when $\alpha \to 0$, such that*

$$
\begin{aligned}
|z(X;\omega)| &\leq \sum_{T \text{ on } X} \prod_{(ij)\in T} \mathrm{e}^{-\frac{m}{2}\mathrm{dist}(v_i,v_j)} \; \lambda(\alpha)^{N(X)} \leq \\
&\leq \mathrm{e}^{-\frac{m}{4}\,\mathrm{diam}(X)} \sum_{T \text{ on } X} \prod_{(ij)\in T} \mathrm{e}^{-\frac{m}{4}\mathrm{dist}(v_i,v_j)} \; \left(\mathrm{e}^{\frac{1}{4}}\lambda(\alpha)\right)^{N(X)}.
\end{aligned}
$$

.

This shows that $z(X;\omega)$ has a small factor per square Δ of volume m^{-2} and a small factor par lenght m^{-1} of each link $(ij) \in T$. Now consider the case $n = 2$ in (3.48). From the fact that $\psi_{\mathrm{c}} = 0$ when $G(X_1, ..., X_k)$ is disconnected we see that $X_1, ..., X_k$ have to form a chain of polymers between the points x_1 and x_2 of lenght at least $|x_1 - x_2|$. The second inequality in the proposition above directly implies both the convergence of the cluster expansion (3.48) and the exponential decay of the two-point function with rate $m' = m/4$. The main theorem will thus be proven.

4 Convergence of the cluster expansion

In this section we shall bound the activity $z(X)$ of a polymer $X = \{\Delta_1, ..., \Delta_{s(X)}, \gamma_1, ..., \gamma_{l(X)}\}$. Our aim is to prove theorem 2, which implies (3.45) and hence the the convergence of the cluster expansion for the correlation function (3.48). Fix a given polymer X and a tree T on it connecting pairs of vertices v_i, v_j. The small field squares Δ_j and the large field regions γ_j will be treated separately. We start by stating a lemma that we shall use over and over again in the rest of this section.

Lemma 1 *(Volume argument) Take a square Δ and a collection of d vertices $\{v_j\}_{j=1}^d$ connected to it by a tree T. Then for any $r > 0$ there exist a $C(r) > 0$ such that*

$$(d!)^r \prod_{j=1}^{d} \mathrm{e}^{-m\,\mathrm{dist}\,(\Delta,v_j)} \leq C(r).$$

Proof: The product above is obiously largest when all v_j's are squares Δ_j and those are arranged as compactly as possible around Δ. Because they are connected to Δ by a tree all the Δ_j's are different and at least $[d/2]$ of them are such that $\mathrm{dist}(\Delta, \Delta_\mathrm{j}) \geq \mathbf{C} m^{-1}\sqrt{d}$. Thus

$$\prod_{j=1}^{d} \mathrm{e}^{-m\,\mathrm{dist}\,(\Delta,v_j)} \leq \left(\mathrm{e}^{-\mathbf{C}\sqrt{d}/2}\right)^d. \tag{4.49}$$

On the other hand one has the trivial bound

$$(d!)^r \leq \left(d^d\right)^r = (d^r)^d.$$

For large d, $\mathrm{e}^{-\mathbf{C}\sqrt{d}/2}$ is smaller that $1/d^r$. □

This means that we shall be able to compensate local factorials to arbitrary powers by a fraction of the exponential decrease of the covariance $C(x, y)$. Indeed one has the following integral representation [10]:

$$C(x,y) = \frac{\mathrm{e}^{m^2/\kappa}}{(2\pi)^2} \int_{1/\kappa}^{\infty} \frac{\mathrm{e}^{-\alpha m^2 - |x-y|^2/4\alpha}}{\alpha}\, \mathrm{d}\alpha \leq \mathbf{C} \left|\log \frac{\kappa}{m^2}\right| \mathrm{e}^{-m|x-y|}. \tag{4.50}$$

Recall that we chose to take the ultraviolet cut-off $\kappa = m^2$, and the prefactor above is just a constant independant of m.

4.1 Small field squares

Fix a square Δ in the chosen polymer X. Denote by d_Δ the incidence number at the vertex Δ of the chosen tree T. We shall prove:

Proposition 1 *With the notation of last section, the following bound holds:*

$$\left| \int_\Delta \mathrm{d}x_1 ... \int_\Delta \mathrm{d}x_{d_\Delta} \frac{\delta^{d_\Delta}}{\delta\phi(x_1)...\delta\phi(x_{d_\Delta})}\, \chi_\Delta[\phi] \mathrm{e}^{-\overline{V}_\Delta[\phi]} \right| \leq \mathbf{C} (d_\Delta!)^{\mathbf{C}} \alpha^{\frac{1}{3}},$$

uniformely in ϕ.

We suposed above that Δ was a vertex of the polymer X, but obviously the proposition applies as well when Δ is a square in the boundary $\Gamma\backslash\gamma$ of a large field region $\gamma \in X$. We use this in next subsection. From (4.50) we have a factor $\mathbf{C}\mathrm{e}^{-m\,\mathrm{dist}(\Delta,v_j)}$ for each line of the tree T emerging from Δ. We save a factor $\mathbf{C}\mathrm{e}^{-\frac{m}{2}\,\mathrm{dist}(\Delta,v_j)}$ in view of theorem 2 that we want to prove and are thus left with a factor $\mathbf{C}\mathrm{e}^{-\frac{m}{4}\,\mathrm{dist}(\Delta,v_j)}$ for both end points of each line $(ij) \in T$. The volume argument above then allows to compensate the factor $(d_\Delta!)^{\mathbf{C}}$ in the proposition, and we have a small factor $\alpha^{\frac{1}{3}}$ per small field square as we wish.

Proof: Leibnitz's formula for functional derivatives gives

$$\begin{aligned}
&\int_\Delta \mathrm{d}x_1 ... \int_\Delta \mathrm{d}x_{d_\Delta} \frac{\delta^{d_\Delta}}{\delta\phi(x_1)...\delta\phi(x_{d_\Delta})}\ \chi_\Delta[\phi] \mathrm{e}^{-\overline{V}_\Delta[\phi]} = \\
&= \sum_{n=0}^{d_\Delta} \binom{d_\Delta}{n} \left(\int_\Delta \mathrm{d}x_1 ... \int_\Delta \mathrm{d}x_n \frac{\delta^n \chi_\Delta[\phi]}{\delta\phi(x_1)...\delta\phi(x_n)} \right) \left(\int_\Delta \mathrm{d}x_{n+1} ... \int_\Delta \mathrm{d}x_{d_\Delta} \frac{\delta^{(d_\Delta - n)} \mathrm{e}^{-\overline{V}_\Delta[\phi]}}{\delta\phi(x_{n+1})...\delta\phi(x_{d_\Delta})} \right).
\end{aligned}$$

We consider the two factors above separately. Define

$$\chi_\Delta^{(n)}[\phi] \equiv \chi^{(n)} \left(\frac{1}{|\Delta|} \int_\Delta g(\phi(x)) \mathrm{d}x \right), \tag{4.51}$$

where $|\Delta| = m^{-2} = \frac{1}{\alpha^2 \varepsilon^2}$ is the volume of the square. As an example for the first two derivatives one gets:

$$\begin{aligned}
\frac{\delta \chi_\Delta[\phi]}{\delta\phi(x)} &= |\Delta|^{-1} \chi'_\Delta[\phi]\, g'(\phi(x)), \\
\frac{\delta^2 \chi_\Delta[\phi]}{\delta\phi(x)\, \delta\phi(y)} &= |\Delta|^{-2} \chi''_\Delta[\phi]\, g'(\phi(x)) g'(\phi(y)) + |\Delta|^{-1} \chi'_\Delta[\phi]\, g''(\phi(x)) \delta(x-y).
\end{aligned}$$

For $g(\phi)$ we choose the following function:

$$g(\phi) \equiv \lambda_1 \alpha^2 \phi^6 + \lambda_2 \mathrm{e}^{-4\alpha\phi}, \tag{4.52}$$

where $\lambda_1 < 1$ and $\lambda_2 < 1$ are numerical constants independant of α and ε to be ajusted later, see remarks below (5.95) and (5.104) in next section. There are two reasons for this choice:

1. Below (see (4.60)) we shall use Schwartz inequality to separate the factors ϕ^3 and $\mathrm{e}^{-2\alpha t \phi}$ in the formula (2.4) for $\overline{V}(\phi)$. To control both of these factors squared we have to introduce the exponents 6 and 4 in (4.52).
2. The coefficient α^2 has the advantage that it leads to the bound $|\overline{V}_\Delta[\phi]| \le \mathbf{C}$ uniformely in α when $\chi_\Delta[\phi] \neq 0$, see the case $n = 0$ in (4.62).

Now $\chi_\Delta^{(n)}[\phi] \neq 0$, for some $n \geq 0$ implies that both

$$\begin{aligned}
|\Delta|^{-1} \int_\Delta \lambda_1 \alpha^2 \phi^6(x) \mathrm{d}x &\le 1, \\
\text{and } |\Delta|^{-1} \int_\Delta \lambda_2 \mathrm{e}^{-4\alpha\phi(x)} \mathrm{d}x &\le 1.
\end{aligned} \tag{4.53}$$

Also we have for the derivatives of g

$$g^{(n)}(\phi) = \begin{cases} \lambda_2(-4\alpha)^n \mathrm{e}^{-4\alpha\phi} + \lambda_1 \alpha^2 [6 \cdot ... \cdot (6-n+1)] \phi^{6-n}, & \text{for } 1 \le n \le 6, \\ \lambda_2(-4\alpha)^n \mathrm{e}^{-4\alpha\phi} & \text{for } n \ge 7, \end{cases} \tag{4.54}$$

and Hölder's inequality for $m \le 5$, (4.53) and $\alpha < 1$ implie

$$|\Delta|^{-1} \int_\Delta \alpha^2 \phi^m(x) \mathrm{d}x \le \alpha^2 \left(|\Delta|^{-1} \int_\Delta \phi^6(x) \mathrm{d}x \right)^{\frac{m}{6}} \le \alpha^2 \left(\lambda_1 \alpha^2 \right)^{-\frac{m}{6}} \le \mathbf{C} \alpha^{\frac{1}{3}}. \tag{4.55}$$

Thus from (4.53), (4.54) and (4.55) we conclude that when $\chi_\Delta[\phi] \neq 0$

$$\left| |\Delta|^{-1} \int_\Delta g^{(n)}(\phi(x))\mathrm{d}x \right| \leq \mathbf{C} 4^n \alpha^{\frac{1}{3}}. \tag{4.56}$$

Notice that the number of integrals which survive the δ functions, is precisely equal to the number of factors $|\Delta|^{-1}$. The number of terms in $\delta^n \chi_\Delta[\phi]/\delta\phi(x_1)...\delta\phi(x_n)$ is bounded by $n!$ and from (3.11) $\chi_\Delta^{(n)}[\phi] \leq \mathbf{C}(n!)^2$. Thus for the first factor one has

$$\left| \int_\Delta \mathrm{d}x_1 ... \int_\Delta \mathrm{d}x_n \frac{\delta^n \chi_\Delta[\phi]}{\delta\phi(x_1)...\delta\phi(x_n)} \right| \leq \mathbf{C}(n!)^{\mathbf{C}} \alpha^{\frac{1}{3}}. \tag{4.57}$$

The other factor in Leibnitz's formula is similar, as an example the first two derivatives are given by

$$\begin{aligned} \frac{\delta \mathrm{e}^{-\overline{V}_\Delta[\phi]}}{\delta\phi(x)} &= -\overline{V}'(\phi(x))\, \mathrm{e}^{-\overline{V}_\Delta[\phi]}, \\ \frac{\delta^2 \mathrm{e}^{-\overline{V}_\Delta[\phi]}}{\delta\phi(x)\,\delta\phi(y)} &= \overline{V}'(\phi(x))\, \overline{V}'(\phi(y))\, \mathrm{e}^{-\overline{V}_\Delta[\phi]} - \overline{V}''(\phi(x))\, \delta(x-y)\, \mathrm{e}^{-\overline{V}_\Delta[\phi]}. \end{aligned} \tag{4.58}$$

The integrals surviving the δ functions are of the following type:

$$\int_\Delta \overline{V}^{(n)}(\phi(x))\mathrm{d}x = \frac{\alpha}{2} \int_0^1 \mathrm{d}t(1-t)^2 \frac{1}{|\Delta|} \int_\Delta \mathrm{d}x \left[\left(\phi^3(x)\mathrm{e}^{-t\alpha\phi(x)}\right)^{(n)} - 4\left(\phi^3(x)\mathrm{e}^{-2t\alpha\phi(x)}\right)^{(n)} \right]. \tag{4.59}$$

Assuming $\chi_\Delta[\phi] \neq 0$, and using successively Schwartz, Hölder and (4.53) we have

$$\begin{aligned} \left| \frac{1}{|\Delta|} \int_\Delta \phi^m(x) \mathrm{e}^{-2t\alpha\phi(x)}\, \mathrm{d}x \right| &\leq \left(\frac{1}{|\Delta|} \int_\Delta \phi^{2m}(x)\, \mathrm{d}x \right)^{\frac{1}{2}} \left(\frac{1}{|\Delta|} \int \mathrm{e}^{-4t\alpha\phi(x)}\, \mathrm{d}x \right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{|\Delta|} \int_\Delta \phi^6(x)\, \mathrm{d}x \right)^{\frac{m}{6}} \left(\frac{1}{|\Delta|} \int \mathrm{e}^{-4\alpha\phi(x)}\, \mathrm{d}x \right)^{\frac{t}{2}} \\ &\overset{(4.53)}{\leq} \left(\frac{1}{\lambda_1 \alpha^2} \right)^{\frac{m}{6}} \left(\frac{1}{\lambda_2} \right)^{\frac{t}{2}} \leq \mathbf{C}\alpha^{-\frac{m}{3}},. \end{aligned} \tag{4.60}$$

The term with $\mathrm{e}^{-\alpha t\phi(x)}$ is similar. Starting with $m = 3$ in (4.60) with see that going from n to $n+1$ in (4.59) either produces a factor $\alpha^{\frac{1}{3}}$ from the last inequality in (4.60) or produces a factor $2\alpha \leq 2\alpha^{\frac{1}{3}}$ from the derivative of the exponential. Altogether we get

$$\left| \frac{1}{|\Delta|} \int_\Delta \left(\phi^3(x)\mathrm{e}^{-2t\alpha\phi(x)}\right)^{(n)} \mathrm{d}x \right| \leq \mathbf{C} 4^n \alpha^{-1+\frac{n}{3}}, \tag{4.61}$$

and thus from (4.59)

$$\left| \int_\Delta \overline{V}^{(n)}(\phi(x))\mathrm{d}x \right| \leq \mathbf{C} \left(4\alpha^{\frac{1}{3}}\right)^n, \ n \geq 0. \tag{4.62}$$

The case $n = 0$ in particular implies

$$K(\Delta) \equiv \int \mathrm{d}\mu(\phi) \chi_\Delta[\phi] \mathrm{e}^{-\overline{V}_\Delta[\phi]} \geq \mathbf{C} \int \mathrm{d}\mu(\phi)\chi_\Delta[\phi] \geq \mathbf{C}, \tag{4.63}$$

if α is small enough. As before the number of terms in the n^{th} derivative of $\mathrm{e}^{-\overline{V}_\Delta[\phi]}$ is bounded by $n!$, thus

$$\left|\int_\Delta \mathrm{d}x_{n+1} \ldots \int_\Delta \mathrm{d}x_{d_\Delta} \frac{\delta^{(d_\Delta-n)}\mathrm{e}^{-\overline{V}_\Delta[\phi]}}{\delta\phi(x_{n+1})\ldots\delta\phi(x_{d_\Delta})}\right| \leq [(d_\Delta-n)!]^{\mathbf{C}}(\alpha^{\frac{1}{3}})^{(d_\Delta-n)}\mathrm{e}^{-\overline{V}_\Delta[\phi]}. \tag{4.64}$$

The sum of the binomial coefficient just gives a factor $2^{d_\Delta} \leq d_\Delta!$. Combining (4.57) and (4.64) proves proposition 1.□

4.2 Large field regions

Fix a large field region γ (we omit the index j of γ_j here) in our polymer X, and denote Γ the fatened large field region associated to it. Let d_Γ denote the incidence number of the tree T at the vertex γ and $\{v_j\}_{j=1}^{d_\Gamma}$ all vertices in X connected to γ by T. We first decompose the integral we have to compute into smaller pieces:

$$\left|\int_\Gamma \mathrm{d}x_1 \ldots \int_\Gamma \mathrm{d}x_{d_\Gamma} \left(\prod_{j=1}^{d_\Gamma} \mathbf{C}\mathrm{e}^{-\frac{m}{4}\mathrm{dist}\,(x_j,v_j)}\right) \frac{\delta^{d_\Gamma}F[\phi;\gamma]}{\delta\phi(x_1)\ldots\delta\phi(x_{d_\Gamma})}\right| =$$

$$= \left|\sum_{\Delta_1\subset\Gamma} \ldots \sum_{\Delta_{d_\Gamma}\subset\Gamma} \left(\prod_{j=1}^{d_\Gamma} \mathbf{C}\mathrm{e}^{-\frac{m}{4}\mathrm{dist}\,(\Delta_j,v_j)}\right) \int_{\Delta_1}\mathrm{d}x_1 \ldots \int_{\Delta_{d_\Gamma}} \mathrm{d}x_{d_\Gamma} \frac{\delta^{d_\Gamma}F[\phi;\gamma]}{\delta\phi(x_1)\ldots\delta\phi(x_{d_\Gamma})}\right| \tag{4.65}$$

The above sum has $N(\Gamma)^{d_\Gamma}$ terms. The naive bound obtained by just bounding each term in the sum by the supremum is not good enough, because the volume argument does not allow to compensate the factor $N(\Gamma)^{d_\Gamma}$. Therefore we reorganize the above sum in the following way. We first fix the number d_Δ of lines arriving in each square $\Delta \subset \Gamma$. In other words we prescribe a distribution $\{d_\Delta\}_{\Delta\subset\Lambda}$ of incidence numbers and we sum over all possible ways to link the d_Γ vertices v_j to Γ with this constraint. This sum is really a sum over all decompositions of the set $\{v_1, \ldots, v_{d_\Gamma}\}$ of all vertices connected to γ into disjoint subsets π_Δ indexed by the squares in Γ and having a prescribed number of elements $|\pi_\Delta| = d_\Delta$. Then we sum over the distributions $\{d_\Delta\}_{\Delta\subset\Lambda}$. Let us denote $\pi_\Delta(j)$ the j^{th} vertex of the subset π_Δ.

$$(4.65) \leq \sum_{\substack{\{d_\Delta\}_{\Delta\subset\Gamma}\\ \sum_\Delta d_\Delta = d_\Gamma}} \sum_{\substack{\pi=\{\pi_\Delta\}_{\Delta\subset\Gamma}\\ \text{of }\{v_1\ldots v_{d_\Gamma}\}\\ |\pi_\Delta|=d_\Delta}} \left(\prod_{\Delta\subset\Gamma}\prod_{j=1}^{d_\Delta} \mathbf{C}\mathrm{e}^{-\frac{m}{4}\mathrm{dist}\,(\Delta,\pi_\Delta(j))}\right)$$

$$\cdot\left(\prod_{\Delta\subset\gamma}\left|\int_\Delta \mathrm{d}x_1\ldots\int_\Delta \mathrm{d}x_{d_\Delta}\frac{\delta^{d_\Delta}(1-\chi_\Delta[\phi])\mathrm{e}^{-\overline{V}_\Delta[\phi]}}{\delta\phi(x_1)\ldots\delta\phi(x_{d_\Delta})}\right|\right)\left(\prod_{\Delta\subset\Gamma\setminus\gamma}\left|\int_\Delta \mathrm{d}x_1\ldots\int_\Delta \mathrm{d}x_{d_\Delta}\frac{\delta^{d_\Delta}\chi_\Delta[\phi]\mathrm{e}^{-\overline{V}_\Delta[\phi]}}{\delta\phi(x_1)\ldots\delta\phi(x_{d_\Delta})}\right|\right).$$

The number of terms in the first sum is bounded by $2^{N(\Gamma)+d_\Gamma}$. A rough bound for the second sum is obtained by replacing it by d_Γ independent sums over all squares in ^{2}. Using a fraction of the exponential decrease to bound each of them by a constant we get:

$$\sum_{\substack{\pi=\{\pi_\Delta\}_{\Delta\subset\Gamma}\\ \text{of }\{v_1\ldots v_{d_\Gamma}\}\\ |\pi_\Delta|=d_\Delta}} \prod_{\Delta\subset\Gamma}\prod_{j=1}^{d_\Delta} \mathbf{C}\mathrm{e}^{-\frac{m}{4}\mathrm{dist}\,(\Delta,\pi_\Delta(j))} \leq \sup_{\substack{\pi=\{\pi_\Delta\}_{\Delta\subset\Gamma}\\ \text{of }\{v_1\ldots v_{d_\Gamma}\}\\ |\pi_\Delta|=d_\Delta}} \prod_{\Delta\subset\Gamma}\prod_{j=1}^{d_\Delta} \mathbf{C}\mathrm{e}^{-\frac{m}{4}\left(1-\frac{1}{4}\right)\mathrm{dist}\,(\Delta,\pi_\Delta(j))}$$

$$= \prod_{\Delta\subset\Gamma}\prod_{j=1}^{d_\Delta} \mathbf{C}\mathrm{e}^{-\frac{m}{4}\left(1-\frac{1}{4}\right)\mathrm{dist}\,(\Delta,v_{\Delta,j})}, \tag{4.66}$$

where $v_{\Delta,j}$ is the decomposition which realizes the maximum for the given $\{d_\Delta\}$. The factor $2^{d_\Gamma}\mathbf{C}^{d_\Gamma} = \prod_{\Delta\in\Gamma}(2\mathbf{C})^{d_\Delta}$ can be compensated by another fraction of the exponential decay, see e.g. (4.49). Thus we are left with:

$$(4.65) \leq 2^{N(\Gamma)} \sup_{\substack{\{d_\Delta\}_{\Delta\subset\Gamma}\\ \sum_\Delta d_\Delta = d_\Gamma}} \left(\prod_{\Delta\subset\Gamma}\prod_{j=1}^{d_\Delta} \mathbf{C}\mathrm{e}^{-\frac{m}{4}\left(1-\frac{2}{4}\right)\mathrm{dist}\,(\Delta,v_{\Delta,j})}\right) \tag{4.67}$$
$$\cdot\left(\prod_{\Delta\subset\gamma}\left|\int_\Delta \mathrm{d}x_1 ... \int_\Delta \mathrm{d}x_{d_\Delta} \frac{\delta^{d_\Delta}(1-\chi_\Delta[\phi])\mathrm{e}^{-\overline{V}_\Delta[\phi]}}{\delta\phi(x_1)...\delta\phi(x_{d_\Delta})}\right|\right)\left(\prod_{\Delta\subset\Gamma\backslash\gamma}\left|\int_\Delta \mathrm{d}x_1 ... \int_\Delta \mathrm{d}x_{d_\Delta} \frac{\delta^{d_\Delta}\chi_\Delta[\phi]\mathrm{e}^{-\overline{V}_\Delta[\phi]}}{\delta\phi(x_1)...\delta\phi(x_{d_\Delta})}\right|\right).$$

The last factor in the formula above is controled by proposition 1 which gives a factor $\mathbf{C}(d_\Delta!)^{\mathbf{C}}\alpha^{\frac{1}{3}}$ per square. Recall that this bound was obtained using only the small field condition $\chi_\Delta[\phi]$. The remaining part of the exponential decay compensates the local factorial. Thus for the squares Δ in the boundary $\Gamma\backslash\gamma$ for which $d_\Delta \neq 0$ we have a small factor $\mathbf{C}\alpha^{\frac{1}{3}}$ as before.

Heuristic discussion

The second factor corresponding to squares in the large field region has to be controled by other methods. Suppose the field $\phi(x)$ in Δ is very negative, then the potential $V_\Delta[\phi]$ will be very large and positive leading to a very small factor $\mathrm{e}^{-V_\Delta[\phi]}$ which compensates any power of $V_\Delta[\phi]$ or its derivatives. If $\phi(x) \geq 0$ and verifies $(1-\chi_\Delta[\phi]) \neq 0$ we distinguish two cases

- Either $\phi(x)$ is more or less constant in Δ and thus from (4.52) ϕ is larger then $\alpha^{-\frac{1}{3}}$. But $\alpha^{-\frac{1}{3}} \leq \alpha^{-1}$ for $\alpha \leq 1$ and $V_\Delta[\phi] = \int_\Delta V(\phi(x))\mathrm{d}x \approx |\Delta|\frac{m^2\phi^2}{2} \approx \alpha^{-\frac{2}{3}}$. Thus we can expect a small factor $\mathrm{e}^{-1/\alpha^{\frac{2}{3}}}$ from $\mathrm{e}^{-V_\Delta[\phi]}$.
- Or $\phi(x)$ is not constant in Δ. In that case we shall prove by means of *Sobolev inequalities* that $\int_\Delta(\nabla\phi)^2(x)\mathrm{d}x \geq \alpha^{-\frac{2}{3}}$. The small factor will then be provided by the measure $\mathrm{e}^{\frac{1}{2}\int_\gamma m^2\phi^2(x)\mathrm{d}x}\mathrm{d}\mu_s^T(\phi)$ which is locally massless in γ and thus formally contains a factor $\mathrm{e}^{-\frac{1}{2}\int_\Delta(\nabla\phi)^2(x)\mathrm{d}x}$ for any square $\Delta\in\Gamma$. Therefore in this case also we can expect a small factor $\mathrm{e}^{-1/\alpha^{\frac{2}{3}}}$ per square.

Altogether we can expect a factor $\mathrm{e}^{-\mathbf{C}N(\gamma)/\alpha^{\frac{2}{3}}}$ for every large field region $\gamma\in X$. To bound products of derivatives of $V_\Delta[\phi]$ we use the fact that the field $\phi(x)$ is small in average in the corridor $B(\gamma)$ around γ and the fact that $\phi(x)$ cannot grow too fast because of the factor $\mathrm{e}^{-\frac{1}{2}\int_\Delta(\nabla\phi)^2(x)\mathrm{d}x}$, see lemma 3 below.

The main tools to make the above discussion rigorous are the following two propositions, proven in next section. The first relies on Sobolev inequalities:

Proposition 2 *Suppose that* $(1-\chi_\Delta[\phi])\neq 0$*, that is either* $\frac{1}{|\Delta|}\int_\Delta \alpha^2\phi^6(x)\mathrm{d}x \geq \frac{1}{4\lambda_1}$ *or* $\frac{1}{|\Delta|}\int_\Delta \mathrm{e}^{-4\alpha\phi(x)}\mathrm{d}x \geq \frac{1}{4\lambda_2}$ *then there exists constants* λ_1 *and* λ_2 *independant of* α *and* ε *such that*

$$\begin{array}{rrcl} \textit{either} & V_\Delta[\phi] & \geq & \mathbf{C}/\alpha^{\frac{2}{3}}, \\ \textit{or} & \displaystyle\int_\Delta (\nabla\phi)^2(x)\mathrm{d}x & \geq & \mathbf{C}/\alpha^{\frac{2}{3}}. \end{array}$$

The second proposition deals with the normalisation problem associated with extracting a factor $\mathrm{e}^{-\frac{1}{2}\int_{\overline{\gamma}}(\nabla\phi)^2(x)\mathrm{d}x}$ from the locally massless measure. Recall that $\overline{\gamma}\equiv\gamma\cup B(\gamma)$ and that $\mathrm{d}\mu_s^T(\phi)$ is a gaussian probability measure on fields $\phi(x)$ defined on the support $\underline{X}$ of the polymer X.

Proposition 3 *With the definitions of subsections 3.2 and 3.3 and assuming L is large enough and* η *is small enough, the following formula defines a new gaussian probability measure* $\mathrm{d}\widetilde{\mu}_s^T(\phi)$

$$\mathrm{e}^{\frac{m^2}{2}\int_\gamma \phi^2(x)\mathrm{d}x}\mathrm{e}^{\frac{1}{2}\int_{\overline{\gamma}}(\nabla\phi)^2(x)\mathrm{d}x}\mathrm{d}\mu_s^T(\phi)\equiv\mathcal{N}_s^T\mathrm{d}\widetilde{\mu}_s^T(\phi),$$

where $\mathrm{d}\widetilde{\mu}_s^T(\phi)$ *has covariance* $\widetilde{C}_s^T \equiv \left[\left(C_s^T\right)^{-1} - m^2\chi_\gamma - \sum_{j=1}^2 \nabla_j^\dagger\chi_{\overline{\gamma}}\nabla_j\right]^{-1} > 0$*, and all operators are on* $L^2({}^2,\mathrm{d}x)$*. Moreover*

$$\mathcal{N}_s^T \leq \mathrm{e}^{\mathbf{C}N(\gamma)}.$$

For small enough α the normalisation factor $\mathcal{N}_s^T$ will be compensated by the volume factor $\mathrm{e}^{-\mathbf{C}N(\gamma)/\alpha^{\frac{2}{3}}}$ expected above.

Let us go back to (4.67) and apply Leibnitz formula for functional derivatives to the second factor:

$$\left|\int_\Delta \mathrm{d}x_1 ... \int_\Delta \mathrm{d}x_{d_\Delta}\frac{\delta^{d_\Delta}(1-\chi_\Delta[\phi])\mathrm{e}^{-\overline{V}_\Delta[\phi]}}{\delta\phi(x_1)...\delta\phi(x_{d_\Delta})}\right| \leq \tag{4.68}$$

$$\leq \sum_{n=0}^{d_\Delta}\binom{d_\Delta}{n}\left|\int_\Delta \mathrm{d}x_1 ... \int_\Delta \mathrm{d}x_n\frac{\delta^n(1-\chi_\Delta[\phi])}{\delta\phi(x_1)...\delta\phi(x_n)}\right|\left|\int_\Delta \mathrm{d}x_{n+1} ... \int_\Delta \mathrm{d}x_{d_\Delta}\frac{\delta^{(d_\Delta-n)}\mathrm{e}^{-\overline{V}_\Delta[\phi]}}{\delta\phi(x_{n+1})...\delta\phi(x_{d_\Delta})}\right|.$$

Using (4.57) to bound the second factor in (4.68), and (4.58) we get

$$(4.68) \leq (d_\Delta!)^{\mathbf{C}} \sup_{\substack{\{n_j\} \\ \sum_{j=1}^k n_j\leq d_\Delta}} \left|\prod_{j=1}^k\int_\Delta \overline{V}^{(n_j)}(\phi(x))\mathrm{d}x\right|\mathrm{e}^{-\overline{V}_\Delta[\phi]}. \tag{4.69}$$

We first prove:

Lemma 2 *For any* $d_\Delta\geq 0$ *and any* $\{n_j\}_{j=1}^k$ *such that* $\sum_{j=1}^k n_j\leq d_\Delta$ *one has*

$$\left|\prod_{j=1}^k\int_\Delta \overline{V}^{(n_j)}(\phi(x))\mathrm{d}x\right|\mathrm{e}^{-\overline{V}_\Delta[\phi]} \leq$$

$$\leq \mathbf{C}^{d_\Delta}\,\mathrm{e}^{\frac{m^2}{2}\int_\Delta\phi^2(x)\mathrm{d}x}\,\mathrm{e}^{\frac{1}{4}\int_\Delta(\nabla\phi)^2(x)\mathrm{d}x}\,\mathrm{e}^{-\mathbf{C}/\alpha^{\frac{2}{3}}}\left[(d_\Delta!)^{\mathbf{C}}+\left(\frac{1}{|\Delta|}\int_\Delta|\phi(x)|\mathrm{d}x\right)^{d_\Delta}\right] \tag{4.70}$$

Proof: Recall that

$$\begin{aligned}\overline{V}(\phi) &= \varepsilon^2\left(-\mathrm{e}^{-\alpha\phi}+\frac{1}{2}\mathrm{e}^{-2\alpha\phi}+\frac{1}{2}\right)-\frac{\varepsilon^2\alpha^2}{2}\phi^2 &(4.71)\\ &= \frac{1}{2}\varepsilon^2\alpha^3\phi^3\int_0^1(1-t)^2\left(\mathrm{e}^{-\alpha t\phi}-4\mathrm{e}^{-2\alpha t\phi}\right)\mathrm{d}t. &(4.72)\end{aligned}$$

First suppose that $\phi \geq 0$. For $n \geq 2$, (4.71) immediately implies $|\overline{V}^{(n)}(\phi)| \leq \mathbf{C}2^n m^2$. For $n = 1, \alpha\phi \leq 1$ and (4.72) implie $|\overline{V}'[\phi]| \leq \mathbf{C}m^2\alpha\phi^2 + \mathbf{C}m^2\alpha^2\phi^3 \leq m^2\phi$. When $\alpha\phi \geq 1$, $\mathrm{e}^{-\alpha\phi} < \alpha\phi$ and (4.71) implie $|\overline{V}'(\phi)| \leq \mathbf{C}m^2\phi$. Next suppose that $\phi \leq 0$. From (4.71) we deduce successively that $|\overline{V}'(\phi)| \leq \mathbf{C}\left|\varepsilon^2\alpha\left(\mathrm{e}^{-2\alpha\phi}-\alpha\phi\right)\right| \leq \mathbf{C}\varepsilon^2\alpha\mathrm{e}^{-2\alpha\phi}$, $|\overline{V}''(\phi)| \leq \mathbf{C}|\varepsilon^2\alpha^2\left(\mathrm{e}^{-2\alpha\phi}-1\right)| \leq \mathbf{C}\varepsilon^2\alpha^2\mathrm{e}^{-2\alpha\phi}$ and $|\overline{V}^{(n)}(\phi)| \leq \mathbf{C}\varepsilon^2 2^n\alpha^n\mathrm{e}^{-2\alpha\phi}$ for $n \geq 3$. Summarizing we have

$$\overline{V}^{(n)} \leq \left\{\begin{array}{lll}\mathbf{C}\,2^n m^2(\phi+1) & \text{if} & \phi \geq 0,\\ \mathbf{C}\,2^n\varepsilon^2\alpha\,\mathrm{e}^{-2\alpha\phi} & \text{if} & \phi \leq 0.\end{array}\right. \tag{4.73}$$

Define

$$\begin{aligned}\Delta^{>} &\equiv \{x \in \Delta | \phi(x) \geq 0\},\\ \Delta^{<} &\equiv \{x \in \Delta | \phi(x) \leq 0\}.\end{aligned}$$

Then

$$\begin{aligned}&\left|\prod_{j=1}^{k}\int_\Delta \overline{V}^{(n_j)}(\phi(x))\mathrm{d}x\right|\mathrm{e}^{-\overline{V}_\Delta[\phi]} \leq\\ &\leq \mathrm{e}^{\frac{m^2}{2}\int_\Delta\phi^2(x)\mathrm{d}x}\prod_{j=1}^{k}\mathrm{e}^{-\frac{V_\Delta[\phi]}{k}}\int_\Delta\left|\overline{V}^{(n_j)}(\phi(x))\right|\mathrm{d}x\\ &\leq \mathbf{C}^{d_\Delta}\mathrm{e}^{\frac{m^2}{2}\int_\Delta\phi^2(x)\mathrm{d}x}\prod_{j=1}^{k}\mathrm{e}^{-\frac{V_\Delta[\phi]}{k}}\left(\varepsilon^2\alpha\int_{\Delta^<}\mathrm{e}^{-2\alpha\phi(x)}\mathrm{d}x+\frac{1}{|\Delta|}\int_{\Delta^>}(\phi(x)+1)\mathrm{d}x\right)\end{aligned} \tag{4.74}$$

But

$$\mathrm{e}^{-2\alpha\phi} = \left(\mathrm{e}^{-\alpha\phi}-1\right)^2+2\left(\mathrm{e}^{-\alpha\phi}-1\right)+1, \tag{4.75}$$

and

$$\begin{aligned}\varepsilon^2\int_{\Delta^<}\left(\mathrm{e}^{-\alpha\phi(x)}-1\right)\mathrm{d}x &\leq \varepsilon^2\int_\Delta\left|\mathrm{e}^{-\alpha\phi(x)}-1\right|\mathrm{d}x = \varepsilon^2|\Delta|\frac{1}{|\Delta|}\int_\Delta|\mathrm{e}^{-\alpha\phi(x)}-1| \leq\\ &\leq \varepsilon^2|\Delta|\left(\frac{1}{|\Delta|}\int_\Delta\left(\mathrm{e}^{-2\alpha\phi(x)}-1\right)^2\right)^{\frac{1}{2}} = \frac{\sqrt{2}}{\alpha}\left(V_\Delta[\phi]\right)^{\frac{1}{2}}. &(4.76)\end{aligned}$$

Combining (4.74),(4.75) and (4.76) and assuming $(1-\chi_\Delta[\phi]) \neq 0$ we get

$$\begin{aligned}&\left|\prod_{j=1}^{k}\int_\Delta\overline{V}^{(n_j)}(\phi(x))\mathrm{d}x\right|\mathrm{e}^{-\overline{V}_\Delta[\phi]} \leq &(4.77)\\ &\leq \mathbf{C}^{d_\Delta}\mathrm{e}^{\frac{m^2}{2}\int_\Delta\phi^2(x)\mathrm{d}x}\left[\mathrm{e}^{-\frac{V_\Delta[\phi]}{k}}\left(2\alpha V_\Delta[\phi]+\sqrt{8}\left(V_\Delta[\phi]\right)^{\frac{1}{2}}+\frac{1}{\alpha}+\frac{1}{|\Delta|}\int_\Delta|\phi(x)|\mathrm{d}x+1\right)\right]^k\end{aligned}$$

-bb-error = =

Figure 3: A partition of $\overline{\gamma}$ in regions Ω. The region Ω is associated to $\Delta' \in B(\gamma)$.

$$
\begin{aligned}
&\leq \mathbf{C}^{d_\Delta} \mathrm{e}^{\frac{m^2}{2}\int_\Delta \phi^2(x)\mathrm{d}x} \left[\mathrm{e}^{-\frac{V_\Delta[\phi]}{2k}} \left(\alpha k + \sqrt{k} + \frac{1}{\alpha} + 1 + \frac{1}{|\Delta|}\int_\Delta |\phi(x)|\mathrm{d}x \right) \right]^k \\
&\leq \mathbf{C}^{d_\Delta} \mathrm{e}^{\frac{m^2}{2}\int_\Delta \phi^2(x)\mathrm{d}x} \mathrm{e}^{\frac{1}{4}\int_\Delta (\nabla\phi)^2(x)\mathrm{d}x} \left(\mathrm{e}^{-\frac{1}{4}\int_\Delta (\nabla\phi)^2(x)\mathrm{d}x} \mathrm{e}^{-\frac{V_\Delta[\phi]}{2}} \right) \left[\left(\frac{d_\Delta}{\alpha} \right)^{d_\Delta} + \left(\frac{1}{|\Delta|}\int_\Delta |\phi(x)|\mathrm{d}x \right)^{d_\Delta} \right] \\
&\overset{\text{Prop.2}}{\leq} \mathbf{C}^{d_\Delta} \mathrm{e}^{\frac{m^2}{2}\int_\Delta \phi^2(x)\mathrm{d}x} \mathrm{e}^{\frac{1}{4}\int_\Delta (\nabla\phi)^2(x)\mathrm{d}x} \mathrm{e}^{-\mathbf{C}/\alpha^{\frac{2}{3}}} \left[\left(\frac{d_\Delta}{\alpha} \right)^{d_\Delta} + \left(\frac{1}{|\Delta|}\int_\Delta |\phi(x)|\mathrm{d}x \right)^{d_\Delta} \right].
\end{aligned}
$$

Using the trivial inequalities $n^n \leq \mathbf{C}(n!)^{\mathbf{C}}$ and $\mathrm{e}^{-\mathbf{C}/\alpha^{\frac{2}{3}}} \frac{1}{\alpha^n} \leq \mathbf{C}(n!)^{\mathbf{C}} \mathrm{e}^{-\mathbf{C}/\alpha^{\frac{2}{3}}}$ we get the desired result (4.70). □ Lemma 2.

To control the second term in the braket in (4.70) we express the field inside γ in terms of the field in the boundary $B(\gamma)$. We assume $\chi_\Delta[\phi] \neq 0$ for all $\Delta \in B(\gamma)$. Take y in the square $\Delta' \in B(\gamma)$ which is closest to $x \in \gamma$.

$$
\begin{aligned}
|\phi(x)| &= \left| \phi(y) + \int_0^1 (x-y)\cdot\nabla\phi(y+t(x-y))\mathrm{d}t \right| \\
&\leq |\phi(y)| + \mathbf{C}\,\mathrm{dist}(\Delta, B(\gamma)) \int_0^1 |\nabla\phi(y+t(x-y))|\mathrm{d}t. \qquad (4.78)
\end{aligned}
$$

For $\Delta \in \gamma$ and $\Delta' \in B(\gamma)$ define the strip $A(\Delta, \Delta')$ by

$$
A(\Delta, \Delta') \equiv \left\{ z \in {}^2 | z = y + t(x-y), x \in \Delta, y \in \Delta', t \in [0,1] \right\}. \qquad (4.79)
$$

Obviously

$$
|A(\Delta, \Delta')| \geq \mathbf{C}|\Delta|\mathrm{dist}(\Delta, B(\gamma))\, m. \qquad (4.80)
$$

Now averaging (4.78) over $x \in \Delta$ and $y \in \Delta'$, using the small field condition in Δ', Hölder's inequality and (4.80) one gets

$$
\begin{aligned}
&\frac{1}{|\Delta|}\int_\Delta |\phi(x)|\mathrm{d}x \leq \\
&\quad\leq \frac{1}{|\Delta'|}\int_{\Delta'} |\phi(y)|\mathrm{d}y + \mathbf{C}\,\mathrm{dist}(\Delta, B(\gamma)) \frac{1}{|\Delta|}\int_\Delta \mathrm{d}x \frac{1}{|\Delta'|}\int_{\Delta'} \mathrm{d}y \int_0^1 \mathrm{d}t |\nabla\phi(y+t(x-y))| \\
&\quad\leq \frac{\mathbf{C}}{\alpha^{\frac{1}{3}}} + \mathbf{C}\,\mathrm{dist}(\Delta, B(\gamma)) \frac{1}{|A(\Delta,\Delta')|} \int_{A(\Delta,\Delta')} |\nabla\phi(x)|\mathrm{d}x \\
&\quad\leq \frac{\mathbf{C}}{\alpha^{\frac{1}{3}}} + \mathbf{C}\,\mathrm{dist}(\Delta, B(\gamma))^{\frac{1}{2}}\, m^{\frac{1}{2}} \left(\int_{A(\Delta,\Delta')} (\nabla\phi)^2(x)\mathrm{d}x \right)^{\frac{1}{2}}. \qquad (4.81)
\end{aligned}
$$

To use (4.81) systematically we make a partition of $\overline{\gamma}$ in regions Ω in the following way. To each square $\Delta' \in B(\gamma)$ we associate the region $\Omega \subset \overline{\gamma}$ made of all squares $\Delta \in \gamma$ for which Δ' is precisely the closest square in $B(\gamma)$, see figure 3. By $\Omega(\Delta)$ we mean the region containing Δ. In particular $A(\Delta, \Delta') \subset \Omega(\Delta)$. We also denote $d_\Omega \equiv \sum_{\Delta\subset\Omega} d_\Delta$ the total number of lines arriving in Ω. Then we have

Lemma 3 *With the above notations and assuming* $\chi_{\Delta'}[\phi] \neq 0$ *for* $\Delta' \in B(\gamma)$ *and* $(1-\chi_{\Delta}[\phi]) \neq 0$ *for* $\Delta \in \gamma$ *one has*

$$\prod_{\Delta\subset\gamma}\left[(d_\Delta!)^{\mathbf{C}} + \left(\frac{1}{|\Delta|}\int_\Delta |\phi(x)|\mathrm{d}x\right)^{d_\Delta}\right] \leq$$
$$\leq \mathrm{e}^{\frac{1}{4}\int_{\overline{\gamma}}(\nabla\phi)^2(x)\mathrm{d}x}\mathrm{e}^{\mathbf{C}N(\gamma)/2\alpha^{\frac{2}{3}}}\left(\prod_{\Delta\subset\gamma}\prod_{j=1}^{d_\Delta}\mathbf{C}\mathrm{e}^{\frac{m}{4}\frac{1}{4}\operatorname{dist}(\Delta,v_{\Delta,j})}\right)\prod_{\Omega\subset\overline{\gamma}} d_\Omega!. \qquad (4.82)$$

Proof:

$$\prod_{\Delta\subset\gamma}\left[(d_\Delta!)^{\mathbf{C}} + \left(\frac{1}{|\Delta|}\int_\Delta |\phi(x)|\mathrm{d}x\right)^{d_\Delta}\right] \leq$$
$$\overset{(4.81)}{\leq} \prod_{\Delta\subset\gamma}\left[(d_\Delta!)^{\mathbf{C}} + \left\{\frac{\mathbf{C}}{\alpha^{\frac{1}{3}}} + \mathbf{C}[m\operatorname{dist}(\Delta,B(\gamma))]^{\frac{1}{2}}\left(\int_{\Omega(\Delta)}(\nabla\phi)^2(x)\mathrm{d}x\right)\right\}^{d_\Delta}\right]$$
$$\leq \mathrm{e}^{\mathbf{C}N(\gamma)/2\alpha^{\frac{2}{3}}}\left(\prod_{\Delta\subset\gamma}\prod_{j=1}^{d_\Delta}\mathrm{e}^{\frac{m}{4}\frac{1}{4}\operatorname{dist}(\Delta,v_{\Delta,j})}\right)$$
$$\cdot\prod_{\Delta\subset\gamma}\left[\left(\prod_{j=1}^{d_\Delta}\mathrm{e}^{-\frac{m}{4}\frac{1}{4}\operatorname{dist}(\Delta,v_{\Delta,j})}\right)(d_\Delta!)^{\mathbf{C}} + \right.$$
$$\left.\left\{\mathrm{e}^{-\mathbf{C}/2d_\Delta\alpha^{\frac{2}{3}}}\frac{\mathbf{C}}{\alpha^{\frac{1}{3}}} + \mathrm{e}^{-\frac{m}{4}\frac{1}{4}\operatorname{dist}(\Delta,B(\gamma))}\mathbf{C}\left[m\operatorname{dist}(\Delta,B(\gamma))\right]^{\frac{1}{2}}\left(\int_{\Omega(\Delta)}(\nabla\phi)^2(x)\mathrm{d}x\right)^{\frac{1}{2}}\right\}^{d_\Delta}\right]$$
$$\leq \mathrm{e}^{\mathbf{C}N(\gamma)/2\alpha^{\frac{2}{3}}}\left(\prod_{\Delta\subset\gamma}\prod_{j=1}^{d_\Delta}\mathrm{e}^{\frac{m}{4}\frac{1}{4}\operatorname{dist}(\Delta,v_{\Delta,j})}\right)$$
$$\cdot\prod_{\Delta\subset\gamma}\left[\mathbf{C} + \left\{\mathbf{C}\sqrt{d_\Delta} + \mathbf{C}\left(\int_{\Omega(\Delta)}(\nabla\phi)^2(x)\mathrm{d}x\right)^{\frac{1}{2}}\right\}^{d_\Delta}\right]$$
$$\leq \mathrm{e}^{\mathbf{C}N(\gamma)/2\alpha^{\frac{2}{3}}}\left(\prod_{\Delta\subset\gamma}\prod_{j=1}^{d_\Delta}\mathrm{e}^{\frac{m}{4}\frac{1}{4}\operatorname{dist}(\Delta,v_{\Delta,j})}\right)\mathrm{e}^{\frac{1}{4}\int_{\overline{\gamma}}(\nabla\phi)^2(x)\mathrm{d}x}$$
$$\cdot\prod_{\Omega\subset\overline{\gamma}}\prod_{\Delta\subset\Omega}\left[\mathbf{C} + \left\{\mathbf{C}\sqrt{d_\Delta} + \mathbf{C}\mathrm{e}^{-\frac{1}{4d_\Omega}\int_\Omega(\nabla\phi)^2(x)\mathrm{d}x}\left(\int_\Omega(\nabla\phi)^2(x)\mathrm{d}x\right)^{\frac{1}{2}}\right\}^{d_\Delta}\right]$$
$$\leq \mathrm{e}^{\mathbf{C}N(\gamma)/2\alpha^{\frac{2}{3}}}\mathrm{e}^{\frac{1}{4}\int_{\overline{\gamma}}(\nabla\phi)^2(x)\mathrm{d}x}\left(\prod_{\Delta\subset\gamma}\prod_{j=1}^{d_\Delta}\mathbf{C}\mathrm{e}^{\frac{m}{4}\frac{1}{4}\operatorname{dist}(\Delta,v_{\Delta,j})}\right)\prod_{\Omega\subset\overline{\gamma}}\prod_{\Delta\subset\Omega}\left(d_\Delta^{\frac{1}{2}} + d_\Omega^{\frac{1}{2}}\right)^{d_\Delta}$$
$$\leq \mathrm{e}^{\mathbf{C}N(\gamma)/2\alpha^{\frac{2}{3}}}\mathrm{e}^{\frac{1}{4}\int_{\overline{\gamma}}(\nabla\phi)^2(x)\mathrm{d}x}\left(\prod_{\Delta\subset\gamma}\prod_{j=1}^{d_\Delta}\mathbf{C}\mathrm{e}^{\frac{m}{4}\frac{1}{4}\operatorname{dist}(\Delta,v_{\Delta,j})}\right)\prod_{\Omega\subset\overline{\gamma}}(d_\Omega!)\,. \qquad (4.83)$$

□ Lemma 3. Theorem 2 now follows from the following remarks:

- The factors $2^{N(\Gamma)}$ in (4.67), $\mathrm{e}^{-\mathbf{C}/\alpha^{\frac{2}{3}}}$ in (4.70) and $\mathrm{e}^{\mathbf{C}N(\gamma)/2\alpha^{\frac{2}{3}}}$ in (4.82) combine to make a small factor $\mathrm{e}^{-\mathbf{C}/\alpha^{\frac{2}{3}}}$ per square again.

- The factors $\mathrm{e}^{\frac{1}{4}\int_{\overline{\gamma}}(\nabla\phi)^2(x)\mathrm{d}x}$ in (4.82), $\mathrm{e}^{\frac{1}{4}\int_{\Delta}(\nabla\phi)^2(x)\mathrm{d}x}$ in (4.70) and $\mathrm{e}^{\frac{m^2}{2}\int_{\Delta}\phi^2(x)\mathrm{d}x}$ combine to make the factor in front of $\mathrm{d}\mu_s^T(\phi)$ in proposition 3.

- The factor $\mathrm{e}^{-\mathbf{C}/\alpha^{\frac{2}{3}}}$ compensates both the normalization factor $\mathcal{N}_s^T$ and the factors $1/K(\Delta)$ per square.

- The factor in parenthesis in (4.82) is compensated by half of the remaing exponential decay in (4.67).

- The factors $(d_\Delta!)^{\mathbf{C}}$ in (4.69), C^{d_Δ} in (4.70) and (4.82) and the factor $d_\Omega!$ in (4.82) are all compensated by the remaining part of the exponential decay in (4.67). To compensate the factors $d_\Omega!$ one has to adapt the volume argument given in Lemma 1. The cube Δ is now replaced by a region Ω but here again a fraction of the d_Ω lines emerging from Ω are longer that $\mathbf{C}\sqrt{d_\Omega}$.

5 Sobolev inequalities

In this last section we prove the propositions 2 and 3 which are at the core of our treatment of the large field regions.

5.1 Proof of proposition 2

The proof rests on three elementary Sobolev inequalities see e.g. [13, 14]. Let $\Omega \subset {}^N$ be an open, connected set whose boundary is piecewise C^1. Let $1 \leq p \leq \infty$ and $p^* \equiv \left(\frac{1}{p} - \frac{1}{N}\right)^{-1} > p$. Suppose $f \in L^p(\Omega)$ and $\frac{\partial f}{\partial x_j} \in L^p(\Omega)$, $\forall j = 1, ..., N$. Then

$$\text{if } 1 \leq p < N \quad \text{then } \|f\|_{L^{p^*}(\Omega)} \leq C(\Omega)\left(\|f\|_{L^p(\Omega)} + \sum_{j=1}^{N}\left\|\frac{\partial f}{\partial x_j}\right\|_{L^p(\Omega)}\right), \tag{5.84}$$

$$\text{if } N = p \leq q < \infty \text{ then } \|f\|_{L^q(\Omega)} \leq C(\Omega)\left(\|f\|_{L^p(\Omega)} + \sum_{j=1}^{N}\left\|\frac{\partial f}{\partial x_j}\right\|_{L^p(\Omega)}\right), \tag{5.85}$$

$$\|f - \overline{f}\|_{L^p(\Omega)} \leq C(\Omega)\|\nabla f\|_{L^p(\Omega)} \qquad \text{where} \quad \overline{f} \equiv \frac{1}{|\Omega|}\int_\Omega f(x)\mathrm{d}x. \tag{5.86}$$

$C(\Omega)$ is a constant depending only on Ω. Recall that $(1 - \chi_\Delta[\phi]) \neq 0$ implies

$$\begin{array}{ll} \text{either} & \dfrac{1}{|\Delta|}\displaystyle\int_\Delta \alpha^2\phi^6(x)\mathrm{d}x \geq \frac{1}{4\lambda_1}, \\ \text{or} & \dfrac{1}{|\Delta|}\displaystyle\int_\Delta \mathrm{e}^{-4\alpha\phi(x)}\mathrm{d}x \geq \frac{1}{4\lambda_2}, \end{array}$$

where λ_1 and λ_2 are fixed numbers.

a) case $\frac{1}{|\Delta|}\int_{\Delta}\alpha^{2}\phi^{6}(\mathbf{x})\mathrm{d}\mathbf{x} \geq \frac{1}{4\lambda_1}$.

Inequality (5.85) for $p = N = 2$ and $q = 6$ implies after a change of variable to eliminate the dependance of $C(\Delta)$ on the size of the square Δ

$$
\begin{aligned}
\left(\frac{1}{4\lambda_1}\right)^{\frac{1}{6}} &\leq \left(\frac{1}{|\Delta|}\int_{\Delta}\alpha^{2}\phi^{6}(x)\mathrm{d}x\right)^{\frac{1}{6}} \\
&\leq \mathbf{C}\left[\left(\frac{1}{|\Delta|}\int_{\Delta}\alpha^{\frac{2}{3}}\phi^{2}(x)\mathrm{d}x\right)^{\frac{1}{2}} + \sum_{j=1}^{2}\left(\int_{\Delta}\alpha^{\frac{2}{3}}\left(\frac{\partial\phi}{\partial x_j}\right)^{2}\mathrm{d}x\right)^{\frac{1}{2}}\right] \\
&\leq \mathbf{C}\left[\left(\frac{1}{|\Delta|}\int_{\Delta}\alpha^{\frac{2}{3}}\phi^{2}(x)\mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_{\Delta}\alpha^{\frac{2}{3}}(\nabla\phi)^{2}(x)\mathrm{d}x\right)^{\frac{1}{2}}\right], \qquad (5.87)
\end{aligned}
$$

where $(\nabla\phi)^2 = \left(\frac{\partial\phi}{\partial x_1}\right)^2 + \left(\frac{\partial\phi}{\partial x_2}\right)^2$ and $\mathbf{C}$ is now independant of the size of the square Δ. There are two possibilities:

$$
\begin{aligned}
\text{either} \quad \int_{\Delta}(\nabla\phi)^{2}(x)\mathrm{d}x &\geq \left(\frac{1}{4\lambda_1}\right)^{\frac{1}{3}}\frac{\mathbf{C}}{\alpha^{\frac{2}{3}}}, \qquad (5.88) \\
\text{or} \quad \int_{\Delta}(\nabla\phi)^{2}(x)\mathrm{d}x &\leq \left(\frac{1}{4\lambda_1}\right)^{\frac{1}{3}}\frac{\mathbf{C}}{\alpha^{\frac{2}{3}}} \text{ and } \frac{1}{|\Delta|}\int_{\Delta}\phi^{2}(x)\mathrm{d}x \geq \left(\frac{1}{4\lambda_1}\right)^{\frac{1}{3}}\frac{\mathbf{C}}{\alpha^{\frac{2}{3}}}. \qquad (5.89)
\end{aligned}
$$

Case (5.88) is one of the conclusions of the proposition 2. Thus we have to prove that the two inequalities in (5.89) imply $V_{\Delta}[\phi] \geq \mathbf{C}/\alpha^{\frac{2}{3}}$. For this let us denote $u \equiv \phi^2$ and $g(t) \equiv \left(\mathrm{e}^{-\sqrt{t}} - 1\right)^2$ which is an increasing concave function of t. With these definitions one has

$$
\begin{aligned}
V(\phi) &\equiv \frac{\varepsilon^2}{2}\left(\mathrm{e}^{-\alpha\phi} - 1\right)^2 \geq \frac{\varepsilon^2}{2}\left(\mathrm{e}^{-\alpha|\phi|} - 1\right)^2 = \frac{\varepsilon^2}{2}g(\alpha^2 u), \qquad (5.90) \\
\overline{u} &\equiv \frac{1}{|\Delta|}\int_{\Delta}u(x)\mathrm{d}x = \frac{1}{|\Delta|}\int_{\Delta}\phi^{2}(x)\mathrm{d}x \geq \left(\frac{1}{4\lambda_1}\right)^{\frac{1}{3}}\frac{\mathbf{C}}{\alpha^{\frac{2}{3}}}, \qquad (5.91) \\
|g(a) - g(b)| &\leq \frac{\mathbf{C}}{1+b}|a - b|, \ \forall a, b \geq 0. \qquad (5.92)
\end{aligned}
$$

Now using (5.86) for $p = 1$ and $N = 2$ we get, with $a = \alpha^2 u(x)$ and $b = \alpha^2\overline{u}$ after a trivial change of variable.

$$
\begin{aligned}
&\left|\frac{1}{|\Delta|}\int_{\Delta}g(\alpha^2 u(x))\mathrm{d}x - g(\alpha^2\overline{u})\right| \leq \frac{1}{|\Delta|}\int_{\Delta}|g(\alpha^2 u(x)) - g(\alpha^2\overline{u})|\mathrm{d}x \overset{(5.92)}{\leq} \\
&\leq \frac{\mathbf{C}\alpha^2}{1+\alpha^2\overline{u}}\frac{1}{|\Delta|}\int_{\Delta}|u(x) - \overline{u}|\mathrm{d}x \overset{(5.86)}{\leq} \frac{\mathbf{C}\alpha^2}{1+\alpha^2\overline{u}}\frac{1}{|\Delta|^{\frac{1}{2}}}\int_{\Delta}|\nabla u(x)|\mathrm{d}x \leq \\
&\leq \frac{\mathbf{C}\alpha^2}{1+\alpha^2\overline{u}}\frac{1}{|\Delta|^{\frac{1}{2}}}\int_{\Delta}|\nabla\phi(x)||\phi(x)|\mathrm{d}x \leq \frac{\mathbf{C}\alpha^2}{1+\alpha^2\overline{u}}\left(\int_{\Delta}(\nabla\phi)^{2}(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\frac{1}{|\Delta|}\int_{\Delta}\phi^{2}(x)\mathrm{d}x\right)^{\frac{1}{2}} \\
&\overset{\substack{(5.89)\\(5.91)}}{\leq} \frac{\mathbf{C}\alpha^2}{1+\alpha^2\overline{u}}\frac{1}{\alpha^{\frac{1}{3}}}\overline{u}^{\frac{1}{2}}. \qquad (5.93)
\end{aligned}
$$

First suppose $\left(\frac{1}{4\lambda_1}\right)^{\frac{1}{3}}\frac{\mathbf{C}}{\alpha^{\frac{2}{3}}} \leq \overline{u} \leq \frac{1}{\alpha^2}$. Then

$$g(\alpha^2\overline{u}) \geq (1-\mathrm{e})^2\alpha^2\overline{u}. \tag{5.94}$$

But

$$\frac{\mathbf{C}\alpha^2}{1+\alpha^2\overline{u}}\frac{1}{\alpha^{\frac{1}{3}}}\overline{u}^{\frac{1}{2}} = \mathbf{C}\frac{\alpha^2\overline{u}}{1+\alpha^2\overline{u}}\frac{1}{\alpha^{\frac{1}{3}}}\frac{1}{\overline{u}^{\frac{1}{2}}} \leq \mathbf{C}\lambda_1^{\frac{1}{6}}\alpha^2\overline{u}. \tag{5.95}$$

Now choose λ_1 such that $\mathbf{C}\lambda_1^{\frac{1}{6}} \leq \frac{1}{2}(1-\mathrm{e})^2$. Then (5.93), (5.94) and (5.95) implie

$$\int_\Delta V(\phi(x))\mathrm{d}x = \frac{\varepsilon^2}{2}|\Delta|\left(\frac{1}{|\Delta|}\int_\Delta g(\alpha^2 u(x))\mathrm{d}x\right) \geq \frac{\varepsilon^2}{2}|\Delta|\frac{1}{2}(1-\mathrm{e})^2\alpha^2\overline{u} \geq \mathbf{C}\overline{u} \geq \frac{\mathbf{C}}{\alpha^{\frac{2}{3}}}. \tag{5.96}$$

Suppose now $\overline{u} \geq \frac{1}{\alpha^2}$. Then $g(\alpha^2\overline{u}) \geq (1-\mathrm{e})^2$. But

$$\frac{\mathbf{C}\alpha^2}{1+\alpha^2\overline{u}}\frac{1}{\alpha^{\frac{1}{3}}}\overline{u}^{\frac{1}{2}} = \frac{\mathbf{C}\alpha\overline{u}^{\frac{1}{2}}}{1+\alpha^2\overline{u}}\alpha^{\frac{2}{3}} \leq \mathbf{C}\alpha^{\frac{2}{3}}. \tag{5.97}$$

And thus again

$$\int_\Delta V(\phi(x))\mathrm{d}x = \frac{\varepsilon^2}{2}|\Delta|\left((1-\mathrm{e})^2 - \mathbf{C}\alpha^{\frac{2}{3}}\right) \geq \frac{\mathbf{C}}{\alpha^2} \geq \frac{\mathbf{C}}{\alpha^{\frac{2}{3}}}. \tag{5.98}$$

The proposition is thus proved in case a).

b) case $\frac{1}{|\boldsymbol{\Delta}|}\int_{\boldsymbol{\Delta}} \mathrm{e}^{-4\alpha\phi(\mathbf{x})}\mathrm{d}\mathbf{x} \geq \frac{1}{4\lambda_2}$.

The inequality (5.84) for $p=1$, $p^*=2$ and $N=2$ allows us to transform the bound on the average of $\mathrm{e}^{-4\alpha\phi(x)}$ into a bound on $\mathrm{e}^{-2\alpha\phi(x)}$ which is the asymptotic behavior of $V(\phi)$ when $\phi \to -\infty$. Denote $v \equiv \mathrm{e}^{-2\alpha\phi}$, then (5.84) gives

$$\begin{aligned}
\left(\frac{1}{4\lambda_2}\right)^{\frac{1}{2}} &\leq \left(\frac{1}{|\Delta|}\mathrm{e}^{-4\alpha\phi(x)}\mathrm{d}x\right)^{\frac{1}{2}} \leq \\
&\leq \mathbf{C}\left[\left(\frac{1}{|\Delta|}\int_\Delta \mathrm{e}^{-2\alpha\phi(x)}\mathrm{d}x\right) + \left(\frac{1}{|\Delta|^{\frac{1}{2}}}\sum_{j=1}^{2}\int_\Delta \mathrm{e}^{-2\alpha\phi(x)}2\alpha\left|\frac{\partial\phi}{\partial x_j}\right|\mathrm{d}x\right)\right] \leq \qquad (5.99)\\
&\leq \mathbf{C}\left[\left(\frac{1}{|\Delta|}\int_\Delta \mathrm{e}^{-2\alpha\phi(x)}\mathrm{d}x\right) + \alpha\left(\frac{1}{|\Delta|}\int_\Delta \mathrm{e}^{-4\alpha\phi(x)}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Delta (\nabla\phi)^2(x)\mathrm{d}x\right)^{\frac{1}{2}}\right].
\end{aligned}$$

Thus

$$\frac{1}{|\Delta|}\int_\Delta \mathrm{e}^{-2\alpha\phi(x)}\mathrm{d}x \geq \left[\mathbf{C} - \alpha\left(\int_\Delta (\nabla\phi)^2(x)\mathrm{d}x\right)^{\frac{1}{2}}\right]\cdot\left(\frac{1}{|\Delta|}\int_\Delta \mathrm{e}^{-4\alpha\phi(x)}\mathrm{d}x\right)^{\frac{1}{2}}. \tag{5.100}$$

There are two possibilities:

$$\text{either} \quad \int_\Delta (\nabla\phi)^2(x)\mathrm{d}x \geq \frac{\mathbf{C}}{\alpha^2} \geq \frac{\mathbf{C}}{\alpha^{\frac{2}{3}}}, \tag{5.101}$$

$$\text{or} \quad \int_\Delta (\nabla\phi)^2(x)\mathrm{d}x \leq \frac{\mathbf{C}}{\alpha^2} \quad \text{and} \quad \frac{1}{|\Delta|}\int_\Delta \mathrm{e}^{-2\alpha\phi(x)}\mathrm{d}x \geq \frac{\mathbf{C}}{\sqrt{\lambda_2}}. \tag{5.102}$$

Case (5.101) is a conclusion of the proposition, hence we have to show that the two inequalities in (5.102) implie $V_\Delta[\phi] \geq \mathbf{C}/\alpha^{\frac{2}{3}}$. Let us denote $v \equiv \mathrm{e}^{-2\alpha\phi}$ and $h(t) \equiv (\sqrt{t}-1)^2$. Then

$$
\begin{aligned}
V(\phi) &\equiv \frac{\varepsilon^2}{2}(\mathrm{e}^{-\alpha\phi}-1)^2 = \frac{\varepsilon^2}{2}h(v), && (5.103)\\
\overline{v} &\equiv \frac{1}{|\Delta|}\int_\Delta v(x)\mathrm{d}x = \frac{1}{|\Delta|}\int_\Delta \mathrm{e}^{-2\alpha\phi(x)}\mathrm{d}x \geq \frac{\mathbf{C}}{\sqrt{\lambda_2}} \geq 2, && (5.104)
\end{aligned}
$$

if λ_2 is chosen small enough. We shall actually choose λ_2 such that also $h(\overline{v}) \geq \frac{\overline{v}}{2}$. If $a \geq 2$ and $b \geq 0$ one has $|h(a)-h(b)| \leq \mathbf{C}|a-b|$. Then using (5.86) for $p=1$ and $N=2$ again we get with $a=\bar{v}$ and $b=v(x)$

$$
\begin{aligned}
\left|\frac{1}{|\Delta|}\int_\Delta h(v(x))\mathrm{d}x - h(\overline{v})\right| &\leq \frac{1}{|\Delta|}\int |h(v(x))-h(\overline{v})|\,\mathrm{d}x \leq \mathbf{C}\frac{1}{|\Delta|}\int_\Delta |v(x)-\overline{v}|\mathrm{d}x\\
&\overset{(5.86)}{\leq} \mathbf{C}\frac{1}{|\Delta|^{\frac{1}{2}}}\int_\Delta |\nabla v(x)|\mathrm{d}x \leq\\
&\leq \mathbf{C}\alpha\left(\int_\Delta (\nabla\phi)^2(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\frac{1}{|\Delta|}\int_\Delta \mathrm{e}^{-4\alpha\phi(x)}\mathrm{d}x\right)^{\frac{1}{2}} \leq\\
&\overset{\substack{(5.100)\\(5.104)}}{\leq} \frac{\mathbf{C}\alpha\left(\int_\Delta (\nabla\phi)^2(x)\mathrm{d}x\right)^{\frac{1}{2}}}{\mathbf{C}-\alpha\left(\int_\Delta (\nabla\phi)^2(x)\mathrm{d}x\right)^{\frac{1}{2}}}\cdot\overline{v}. \qquad (5.105)
\end{aligned}
$$

Now choose the numerical constant $\mathbf{C}$ in (5.102) small enough so that the coefficient of $\overline{v}$ in (5.105) is less than say $\frac{1}{4}$. Then, recalling that $\overline{v} \geq 2$

$$
\int_\Delta V(\phi(x))\mathrm{d}x = \frac{\varepsilon^2}{2}|\Delta|\left(\frac{1}{|\Delta|}\int_\Delta h(v(x))\mathrm{d}x\right) \overset{\substack{(5.104)\\(5.105)}}{\geq} \frac{\varepsilon^2}{2}|\Delta|\left(\frac{\overline{v}}{2}-\frac{\overline{v}}{4}\right) \overset{(5.104)}{\geq} \mathbf{C}\frac{1}{\alpha^2}, \qquad (5.106)
$$

and proposition 2 is proved in case b) as well. □

5.2 Proof of proposition 3

Our aim is to show that $\mathcal{N}_s^T$ is finite and does not grow faster than exponentially in the number of squares $N(\gamma)$ in the total large field region $\gamma = \cup_{j=1}^{l(X)}\gamma_j \subset X$. Recall the basic formulas defining the covariances C and C_s^T as operators on $L^2({}^2, \mathrm{d}x)$:

$$
\begin{aligned}
C(x,y) &\equiv \frac{1}{(2\pi)^2}\int_2 \mathrm{e}^{ip\cdot(x-y)}\widehat{C}(p)\mathrm{d}p, \quad \text{where} \quad \widehat{C}(p) \equiv \frac{\mathrm{e}^{-p^2/\kappa}}{p^2+m^2}, && (5.107)\\
C_s^T(x,y) &\equiv \begin{cases} [1-s_{\max}^T(ij)]C(x,y) & \text{if} \quad x \in v_i \text{ and } y \in v_j, i \neq j,\\ C(x,y) & \text{if} \quad x \text{ and } y \in v_i. \end{cases} && (5.108)\\
&= \sum_{\substack{\text{partitions}\\ \pi=\{X_1,\ldots,X_k\}\\ \text{of } X}} a_s^T(\pi)\sum_{\ell=1}^r \chi_{X_\ell}(x)C(x,y)\chi_{X_\ell}(y)\\
&\text{where} \quad a_s^T(\pi) > 0 \quad \text{and} \quad \sum_\pi a_s^T(\pi) = 1. && (5.109)
\end{aligned}
$$

One has the obvious inequalities (operator positivity and pointwise positivity)

$$C > 0, \quad C_s^T \geq 0, \tag{5.110}$$
$$C(x,y) \geq C_s^T(x,y) \geq 0. \tag{5.111}$$

However it is *wrong* that "$C \geq C_s^T$", and this is the origin of the pain. The normalisation factor is given by

$$\begin{aligned} \mathcal{N}_s^T &\equiv \int \mathrm{d}\mu_s^T(\phi)\, \mathrm{e}^{\frac{m^2}{2}\int_\gamma \phi^2(x)\mathrm{d}x} \mathrm{e}^{\frac{1}{2}\int_{\overline{\gamma}}(\nabla\phi)^2(x)\mathrm{d}x} \leq \\ &\leq \int \mathrm{d}\mu_s^T(\phi)\, \mathrm{e}^{\frac{m^2}{2}\int_{\overline{\gamma}} \phi^2(x)\mathrm{d}x} \mathrm{e}^{\frac{1}{2}\int_{\overline{\gamma}}(\nabla\phi)^2(x)\mathrm{d}x} = \int \mathrm{d}\mu_s^T(\phi)\mathrm{e}^{\frac{1}{2}(\phi, A\phi)}, \end{aligned} \tag{5.112}$$

where

$$A \equiv m^2\chi_{\overline{\gamma}} + \sum_{j=1}^{2} \nabla_j^\dagger \chi_{\overline{\gamma}} \nabla_j \geq 0, \tag{5.113}$$

and $\chi_{\overline{\gamma}}$ is the characteristic function of the support of $\overline{\gamma}$, $\nabla_j \equiv \frac{\partial}{\partial x_j}$ and † is the adjoint in $L^2({}^2, \mathrm{d}x)$ with respect to the usual scalar product $(.,.)$. Now if $A < \left(C_s^T\right)^{-1}$ and $\left(C_s^T\right)^{\frac{1}{2}} A \left(C_s^T\right)^{\frac{1}{2}}$ is trace-class one has, see e.g. [12]

$$\begin{aligned} \int \mathrm{d}\mu_s^T(\phi)\mathrm{e}^{\frac{1}{2}(\phi, A\phi)} &= \det\left[1 - \left(C_s^T\right)^{\frac{1}{2}} A \left(C_s^T\right)^{\frac{1}{2}}\right]^{-\frac{1}{2}} = \\ &= \exp\left\{-\frac{1}{2}\mathrm{Tr}\,\log\left[1 - \left(C_s^T\right)^{\frac{1}{2}} A \left(C_s^T\right)^{\frac{1}{2}}\right]\right\} = \\ &= \exp\left\{\frac{1}{2}\sum_{n\geq 1}\frac{1}{n}\left(AC_s^T\right)^n\right\}. \end{aligned} \tag{5.114}$$

In our case we don't know a priori that $A < \left(C_s^T\right)^{-1}$, however we shall prove that the above series is absolutely convergent, this then implies $\mathcal{N}_s^T < \infty$ and therefore $A < \left(C_s^T\right)^{-1}$. Proposition 2 will follow from (5.114) and lemma 4 below. The trace class property follows from lemma 4 as well.

Lemma 4 *Under the hypothesis of proposition 3 one has*

$$\mathrm{Tr}\left(AC_s^T\right)^n \leq \frac{\mathbf{C}}{n}\kappa|\gamma|. \tag{5.115}$$

Remark: From the above bound we can see that the ultraviolet cut-off is essential for $\mathcal{N}_s^T$ to be finite. Indeed removing the cut-off means $\kappa \to \infty$. For $\kappa = m^2$, one has $\kappa|\gamma| = |\gamma|/|\Delta| = N(\gamma) \leq \mathbf{C}N(\Gamma)$.

Proof: Let $\chi(x)$ be any real function on ^{2}. One has the trivial identity

$$B \equiv \sum_{j=1}^{2}\left[\frac{im}{\sqrt{2}} + (-i\nabla_j)\right]^\dagger \chi \left[\frac{im}{\sqrt{2}} + (-i\nabla_j)\right] =$$

$$\begin{aligned} &= m^2\chi + \sum_{j=1}^{2} \nabla_j^\dagger \chi \nabla_j - \frac{im}{\sqrt{2}} \sum_{j=1}^{2} [\chi, (-i\nabla_j)] = \\ &= m^2\chi + \sum_{j=1}^{2} \nabla_j^\dagger \chi \nabla_j + \frac{m}{\sqrt{2}} \sum_{j=1}^{2} \frac{\partial \chi}{\partial x_j}. \end{aligned} \tag{5.116}$$

For $\chi = \chi_{\overline{\gamma}}$ the first two terms in (5.116) reproduce A. The last term corresponds to the boundary effects. The general strategy of the proof is to replace in a first step A by B, for which the analog of (5.115) is "easy" to prove, and then in a second step to show that the boundary effects are not too large when the region $\overline{\gamma}$ is sufficiently far away from the boundary of the polymer X. This is the reason for defining the boundary $\widetilde{B}(\gamma)$ with r_2 large enough. From now on define B as in (5.116) with $\chi = \chi_{\tilde{\gamma}}$, see section 3.2 for the notations. Then we have the following inequality between A and B:

$$\begin{aligned} A \quad &\equiv \quad m^2\chi_{\overline{\gamma}} + \sum_{j=1}^{2} \nabla_j^\dagger \chi_{\overline{\gamma}} \nabla_j \le \\ &\le \quad m^2\chi_{\overline{\gamma}} + \sum_{j=1}^{2} \nabla_j^\dagger \chi_{\overline{\gamma}} \nabla_j + \overbrace{\frac{m}{\sqrt{2}} \sum_{j=1}^{2} \left| \frac{\partial \chi_{\tilde{\gamma}}}{\partial x_j} \right| - \frac{im}{\sqrt{2}} \sum_{j=1}^{2} [\chi_{\tilde{\gamma}}, (-i\nabla_j)]}^{\geq 0} = \\ &\overset{(5.116)}{=} \quad \sum_{j=1}^{2} \left[\frac{im}{\sqrt{2}} + (-i\nabla_j) \right]^\dagger \chi_{\tilde{\gamma}} \left[\frac{im}{\sqrt{2}} + (-i\nabla_j) \right] \\ &\qquad - \left(m^2 \chi_{\widetilde{B}(\gamma)} + \sum_{j=1}^{2} \nabla_j^\dagger \chi_{\widetilde{B}(\gamma)} \nabla_j - \frac{m}{\sqrt{2}} \sum_{j=1}^{2} \left| \frac{\partial \chi_{\tilde{\gamma}}}{\partial x_j} \right| \right) \\ &\equiv \quad B - \delta. \end{aligned} \tag{5.117}$$

For technical reasons which will become clear soon we introduce a cartesian coordinate system $(\bar{x}_1, \bar{x}_2)$ on ^{2}, rotated by an angle $\frac{\pi}{4}$ with respect to the original (x_1, x_2), i.e. $x_1 = \frac{1}{\sqrt{2}}(\bar{x}_1 - \bar{x}_2)$ and $x_2 = \frac{1}{\sqrt{2}}(\bar{x}_1 + \bar{x}_2)$. Denote $\overline{\nabla}_j \equiv \frac{\partial}{\partial \bar{x}_j}$. For any $\phi \in L^2(^2)$ and any $\Lambda \subset {}^2$ one has the trivial equality

$$\int_\Lambda \left[\left(\frac{\partial \phi}{\partial x_1} \right)^2 + \left(\frac{\partial \phi}{\partial x_2} \right)^2 \right] \mathrm{d}x = \int_\Lambda \left[\left(\frac{\partial \phi}{\partial \bar{x}_1} \right)^2 + \left(\frac{\partial \phi}{\partial \bar{x}_2} \right)^2 \right] \mathrm{d}x,$$

from which one concludes that $\sum_{j=1}^2 \nabla_j^\dagger \chi_\Lambda \nabla_j = \sum_{j=1}^2 \overline{\nabla}_j^\dagger \chi_\Lambda \overline{\nabla}_j$. These remarks allow us to write an inequality which is the analog of (5.117):

$$\begin{aligned} A &\le \bar{B} - \bar{\delta} \quad \text{where} \\ \bar{B} &\equiv \sum_{j=1}^{2} \left[\frac{im}{\sqrt{2}} + \left(-i\overline{\nabla}_j\right) \right]^\dagger \chi_{\tilde{\gamma}} \left[\frac{im}{\sqrt{2}} + \left(-i\overline{\nabla}_j\right) \right] \quad \text{and} \\ \bar{\delta} &\equiv m^2 \chi_{\widetilde{B}(\gamma)} + \sum_{j=1}^{2} \nabla_j^\dagger \chi_{\widetilde{B}(\gamma)} \nabla_j - \frac{m}{\sqrt{2}} \sum_{j=1} \left| \frac{\partial \chi_{\tilde{\gamma}}}{\partial \bar{x}_j} \right|. \end{aligned} \tag{5.118}$$

Combining (5.117) and (5.118) gives

$$A \le \frac{1}{2}(B + \bar{B}) - \frac{1}{2}(\delta + \bar{\delta}). \tag{5.119}$$

Next we use Peierls-Bogoliubov inequality which says that for any convex function $f(x)$ the mapping $B \to \mathrm{Tr}\, f(B)$ is convex on operators. For $f(x) = x^n$ which is convex on $^+, \forall n \geq 1$ we get:

$$\begin{aligned}
\mathrm{Tr}\left[\frac{1}{2}\left(B+\bar{B}\right)C_s^T\right]^n &= \mathrm{Tr}\left[\frac{1}{2}\left(C_s^T\right)^{\frac{1}{2}} B \left(C_s^T\right)^{\frac{1}{2}} + \frac{1}{2}\left(C_s^T\right)^{\frac{1}{2}} \bar{B} \left(C_s^T\right)^{\frac{1}{2}}\right]^n \leq \\
&\leq \frac{1}{2}\mathrm{Tr}\left[\left(C_s^T\right)^{\frac{1}{2}} B \left(C_s^T\right)^{\frac{1}{2}}\right]^n + \frac{1}{2}\mathrm{Tr}\left[\left(C_s^T\right)^{\frac{1}{2}} \bar{B} \left(C_s^T\right)^{\frac{1}{2}}\right]^n = \\
&= \frac{1}{2}\mathrm{Tr}\,\left(BC_s^T\right)^n + \frac{1}{2}\mathrm{Tr}\,\left(\bar{B}C_s^T\right)^n . \qquad (5.120)
\end{aligned}$$

Lemma 4 then follows from (5.119), (5.120), the elementary property of the trace which says that $0 \leq A \leq B$ and $C \geq 0$ implie $\mathrm{Tr}(AC)^n \leq \mathrm{Tr}(BC)^n$ and from the following two results:

Lemma 5 *(Bulk effects.) The following inequalities hold:*

$$\begin{aligned}
\mathrm{Tr}\,\left(BC_s^T\right)^n &\leq \frac{\mathbf{C}}{n}\kappa|\gamma|, \qquad \textit{and} \\
\mathrm{Tr}\,\left(\bar{B}C_s^T\right)^n &\leq \frac{\mathbf{C}}{n}\kappa|\gamma|.
\end{aligned}$$

Lemma 6 *(Boundary effects.) Under the hypothesis of proposition 3 the following inequality holds:*

$$\delta + \bar{\delta} \geq 0. \qquad (5.121)$$

Remark: The reason for introducing the coordinate system $(\bar{x}_1, \bar{x}_2)$ is that it turns out to be much easier to prove that $\delta + \bar{\delta} \geq 0$ than just $\delta \geq 0$.

Proof of lemma 5

From the definitions

$$\begin{aligned}
\mathrm{Tr}\,\left(BC_s^T\right)^n = \sum_{j_1 \cdots j_n} \mathrm{Tr}\, \Big\{ &\left[im/\sqrt{2} + (-i\nabla_{j_1})\right]^\dagger \chi_{\tilde{\gamma}} \left[im/\sqrt{2} + (-i\nabla_{j_1})\right] C_s^T \\
&\left[im/\sqrt{2} + (-i\nabla_{j_2})\right]^\dagger \chi_{\tilde{\gamma}} \left[im/\sqrt{2} + (-i\nabla_{j_2})\right] C_s^T \\
&\vdots \\
&\left[im/\sqrt{2} + (-i\nabla_{j_n})\right]^\dagger \chi_{\tilde{\gamma}} \left[im/\sqrt{2} + (-i\nabla_{j_n})\right] C_s^T \Big\} . \qquad (5.122)
\end{aligned}$$

Note that $\chi_{\tilde{\gamma}}(x) = 0$ when x is on the boundary $\partial\Gamma$ of the polymer, therefore we can pass the operators ∇_j and $\nabla_j^\dagger$ through the characteristic functions χ_{X_ℓ} in the formula (5.109) for the propagator C_s^T. Namely:

$$\begin{aligned}
&\chi_{\tilde{\gamma}} \left[\frac{im}{\sqrt{2}} + (-i\nabla_j)\right] C_s^T \left[\frac{im}{\sqrt{2}} + (-i\nabla_k)\right]^\dagger \chi_{\tilde{\gamma}} = \\
&= \chi_{\tilde{\gamma}} \left[\frac{im}{\sqrt{2}} + (-i\nabla_j)\right] \left[\sum_{\pi \text{ of } X} a_s^T(\pi) \sum_{\ell=1}^{r} \chi_{X_\ell} \left(\frac{\mathrm{e}^{-p^2/\kappa}}{p^2+m^2}\right)^{\check{}} \chi_{X_\ell}\right] \left[\frac{im}{\sqrt{2}} + (-i\nabla_k)\right]^\dagger \chi_{\tilde{\gamma}}
\end{aligned}$$

$$
\begin{aligned}
&= \chi_{\tilde\gamma}\left[\sum_{\pi \,\mathrm{of}\, X} a_s^T(\pi)\sum_{\ell=1}^{r} \chi_{X_\ell}\left(\left(im/\sqrt{2}+p_j\right)\left(-im/\sqrt{2}+p_k\right)\frac{\mathrm{e}^{-p^2/\kappa}}{p^2+m^2}\right)^{\check{}}\chi_{X_\ell}\right]\chi_{\tilde\gamma}\\
&\equiv \chi_{\tilde\gamma}C^{T,s}_{jk}\chi_{\tilde\gamma}. \qquad (5.123)
\end{aligned}
$$

Denote by $\underline{C}^{T,s}$ the operator on $L^2(^2)\otimes{}^2$ defined by $\left(\underline{C}^{T,s}\Phi\right)_j \equiv \sum_{k=1}^2 C^{T,s}_{jk}\phi_k$, for $\Phi = (\phi_1,\phi_2) \in L^2(^2)\otimes{}^2$. Let $|p(m)\rangle \equiv \begin{pmatrix} im/\sqrt{2}+p_1 \\ im/\sqrt{2}+p_2 \end{pmatrix} \in {}^2$ and $|p(m)\rangle\langle p(m)|$ be the orthogonal projection on this vector in ^{2}. We rewrite $\underline{C}^{T,s}$ as

$$
\underline{C}^{T,s} = \sum_{\pi \,\mathrm{on}\, X} a_s^T(\pi)\sum_{\ell=1}^{r}\chi_{X_\ell}\left(|p(m)\rangle\langle p(m)|\otimes\frac{\mathrm{e}^{-p^2/\kappa}}{p^2+m^2}\right)^{\check{}}\chi_{X_\ell}, \qquad (5.124)
$$

and now the initial trace on L^2 can be written as a trace on $L^2\otimes{}^2$:

$$
\begin{aligned}
\mathrm{Tr}\left(BC_s^T\right)^n &= \mathrm{Tr}\sum_{j_1\dots j_n}\chi_{\tilde\gamma}\,C^{T,s}_{j_1j_2}\,\chi_{\tilde\gamma}\,C^{T,s}_{j_2j_3}\,\chi_{\tilde\gamma}\dots\chi_{\tilde\gamma}\,C^{T,s}_{j_nj_1}\\
&= \mathrm{Tr}_{L^2\otimes^2}\,\chi_{\tilde\gamma}\,\underline{C}^{T,s}\,\chi_{\tilde\gamma}\,\underline{C}^{T,s}\,\chi_{\tilde\gamma}\dots\chi_{\tilde\gamma}\,\underline{C}^{T,s}. \qquad (5.125)
\end{aligned}
$$

Remark that $\langle p(m)|p(m)\rangle = p^2+m^2$ implies $|p(m)\rangle\langle p(m)| \le (p^2+m^2)_2$. Next define $\underline{\tilde C}^{T,s} \ge \underline{C}^{T,s}$ by replacing $|p(m)\rangle\langle p(m)|$ by $(p^2+m^2)_2$:

$$
\underline{C}^{T,s} \le \underline{\tilde C}^{T,s} \equiv \sum_{\pi \,\mathrm{on}\, X} a_s^T(\pi)\sum_{\ell=1}^{r}\chi_{X_\ell}{}_2\otimes\left(\mathrm{e}^{-p^2/\kappa}\right)^{\check{}}\chi_{X_\ell}. \qquad (5.126)
$$

The aim of all these manipulations was to produce the operator $\left(\mathrm{e}^{-p^2/\kappa}\right)^{\check{}}$, which has a positive kernel $\left(\mathrm{e}^{-p^2/\kappa}\right)^{\check{}}(x,y) > 0$. This allows us to remove the nasty s-parameters! Namely define

$$
\underline{\tilde C} \equiv {}_2\otimes\left(\mathrm{e}^{-p^2/\kappa}\right)^{\check{}} \ge 0. \qquad (5.127)
$$

From $\sum_{\pi \,\mathrm{of}\, X} a_s^T(\pi) = 1$ we deduce the pointwise inequality:

$$
0 \le \underline{\tilde C}^{T,s}(x,y) \le \underline{\tilde C}(x,y). \qquad (5.128)
$$

Now from (5.125) and using successively (5.126) and (5.128) we obtain

$$
\begin{aligned}
\mathrm{Tr}\left(BC_s^T\right)^n &\overset{(5.126)}{\le} \mathrm{Tr}_{L^2\otimes^2}\,\chi_{\tilde\gamma}\underline{\tilde C}^{T,s}\chi_{\tilde\gamma}\underline{\tilde C}^{T,s}\cdot\ldots\cdot\chi_{\tilde\gamma}\underline{\tilde C}^{T,s} \le\\
&\overset{(5.128)}{\le} \mathrm{Tr}_{L^2\otimes^2}\,\chi_{\tilde\gamma}\underline{\tilde C}\,\chi_{\tilde\gamma}\underline{\tilde C}\,\cdot\ldots\cdot\chi_{\tilde\gamma}\underline{\tilde C} \le\\
&\overset{(5.128)}{\le} \mathrm{Tr}_{L^2\otimes^2}\,\chi_{\tilde\gamma}\underline{\tilde C}^n = 2\mathrm{Tr}\,\chi_{\tilde\gamma}\left(\mathrm{e}^{-np^2/\kappa}\right)^{\check{}} \le\\
&= 2\int_{\tilde\gamma}\left(\mathrm{e}^{-np^2/\kappa}\right)^{\check{}}(x,x)\mathrm{d}x = 2|\tilde\gamma|\frac{1}{(2\pi)^2}\int_{^2}\mathrm{e}^{-np^2/\kappa}\mathrm{d}p \le\\
&= \frac{1}{2\pi}|\tilde\gamma|\frac{\kappa}{n} \le \frac{\mathbf{C}}{n}\kappa|\Gamma|. \qquad (5.129)
\end{aligned}
$$

-bb-error = =

Figure 4: An edge and a an exterior corner of $\tilde{B}(\gamma)$.

The case where $\bar{B}$ replaces B is identical. The components of the vector $|p(m)\rangle$ are now $\bar{p}_j$ in the new coordinate system but obviously $\sum_{j=1}^{2}(\bar{p}_j)^2 = \sum_{j=1}^{2}(p_j)^2 \equiv p^2$. □ Lemma 5.

Proof of lemma 6

We have to show that $(\phi, (\delta + \bar{\delta})\phi) \geq 0 \; \forall \phi \in L^2(^2, \mathrm{d}x)$ when L is large enough and η is small enough. We decompose the operators δ and $\bar{\delta}$ in contributions from the edges and the corners of $\bar{\gamma}$ in an obvious way see figure 4:

$$\delta = \sum_{\text{edges}} \delta_{\mathrm{e}} + \sum_{\text{corners}} \delta_{\mathrm{c}}, \tag{5.130}$$

and likewise for $\bar{\delta}$. Consider first an edge of lenght l/m, and of width $r_2 > L/m > 0$. From the definition (5.117) and supposing that $x = r_2$ corresponds to the edge of $\widetilde{B}(\gamma)$ we have (with the trivial change of variable $x' = m(x - r_2) + L$, $y' = my$ and $\phi'(x', y') = \phi(x, y)$):

$$\begin{aligned}
(\phi, \delta_{\mathrm{e}}\phi) &= \int_0^{\frac{l}{m}} \mathrm{d}y \left\{ \int_0^{r_2} \mathrm{d}x \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + m^2\phi^2(x,y) \right] - \frac{m}{\sqrt{2}}\phi^2(r_2, y) \right\} = \\
&\geq \int_0^{l} \mathrm{d}y' \left\{ \int_0^{L} \mathrm{d}x' \left[\left(\frac{\partial\phi'}{\partial x'}\right)^2 + \left(\frac{\partial\phi'}{\partial y'}\right)^2 + \phi'^2(x',y') \right] - \frac{1}{\sqrt{2}}\phi'^2(L, y') \right\} \geq \\
&\geq \int_0^{l} \mathrm{d}y' \left\{ \int_0^{L} \mathrm{d}x' \left[\left(\frac{\partial\phi'}{\partial x'}\right)^2 + \phi'^2 \right] - \frac{1}{\sqrt{2}}\phi'^2(L, y') \right\}.
\end{aligned} \tag{5.131}$$

Now fix $\phi'(L, y')$ for all $y' \in [0, l]$ and consider

$$\inf_{\phi:\phi(L,y')\text{ fixed}} \int_0^L \left[\left(\frac{\partial\phi}{\partial x'}\right)^2(x', y') + \phi^2(x', y') \right] \mathrm{d}x' \equiv \inf_{\phi:\phi(L,y')\text{ fixed}} F[\phi]. \tag{5.132}$$

The infimum $\bar{\phi}$ is in particular a stationary point and thus verifies:

$$0 = \frac{\delta F}{\delta\phi(x', y')}[\bar{\phi}] = -\frac{\partial^2\bar{\phi}}{\partial x'^2}(x', y') + \bar{\phi}^2(x', y'), \tag{5.133}$$

whose solutions are $\phi'(L, y')\mathrm{e}^{\pm(L-x')}$. By inspection one verifies that $\phi'(L, y')\mathrm{e}^{(L-x')}$ is not a minimum. In conclusion

$$\bar{\phi}(x', y') = \phi'(L, y')\mathrm{e}^{-(L-x')}. \tag{5.134}$$

We can thus bound the x'-integral in (5.131)

$$\begin{aligned}
F[\phi'] - \frac{1}{\sqrt{2}}\phi'^2(L, y') &\geq \phi'^2(L, y') \left\{ \int_0^L \left[\mathrm{e}^{-2(L-x')} + \mathrm{e}^{-2(L-x')} \right] \mathrm{d}x' - \frac{1}{\sqrt{2}} \right\} = \\
&= \phi'^2(L, y') \left(1 - \mathrm{e}^{-2L} - \frac{1}{\sqrt{2}} \right) > 0.
\end{aligned} \tag{5.135}$$

The case $\bar{\delta}$ is completely' similar except for the term $\frac{1}{\sqrt{2}}$ in (5.131) which is replaced by 1 because on an edge of $\tilde{\gamma}$ one has

$$\sum_{j=1}^{2}\left|\frac{\partial\chi_{\tilde{\gamma}}}{\partial\bar{x}_j}\right| = \sqrt{2}\sum_{j=1}^{2}\left|\frac{\partial\chi_{\tilde{\gamma}}}{\partial x_j}\right| .$$

Collecting all results above we conclude that if L is large enough

$$(\phi, \left(\delta_{\mathrm{e}}+\bar{\delta}_{\mathrm{e}}\right)\phi) \geq m\int_0^{l/m}\mathrm{d}y\,\phi^2(r_2,y)\left[1-2\mathrm{e}^{-2L}-\frac{1}{\sqrt{2}}\right] \geq 0. \tag{5.136}$$

The treatment of corners is similar if we use polar coordinates. Consider first an external corner and suppose $r=r_2$ corresponds to the edge of $\widetilde{B}(\gamma)$ and recall that $r_2=r_1+L/m=(1+\eta)r_1$. Using the change of variables $r'=m(r-r_2)+L$, $\phi'(r',\varphi)=\phi(r,\varphi)$ we have

$$\begin{aligned}
(\phi,\delta_{\mathrm{c}}\phi) \;\geq\; & \int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\left\{\int_{r_1}^{r_2}\mathrm{d}r\,r\left[\left(\frac{\partial\phi}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\phi}{\partial\varphi}\right)^2+m^2\phi^2\right]\right. \\
& \left. -\frac{m}{\sqrt{2}}r_2(|\cos\varphi|+|\sin\varphi|)\phi^2(r_2,\varphi)\right\} \\
\geq\; & mr_1\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\left\{\int_0^{L}\mathrm{d}r'\left[\left(\frac{\partial\phi'}{\partial r'}\right)^2+\phi'^2\right]\right. \\
& \left. -\frac{1+\eta}{\sqrt{2}}(|\cos\varphi|+|\sin\varphi|)\phi'^2(L,\varphi)\right\}.
\end{aligned} \tag{5.137}$$

As before for a fixed $\phi'(L,\varphi)$ the r'-integral is minimized by $\bar{\phi}(r',\varphi)\equiv\bar{\phi}(L,\varphi)\mathrm{e}^{-(L-r')}$ from which we conclude that

$$\{...\} \geq \phi'^2(L,\varphi)\left[1-\mathrm{e}^{-2L}-\frac{1+\eta}{\sqrt{2}}(|\cos\varphi|+|\sin\varphi|)\right]. \tag{5.138}$$

For $\bar{\delta}_{\mathrm{c}}$ the function $|\cos\varphi|+|\sin\varphi|$ is replaced by $\left|\cos\left(\varphi-\frac{\pi}{4}\right)\right|+\left|\sin\left(\varphi-\frac{\pi}{4}\right)\right|$, because of the change of orientation of the $(\bar{x}_1,\bar{x}_2)$ coordinate system. Now it is clear that

$$f(\varphi)\equiv\frac{1}{2}\left[|\cos\varphi|+|\sin\varphi|+\left|\cos\left(\varphi-\frac{\pi}{4}\right)\right|+\left|\sin\left(\varphi-\frac{\pi}{4}\right)\right|\right] \leq a < \sqrt{2}. \tag{5.139}$$

Thus provided L is large enough and η small enough we have

$$(\phi,\left(\delta_{\mathrm{c}}+\bar{\delta}_{\mathrm{c}}\right)\phi) \geq mr_1\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\,\phi^2(r_2,\varphi)\left[1-\mathrm{e}^{-2L}-\frac{(1+\eta)}{\sqrt{2}}f(\varphi)\right] \geq 0. \tag{5.140}$$

For an internal corner there is no boundary term like $\phi^2(r_2,\varphi)$ and thus δ and $\bar{\delta}$ are both obviously positive. □ Lemma 6. □ Lemma 4.

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