On Hyperbolic Flows and the Problem of Chaos in Quantum Systems*

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Abstract

We briefly review the non-commutative generalisation, presented in [1], of the theory of hyperbolic dynamical systems; and then prove that hyperbolicity cannot be a paradigm for quantum chaos, except possibly in a certain asymptotic sense.

1. Introduction

In a recent work [1], the theory of hyperbolic, or Anosov, flows [2,3], which provides a paradigm for classical chaos, was generalised to non-commutative dynamical systems. The question naturally arises of whether such flows can prevail in the standard Von Neumann model [4] of (finite) quantum systems, and thus provide a paradigm for quantum chaos too. The object of this note is to demonstrate that they cannot do so, except possibly in an asymptotic sense, corresponding to the proximity of a classical limit. This is quite in line with results obtained on rather different bases about quantum chaos [5-7].

In Section 2, I shall outline the generalised theory [1] of hyperbolic flows, and in Section 3 I shall establish the above-described results concerning quantum chaos. It will be seen that the main result, i.e. Theorem 3, generalises one obtained in [1], under special conditions, for the particular case of the free dynamics of a particle on a manifold of constant negative curvature.

2. Hyperbolic Flows

2.1. The Classical Model. In a standard way (cf. [3]), we take our classical dynamical model, Σ_c, to be given by a triple (X, ϕ, μ), where X, the phase space, is a compact, differentiable, Riemannian manifold, $\{\phi_t | t \in \mathbf{Z} \ or \ \mathbf{R}\}$ is a one-parameter group of diffeomorphisms of X, representing the dynamics of the system, and μ is a smooth, $\phi-$invariant probability measure on X, corresponding to a stationary state. We denote the tangent space at $x(\in X)$ by $T(x)$ and, for fixed t, we define $\phi_t^{\star} := d\phi_t : T(x) \rightarrow T(\phi_t x)$. We shall

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employ the following definition of hyperbolic flows, which is a restricted version of that of Anosov [2,3].

Def. 2.1. We term the dynamics of Σ_c hyperbolic if there are unit vector fields $V_1, .., V_m$; $V_{m+1}, .., V_n$ over X, which are linearly independent at each point of X and satisfy the following conditions.

(a) $0 < m < n = Dim(X)$ *or* $Dim(X) - 1$, according to whether the time scale is discrete or continuous.

(b)

$$\phi_t^{\star} V_j(x) = V_j(\phi_t x)\exp(\lambda_j t) \tag{2.1}$$

where the Liapounov exponents λ are constants, with λ_j positive for $j \leq m$ and negative for $j > m$.

(c) The vector fields V_j have globally integral curves, generated by the action of one-parameter groups $\{\theta_j(s) | s \in \mathbf{R}\}$ of diffeomorphisms of X. Thus,

$$x_j(s) = \theta_j(s)x \tag{2.2}$$

is the unique global solution of the equation

$$x_j'(s) = V_j(x_j(s)) \ \forall s \in \mathbf{R};\ x(0) = x \tag{2.3}$$

Comments. (1) It follows immediately from the condition (b) that, generically, orbits emanating from neighbouring points separate exponentially fast from one another. Hence the dynamics is extremely unstable, i.e. chaotic.

(2) As prototype examples of hyperbolic systems, we cite (cf. [3])

(a) the 'Arnold Cat' model, whose dynamics corresponds to iterations of an automorphism of the two-dimensional torus; and

(b) geodesic flow over a compact manifold of negative curvature.

The following key lemma, concerning hyperbolic systems, will be proved in the Appendix.

Lemma 2.2. *For any hyberbolic system,* Σ_c,

$$\phi_t \theta_j(s) \phi_t^{-1} = \theta_j(s.\exp(\lambda_j t)) \tag{2.4}$$

2.2. Algebraic Formulation of Σ_c. As a first step towards a quantum generalisation of the above model, we reformulate it as an *abelian* $W^{\star}-$dynamical system $(\mathcal{A}_c, \alpha_c, \omega_c)$, where $\mathcal{A}_c$, the algebra of observables, is $L^{\infty}(X, d\mu)$, $\{\alpha_c(t) | t \in \mathbf{Z}$ *or* $\mathbf{R}\}$ is the one-parameter group of automorphisms of $\mathcal{A}_c$ induced by ϕ, i.e.

$$(\alpha_c(t)f)(x) \equiv f(\phi_t^{-1}x) \tag{2.5}$$

and ω_c is the normal state on $\mathcal{A}$ corresponding to μ, i.e.

$$\omega_c(f)\equiv\int f d\mu \tag{2.6}$$

Further, we denote by $\sigma_{c,j}(s)$ the automorphism of $\mathcal{A}$ induced by θ_j, i.e.

$$(\theta_{c,j}(s)f)(x)\equiv f(\theta_j(-s)x) \tag{2.7}$$

It follows immediately from (2.5) and (2.7) that the hyperbolicity condition (2.4) is equivalent to

$$\alpha_c(t)\sigma_{c,j}(s)\alpha(-t) = \sigma_{c,j}(s.\exp(\lambda_j t)) \tag{2.8}$$

2.3. Generalisation to Quantum Systems [1]. Now let $\Sigma = (\mathcal{A}, \alpha, \omega)$ be an arbitrary $W^\star$-dynamical system, with $\mathcal{A}$ an algebra of observables, $\{\alpha(t)|t\in\mathbf{Z}\ or\ \mathbf{R}\}$ a one-parameter group of automorphisms of $\mathcal{A}$, and ω a normal, $\alpha-$invariant state on $\mathcal{A}$. Thus, Σ provides a generalisation of the model Σ_c to quantum systems. For these, $\mathcal{A}$ is non-abelian and conforms to the canonical commutation relations.

Def. 2.3. We term the system Σ *hyperbolic* if $\mathcal{A}$ is equipped with weakly continuous one-parameter groups of automorphisms $\sigma_1(\mathbf{R}), ..\ ., \sigma_m(\mathbf{R});\ \sigma_{m+1}(\mathbf{R}), ..\ ., \sigma_n(\mathbf{R})$, with $0 < m < n$, such that

(a) the generators, δ_j of these groups are linearly independent, and

(b) the condition corresponding to (2.8) is fulfilled, i.e.

$$\alpha(t)\sigma_j(s)\alpha(-t) = \sigma_j(s.\exp(\lambda_j t)) \tag{2.9}$$

where the λ's are constants, with λ_j positive for $j\leq m$ and negative for $j > m$.

Comments. (1) The present definition is less restrictive than that of [1] in that it does not require the σ_j-invariance of ω.

(2) In the classical case, the derivations δ_j are those corresponding to the vector fields V_j.

(3) In general, by contrast with the classical case, Def. 2.3 does not provide a specification of n in terms of the structure of $\mathcal{A}$. We would hope that this deficiency could be remedied in the future. For present purposes, however, it suffices that $0 < m < n$.

Proposition 2.4. [1] *If Σ is hyperbolic, then the domain $D(\delta_j)$ of the derivation δ_j is stable under $\alpha(\mathbf{R})$,*

$$\alpha(t)\delta_j\alpha(-t) = \delta_j\exp(\lambda_j t) \tag{2.10}$$

and

$$\|\delta_j\alpha(t)A\| = \|\delta_j A\|\exp(-\lambda_j t)\ \forall A\in D(\delta_j),\ t\in\mathbf{R} \tag{2.11}$$

Comments. (1) Equation (2.10) is the differential form of (2.9), and reveals the exponential time dependence of the evolute $\alpha(t)\delta_j\alpha(-t)$ of δ_j, with Liapounov exponent λ_j.

(2) Since $\delta_j A$ represent the rate at which A moves along the orbit generated by the action of σ_j, this Proposition establishes a precise sense in which the dynamics is hyperbolic.

Examples of Hyperbolic Quantum Systems. (cf. [1].)

(1) A quantum version of the 'Arnold Cat'.

(2) A Wightman relativistic quantum field in Minkowski space, as viewed by a uniformly accelerated observer.

3. Suppression of Chaos by Quantisation

In view of the extreme sensitivity of hyperbolic systems to perturbations of initial conditions (cf. Prop. 2.4), we shall take these as our paradigm of chaotic dynamics. The examples cited at the end of §2 thus provide us with specific models of non-commutative chaotic systems.

However, neither of these corresponds to a finite quantum system, in the standard sense [4], for the following reasons.

(1) Ex.1 is a quantised version [8] of the classical hyperbolic system given by iterations of the automorphisms of a torus, T^2. Specifically, the quantisation amounts to the replacement of T^2 by a *non-commutative* torus, in the sense of Connes [9], as defined by a Weyl algebra, $\mathcal{A}$, over the dual group, $\mathbf{Z}^2$, of T^2. The dynamics is quasi-free, and induced by the same transformation of $\mathbf{Z}^2$ as in the classical case, and the equilibrium state is the tracial one on $\mathcal{A}$. Now this model does *not* correspond a standard finite quantum system, since $\mathcal{A}$ is not a type I factor. In fact, it is either of type II_1 or a tensor product of $L^\infty(T^2)$ and the algebra of $n-by-n$ matrices, with n finite, according to whether the parameter representing the non-commutativity of the torus is irrational or rational. Thus, in the first case, $\mathcal{A}$ is isomorphic with the algebra of an infinitely extended Heisenberg ferromagnet in its infinite temperature state; and, in the second case, it is essentially classical.

(2) Ex.2 is manifestly a model of a system with an infinite number of degrees of freedom, and its algebra of observables is of type III.

In view of these observations, it is natural to ask whether chaos, as characterised by hyperbolicity, can occur in standard finite quantum systems. For example, does the hyperbolic property survive canonical quantisation in the case of the free motion of a particle on a compact manifold of constant negative curvature? The answer we shall provide to these questions is negative. In other words, we shall establish that quantum mechanics suppresses chaos. This does *not* mean that it destroys it, but simply that a weaker condition than strict hyperbolicity is needed to represent any viable definition of quantum chaos. We shall discuss this matter further in the Comment following Theorem 3.

We base our treatment of finite quantum systems on the standard model $\Sigma = (\mathcal{A}, \alpha, \omega)$, where $\mathcal{A}$ is the set, $\mathcal{L}(\mathcal{H})$, of bounded operators in a separable Hilbert space, $\mathcal{H}$, $\alpha(\mathbf{R})$ is implemented by a unitary representation of $\mathbf{R}$ in $\mathcal{H}$, generated by a Hamiltonian operator,

H, with discrete spectrum, i.e.,

$$\alpha(t)A = \exp(iHt/\hbar)A\exp(-iHt/\hbar)\ \forall A\in\mathcal{A}, t\in\mathbf{R} \tag{3.1}$$

and ω is a normal, $\alpha-$invariant state on $\mathcal{A}$.

Note. The above conditions, including the discreteness of the spectrum of H, are fulfilled not only by finite systems of particles, with realistic interactions, in bounded regions of a Euclidean space or on a lattice [10, P.65], but also by the canonically quantised version of the classical hyperbolic system, given by the free dynamics of a particle on a compact manifold of constant negative curvature [11]. It was proved by [1] that the latter model is not hyperbolic, subject to the assumption that at least one of the automorphism groups σ is unitarily implementable in $\mathcal{H}$. The following theorem generalises this result to all finite systems, without any such assumption.

Theorem 3. *The above-specified model of a finite system is never hyperbolic.*

Comment. This does not rule out the possibility of a quantum chaos that is represented by a weakened form of the hyperbolicity condition of Def. 2.3. For example, it remains an open question whether the free quantum dynamics of a particle on a compact manifold of constant negative curvature fulfills the condition (2.9) in an appropriate classical limit. Since the dimensionless constant representing quantum effects in this model is $\gamma := \hbar/(mE)^{\frac{1}{2}}R$, where m and E are the mass and energy, respectively, of the particle, and R is the radius of curvature of the manifold, such a classical limit would imply that the dynamics was close to hyperbolicity when $\gamma << 1$. In this case, it could well be that the motion of the particle could still appear extremely unstable, and thus be deemed to be chaotic, over physically relevant time scales.

Proof of Theorem 3. We shall employ a *reductio ad absurdum* argument to prove that the hyperbolic property (2.9) is incompatible with the discreteness of the spectrum of H. Thus, we suppose that (2.9) is valid for one of the σ's, i.e. that

$$\alpha(t)\sigma(s)\alpha(-t) = \sigma(s.\exp(\lambda t)) \tag{3.2}$$

where λ is a real, non-zero constant, and the suffix j has been dropped. Hence, it follows from the $\alpha-$invariance of ω that

$$\int dsf(s)\langle\omega;\sigma(s)\alpha(-t)A\rangle = \int dsf(s)\langle\omega;\sigma(s.\exp(\lambda t))A\rangle\ \forall A\in\mathcal{A},\ f\in\mathbf{C}_0(\mathbf{R}),\ t\in\mathbf{R} \tag{3.3}$$

where $\mathbf{C}_0(\mathbf{R})$ denotes the continuous functions on $\mathbf{R}$ with compact support. By the dominated convergence theorem, we see that

$$R.H.S.\ of\ (3.3)\to c\omega(A),\ with\ c = \int dsf(s),\ as\ \lambda t\to -\infty \tag{3.4}$$

On the other hand,

$$L.H.S.\ of\ (3.3)\equiv\psi(\alpha(-t)A) \tag{3.5}$$

where

$$\psi = \int ds \sigma^{\star}(s)\omega, \tag{3.6}$$

and $\sigma^{\star}(s)$ is the dual of $\sigma(s)$. Thus ψ, like ω, is normal.

Now let $\{\phi_r | r\in\mathbf{N}\}$ be an orthogonal basis in $\mathcal{H}$, consisting of eigenvectors of H, and let $\{F_{mn} | m, n\in\mathbf{N}\}$ be the operators in this space defined by

$$F_{mn}\phi_r := \delta_{nr}\phi_m \tag{3.7}$$

These operators thus form a basis set for $\mathcal{A}$. Moreover, on applying (3.1) to the case where $A = F_{mn}$, it follows from (3.5) that

$$L.H.S.\ of\ (3.3) = \psi(F_{lm})\exp(-i\nu_{mn}t) \tag{3.8}$$

where ν_{mn} is the difference in the energy levels of ϕ_m and ϕ_n, in units of Planck's constant. Further, it follows from (3.1), (3.7) and the $\alpha-$invariance of ω that, if $\nu_{mn}\neq 0$, then $\omega(F_{mn}) = 0$. Consequently, the compatibility of (3.4) with (3.8) implies that

$$\psi(F_{mn}) = 0\ when\ \nu_{mn}\neq 0$$

By (3.1) and the normality of ψ, this signifies that ψ is $\alpha-$invariant. Hence, by(3.6), together with the $\alpha-$invariance of ω and the hyperbolicity condition (3.2),

$$\int ds f(s)\langle\omega;\sigma(s)A\rangle = \int ds f(s)\langle\omega;\sigma(s)\alpha(t)A\rangle$$

$$\equiv \int ds f(s)\langle\omega;\alpha(t)\sigma(se^{\lambda t})A\rangle \equiv \int ds f(s)\langle\omega;\sigma(se^{\lambda t})A\rangle\ \forall A\in\mathcal{A},\ f\in\mathbf{C}_0(\mathbf{R})\ t\in\mathbf{R} \tag{3.9}$$

Now let $\{f_n\}$ be a positive sequence in $\mathbf{C}_0(\mathbf{R})$, whose integrals are all unity, and whose supports are all contained in a single compact and converge to some point u. Then, on replacing f by f_n in (3.9), and passing to the limit $n\to\infty$, we obtain the equation

$$\omega(\sigma(u)A) = \omega(\sigma(ue^{\lambda t})A)\ \forall A\in\mathcal{A},\ t, u\in\mathbf{R}$$

Hence, replacing A by $\sigma(-u)A$ here, and defining

$$v := u(e^{\lambda t} - 1),$$

$$\omega(A) = \omega(\sigma(v)A)\ \forall A\in\mathcal{A}\ v\in\mathbf{R}, \tag{3.10}$$

which signifies that ω is $\sigma-$invariant. Consequently, since this state is also $\alpha-$invariant, it follows from [1, Th.3.3] that the spectrum of $\mathcal{H}$ covers the real line. However, this result, which stems from the supposition that Σ is hyperbolic, conflicts with the specification that the spectrum of H is discrete. We conclude therefore that Σ cannot be hyperbolic.

Appendix A

Proof of Lemma 2.2. In order to simplify the notation, we shall drop the suffixes t, j from the symbols ϕ, V, θ, λ, and define

$$\kappa = \exp(\lambda t) \tag{A.1}$$

for fixed t. Thus, by equations (2.1)-(2.3),

$$\phi^{\star}V(x) = \kappa V(\phi x); \tag{A.2}$$

and

$$x(s) = \theta(s)x \tag{A.3}$$

is the unique global solution of

$$x'(s) = V(x(s)), \ \forall s \in \mathbf{R}; \ x(0) = x \tag{A.4}$$

Hence, defining

$$x_{\phi}(s) := \phi\theta(\kappa^{-1}s)\phi^{-1}x, \tag{A.5}$$

it follows from (A.2-4) that

$$x'_{\phi}(s) = \kappa^{-1}\phi^{\star}(\phi^{-1}x)'[\kappa^{-1}s] =$$

$$\kappa^{-1}\phi^{\star}V((\phi^{-1}x)[\kappa^{-1}s]) = V(\phi(\phi^{-1}x)[\kappa^{-1}s]) = V(\phi\theta[\kappa^{-1}s]\phi^{-1}x)$$

i.e.

$$x'_{\phi}(s) = V(x_{\phi}(s)),$$

which signifies that $x_{\phi}(.)$ satisfies (A.4). Hence, by the uniqueness of the solution of that equation,

$$x_{\phi}(s) = x(s) \ \forall s \in \mathbf{R},$$

i.e., by (A.3) and (A.5),

$$\phi\theta(s)\phi^{-1} \equiv \theta(\kappa s),$$

which is the required result.

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