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ON LIMIT'S THEOREM FOR CLASS H^1.

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Abstract

In a recent paper by P.W.Jones and J.Journé (*On weak convergence in* $H^1(R^d)$, Proc. Amer. Math. Soc. 120 (1994) , 137 - 138.) , there are some inaccuracies. In this note, we give a counterexample to his theorem and a corrected form of the limit's theorem.

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Let $H^1(R^d)$ denote the Hardy space of functions on R^d whose Poisson maximal function belong to L^1, and let VMO denote the closure of the Schwartz space in $BMO(R^d)$. The following example shows that the P.W.Jones and J.Journé theorem concerning the limits on H^1 is not correct.

Let φ be an unbounded functions belonging to VMO. We try a sequence of functions $\{f_n\}$ from $H^1 = H^1(R)$ which converges almost everywhere to $f \in H^1$ and such that

$$\lim_{n\to\infty} \| f_n - f \|_{H^1} = 0.$$

But, contrarity to the assertion given in [2]

$$\int_R f_n \varphi dx \not\longrightarrow \int_R f \varphi dx,$$

since $f\varphi \notin L^1$.

Indeed, if locally integrable function φ belong to VMO and $\varphi \notin L^\infty$, then for almost every $x \in R$

$$| \varphi(x) | = \lim_{\delta\to 0} \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} | \varphi(t) | \, dt. \tag{1}$$

Consider the sequence $\{x_n\}$ with the following properties

$$(i) \qquad | \varphi(x_n) | = \lim_{\delta\to 0} \frac{1}{2\delta} \int_{x_n-\delta}^{x_n+\delta} | \varphi(t) | \, dt \quad ,$$

$$(ii) \quad | \varphi(x_n) | > 2 \, | \varphi(x_{n-1}) | > ... > 2^{n-1} \, | \varphi(x_1) | > 2^n.$$

Then for every n there exists $\delta_n > 0$ such that

$$\delta_n < \frac{1}{3} \inf_{1\le k\le\infty, n\neq k} | x_n - x_k | \qquad \text{and} \qquad 2^n < \frac{1}{2\delta_n} \int_{x_n-\delta_n}^{x_n+\delta_n} | \varphi(t) | \, dt.$$

For each $n \in N$ put

$$f_n(x) = \sum_{k=1}^{n} 2^{-k} a_k(x), \qquad f(x) = \sum_{k=1}^{\infty} 2^{-k} a_k(x).$$

Where

$$a_k(x) = \begin{cases} \delta_k^{-1} & , \quad x \in (x_k - \delta_k, x_k), \\ -\delta_k^{-1} & , \quad x \in (x_k, x_k + \delta_k), \\ 0 & , \quad x \notin (x_k - \delta_k, x_k + \delta_k). \end{cases}$$

It is known [1] that $f \in H^1$, $\| f_n \|_{H^1} < C$ and

$$\lim_{n\to\infty} \| f_n - f \|_{H^1} = 0. \tag{2}$$

¿From (2) and by the duality H^1 - BMO (cf. R.R.Coifman and G.Weiss [1]) we obtain, that

$$\lim_{n\to\infty} \int_R f_n \varphi dx$$

exists.

In the other hand a simple calculation gives

$$\int_R \mid f\varphi \mid dx = \int_R \mid \sum_{k=1}^{\infty} 2^{-k} a_k(x)\varphi(x) \mid dx =$$

$$2\sum_{k=1}^{\infty} 2^{-k} \frac{1}{2\delta_n} \int_{x_n-\delta_n}^{x_n+\delta_n} \mid \varphi(x) \mid dx > 2\sum_{k=1}^{\infty} 2^{-k} 2^k = \infty$$

Consequently, $f\varphi \notin L^1$. Moreover, in [3] one exhibit examples of functions $\varphi \in BMO$ and $f \in H^1$ such that $f\varphi$ is not A - integrable and even is not Q - integrable. A simple modifications on these examples show that the product $f\varphi$, if $f \in H^1$ and $\varphi \in VMO$, may be not Q - integrable too.

Therefore a correct version of the theorem given by P.W.Jones and J.Journé is:

Theorem 1. *Suppose $\{f_n\}$ is a sequence of functions belonging to $H^1(R^d)$ such that $\| f_n \|_{H^1} \leq 1$ for all n and such that $f_n(x) \to f(x)$ for almost every $x \in R^d$. Then $f \in H^1(R^d)$, $\| f \|_{H^1} \leq 1$, and*

$$\int_{R^d} f_n \varphi dx \longrightarrow \int_{R^d} f\varphi dx$$

for all $\varphi \in C_0$.

Here C_0 is the set of all continuous functions with compact support. This corresponds to the analogous result given by R.R.Coifman and G.Weiss [1, pp.638].

It is known [1] that every $\varphi \in BMO(R^d)$ determines a bounded linear functional on $H^1(R^d)$ and the dual form is given by

$$\langle f, \varphi \rangle = \sum_{k=1}^{\infty} \lambda_k \int_{R^d} a_k(x)\varphi(x)dx,$$

where $\sum_{k=1}^{\infty} \lambda_k a_k$ is any atomic decomposition of f. By means of a representation of a bounded linear functionals on $H^1(R^d)$ we can obtain the similar result:

Theorem 2. *Suppose $\{f_n\}$ is a sequence of functions belonging $H^1(R^d)$ such that $\| f_n \|_{H^1} \leq 1$ for all n and such that $f_n(x) \to f(x)$ for almost every $x \in R^d$. Then $f \in H^1(R^d)$, $\| f \|_{H^1} \leq 1$, and*

$$\langle f_n, \varphi \rangle \longrightarrow \langle f, \varphi \rangle$$

for all $\varphi \in VMO$.

Theorems 1 and 2 give the correct version of the limit's for the class $H^1(R^d)$.

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