Classical Mechanics as Quantum Mechanics with Infinitesimal $\hbar$

by

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Abstract. We develop an approach to the classical limit of quantum theory using the mathematical framework of nonstandard analysis. In this framework infinitesimal quantities have a rigorous meaning, and the quantum mechanical parameter $\hbar$ can be chosen to be such an infinitesimal. We consider those bounded observables which are transformed continuously on the standard (non-infinitesimal) scale by the phase space translations. We show that, up to corrections of infinitesimally small norm, such continuous elements form a commutative algebra which is isomorphic to the algebra of classical observables represented by functions on phase space. Commutators of differentiable quantum observables, divided by $\hbar$, are infinitesimally close to the Poisson bracket of the corresponding functions. Moreover, the quantum time evolution is infinitesimally close to the classical time evolution. Analogous results are shown for the classical limit of a spin system, in which the half-integer spin parameter, i.e. the angular momentum divided by $\hbar$, is taken as an infinite number.

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1 Introduction

The classical limit of quantum mechanics is often identified with the WKB [Mas, BS] method. While this approach gives a good picture of the asymptotic behaviour of solutions of the Schrödinger equation as $\hbar \to 0$, it does not give a satisfactory explanation why in this limit the non-commutativity of quantum observables suddenly turns into the commutativity of classical observables. The same is true of approaches based on Feynman integrals [AHK], and on the limits of coherent states [Hep, Hag]. An approach to the classical limit emphasizing the limit of observables and their algebraic structure has recently been developed in [We2] (compare also [Rie, Em1]). This approach makes rigorous the intuitive criterion for deciding which observables in quantum theory may effectively be treated classically: *classical observables should not change too much under small position or momentum translations*, where, due to the relation $p = \hbar k$, a small momentum translation might still correspond to a large translation in terms of wave numbers.

The aim of this paper is to show that a full theory of the classical limit can be based on this single physical idea. We make use of nonstandard analysis [Rob, AFHL] because it allows us to describe an infinite separation of scales inside a single mathematical structure. The main idea is that our usual "standard" quantities can be embedded into a larger structure containing also infinitely small and infinitely large quantities (this embedding is to a certain extent comparable to the embedding of the reals into the complex numbers). Like almost all results of nonstandard analysis our results can be translated into statements about ordinary limits (in an abstract sense), e.g. an infinitesimal (i.e. an infinitely small) number can be seen as nothing but a maximally detailed description of how a sequence of real numbers can go to zero. The actual construction of quantum theory with infinitesimal $\hbar$ is completely trivial, thanks to a powerful principle of nonstandard analysis, called the Transfer Principle. It states that whatever can be formulated correctly in standard terms is immediately true or defined, respectively, in the nonstandard world. In contrast to standard analysis the key here is not to go to the limit, but to extract from the limiting theory the "standard part". Here the above mentioned physical idea gives an immediate criterion: we only need to restrict the theory to observables which are *continuous on the standard scale*, and then to neglect infinitesimal terms.

This formulation corresponds completely to the physical intuition. Moreover it is much more compact than the formulation in conventional mathematical terms [We2] on which it is based. At the same time it retains full mathematical rigour. Due to its extreme simplicity the nonstandard formulation is also more suggestive of further generalizations. Another bonus is that some proofs are simplified, but this is not our main point, and, in fact, we draw heavily on the standard techniques of proof.

We briefly review some ideas and notation relating to nonstandard analysis. There are quite good introductions to the subject [Lin, AFHL, HL], including undergraduate calculus courses based on it [Kei], and we refer to these for more detailed information. Whenever it is convenient we denote an entitiy of our standard mathematical world by a prefix “ * ” if it is considered in the nonstandard universe. For example the real line is denoted by ${}^*\mathbb{R}$. It contains elements of the form ${}^*r$, where $r \in \mathbb{R}$ is an ordinary real number, but by far not all elements of ${}^*\mathbb{R}$ are of this form. In particular, there are infinitesimals $\varepsilon \in {}^*\mathbb{R}$ with the property that $\varepsilon > 0$, but $\varepsilon < {}^*r$ for every $r > 0$. When $x, y \in {}^*\mathbb{R}$, we write “$x \approx y$” for “$x - y$ is infinitesimal”. The infinitesimals have a decent arithmetic and, for example, the inverse of an infinitesimal is an infinite number, which is larger than any standard real ${}^*r$ in accordance with our intuition.

2 Classical limit for phase space observables

The first structure to which we will apply the Transfer Principle is an irreducible representation of the canonical commutation relations in Weyl form with $d < \infty$ degrees of freedom, i.e. we consider on the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^d)$ the unitary operators $W(x,p)$ of phase space translations given by

$$\Big(W(x,p)\psi\Big)(y) = \exp\left\{-i\frac{x \cdot p}{2\hbar} + i\frac{p \cdot y}{\hbar}\right\}\psi(y-x) \quad . \tag{1}$$

Alternatively, these operators can be written as

$$W(x,p) = e^{ix\cdot\mathbf{P}+iq\cdot\mathbf{Q}} \quad , \tag{2}$$

where $\mathbf{P}$ and $\mathbf{Q}$ denote the usual momentum and position operators, and the dot stands for the scalar product in $\mathbb{R}^d$. It will often be convenient to denote phase space points (x,p) by a single letter ξ. The Euclidean length of ξ is written as $|\xi| = (x \cdot x + p \cdot p)^{1/2}$. Observables are described by bounded operators A on the Hilbert space, written as $A \in \mathcal{B}(\mathcal{H})$. On the observables the Weyl operators implement the phase space translations α_ξ via

$$\alpha_\xi(A) = W(\xi)\, A\, W(\xi)^* \quad . \tag{3}$$

By the transfer principle all these formulas make sense also in the nonstandard world, i. e. for observables $A \in {}^*\mathcal{B}(\mathcal{H})$, for phase space points $\xi = (x,p) \in {}^*\mathbb{R}^d$, and for any value of the constant $\hbar \in {}^*\mathbb{R}$, including infinitesimal or infinite numbers. (We refrain from taking the number d of degrees of freedom to be an infinite number in ${}^*\mathbb{N}$). The norm is a well-defined ${}^*\mathbb{R}$-valued function on ${}^*\mathcal{B}(\mathcal{H})$, and, of course, it can be infinitesimal, as well as infinite. We will call $A \in {}^*\mathcal{B}(\mathcal{H})$ *infinitesimal*, writing $A \approx 0$, if $\|A\| \approx 0$, and we will say that $A \in {}^*\mathcal{B}(\mathcal{H})$ is *finite*, if there is an ordinary real number $r \in {}^*\mathbb{R}$ such that $\|A\| \leq {}^*r$.

We can now state the basic idea of this paper: inside this monstrously large object ${}^*\mathcal{B}(\mathcal{H})$ we will single out those operators that are well-behaved in the sense of the classical limit. The Weyl operators themselves are an example of badly behaved operators, since they oscillate wildly on an infinitesimal scale. In contrast, operators that we can imagine as "observables" ought to depend continuously on $\mathbf{P}$ and $\mathbf{Q}$. We define the algebra $\mathcal{D} \subset {}^*\mathcal{B}(\mathcal{H})$ of "good" observables as those finite elements $A \in {}^*\mathcal{B}(\mathcal{H})$, such that $\alpha_\xi(A)$ is *continuous on the standard scale.* Nonstandard analysis offers two equivalent ways of making this phrase precise. The first is to say that for all standard $\varepsilon \in \mathbb{R}$ we can find a standard $\delta \in \mathbb{R}$ such that $|\xi| < {}^*\delta$ implies $\|\alpha_\xi(A) - A\| \leq {}^*\varepsilon$. The second is to say that

$$\xi \approx 0 \qquad \text{implies} \qquad \alpha_\xi(A) \approx A \quad . \tag{4}$$

For example, it is seen immediately that the Weyl operators fail this definition, whereas, if we scale down their oscillations by a factor $\hbar$, they do become continuous. Indeed, due to the commutation relations of the Weyl operators, we have

$$\alpha_\xi(W(\hbar\eta)) = e^{i\sigma(\xi,\eta)}\, W(\hbar\eta) \quad , \tag{5}$$

where $\sigma((x,p),(x',p')) = p \cdot x' - x \cdot p'$ is the symplectic form on phase space. Since the exponential factor is continuous in ξ on the standard scale, we have $W(\hbar\eta) \in \mathcal{D}$, if η is finite. Also, every infinitesimal operator is in $\mathcal{D}$. Thus $\mathcal{D}$ is every bit as non-commutative as ${}^*\mathcal{B}(\mathcal{H})$. The surprise is, however, that we only need to neglect infinitesimal terms to see in it the observable algebra of classical mechanics.

Theorem. *Let $\mathcal{D} \subset {}^*\mathcal{B}(\mathcal{H})$ be the algebra of continuous observables defined above, and let $\mathcal{D}_\approx$ denote the same algebra, but with all infinitesimal elements identified with 0. Then $\mathcal{D}_\approx$ is canonically isomorphic to the algebra of bounded uniformly continuous functions on phase space.*

We sketch the rather simple proof because it uses only ideas well-known from the physics literature, and gives an explicit description of the isomorphism claimed in the Theorem. Let Ω denote the ground state wave function of the oscillator Hamiltonian $H = (\mathbf{P}^2 + \mathbf{Q}^2)/2 = (-\hbar^2\Delta + \mathbf{Q}^2)/2$. Then for any $A \in \mathcal{B}(\mathcal{H})$, we define a function on phase space by

$$\begin{aligned}(S^\uparrow A)(\xi) &= \langle W(\xi)\Omega,\, A\,W(\xi)\Omega\rangle \\ &= \langle \Omega,\, \alpha_{-\xi}(A)\Omega\rangle \quad .\end{aligned} \tag{6}$$

This is variously called the lower symbol [Sim], a smeared Wigner function [Car], the Husimi function [Tak], or the convolution with a coherent state [We1] of the operator A. In the other direction, we have the upper symbol [Sim], or P-representation [KS] (also going by many other names) which assigns an operator to each bounded measurable function f via

$$S^\downarrow f = \int \frac{dx\,dp}{(2\pi\hbar)^d}\; f(x,p)\, \alpha_\xi(|\Omega\rangle\langle\Omega|) \quad . \tag{7}$$

Unlike the Wigner-Weyl isomorphisms between operators and phase space functions, these operators map positive elements into positive elements. Moreover, $S^\uparrow S^\downarrow$ and $S^\downarrow S^\uparrow$ are both operators which just average over translations with a Gaussian weight. Explicitly,

$$S^\downarrow S^\uparrow(A) = \int \frac{dx\,dp}{(2\pi\hbar)^d}\; e^{-\xi^2/(2\hbar)}\; \alpha_\xi(A) \quad . \tag{8}$$

By transfer, all these formulas remain valid in the nonstandard case. However, since $\hbar$ is infinitesimal, the Gaussian factor is a nonstandard representation of a Dirac δ-Function. Hence for continuous operators $A \in \mathcal{D}$, $S^\downarrow S^\uparrow(A) \approx A$. Identifying infinitesimally close elements we find that $S^\uparrow$ and $S^\downarrow$ become inverses of each other. On the other hand, it is clear that, for $A \in \mathcal{D}$, the function $S^\uparrow(A)$ is also finite and continuous in the sense of equation (4), α_ξ being interpreted as the phase space translation of functions. This is the same as saying that α_ξ is uniformly continuous up to infinitesimals. Hence $\mathcal{D}_\approx$ is isomorphic to the space of bounded uniformly continuous functions on phase space. Because $S^\uparrow$ and $S^\downarrow$ both preserve positivity, these isomorphisms respect the ordering as well. This implies that they are also algebraic isomorphisms.

Since the product of functions is commutative, we have that $AB-BA \approx 0$ for $A, B \in \mathcal{D}$. For sufficiently smooth observables we can even determine the precise order of this infinitesimal. We say that $A \in \mathcal{D}$ is twice differentiable, if it has "partial derivatives" $A_i, A_{ij} \in \mathcal{D}$ such that $\xi \approx 0$ implies

$$\alpha_\xi(A) = A + \sum_1^{2d} \xi_i A_i + \sum_{i,j=1}^{2d} \xi_i \xi_j A_{ij} + \xi^2 D(\xi) \quad , \tag{9}$$

where $D(\xi) \approx 0$. Here ξ_i denotes the components of ξ, i.e. the d position and the d momentum coordinates. Applying $S^\uparrow$ and $\alpha_{\xi+\eta} = \alpha_\xi \alpha_\eta$ we see that $S^\uparrow(A)$ is twice differentiable with uniformly continuous bounded derivatives. Then, if $A, B \in \mathcal{D}$ are twice differentiable, we get

$$\frac{i}{\hbar}[A, B] \approx S^\downarrow\{S^\uparrow(A), S^\uparrow(B)\} \quad , \tag{10}$$

where the braces on the right hand side denote the Poisson bracket of the phase space functions. The operators $S^\uparrow$ and $S^\downarrow$ just effect the isomorphism between functions and operators given by the Theorem. The proof of formula (10) is rather involved, but completely parallel to the standard proof of the analogous result in [We2], so we will omit it.

An instructive example is the case of Weyl operators with slowed-down oscillation, i.e. $W(\hbar\eta)$ for finite η. Their classical limits are the exponential functions

$$(S^\uparrow W(\hbar\eta))(\xi) = e^{i\{\xi,\eta\} - \hbar\eta^2/4} \approx e^{i\{\xi,\eta\}} \quad , \tag{11}$$

with the "symplectic form" $\{(x_1, p_1),\, (x_2, p_2)\} = p_1 \cdot x_2 - p_2 \cdot x_1$. Commutators become

$$\frac{i}{\hbar}[W(\hbar\xi),\, W(\hbar\eta)] = \frac{2}{\hbar}\sin\!\left(\frac{\hbar}{2}\{\xi, \eta\}\right) W(\hbar(\xi+\eta)) \quad . \tag{12}$$

Because $\hbar \approx 0$, and ξ, η are finite, the factor is $\approx \{\xi, \eta\}$, from which equation (10) is verified by applying $S^{\uparrow}$ to each Weyl operator.

By equation (10) the quantum mechanical equations of motion are infinitesimally close to the classical ones. The same is also true for the *solutions* of these equations: If $H \in \mathcal{D}$ is a hermitian operator, we define the time evolution of an observable $A \in \mathcal{D}$ as usual by

$$A(t) = e^{itH/\hbar} A e^{itH/\hbar} \quad . \tag{13}$$

Then one can prove as in the standard case [We2] that

$$(S^{\uparrow} A(t))(\xi) \approx (S^{\uparrow} A)(\mathcal{F}_t \xi) \quad . \tag{14}$$

Here $\mathcal{F}_t \xi$ denotes the solution of Hamilton's equation of motion with Hamiltonian function $S^{\uparrow} H$ for the time interval t with initial condition ξ.

3 Classical limit for spin systems

The same basic idea for obtaining a classical limit works in a number of other contexts. We present here the case of spin systems, restricting attention to a single spin, for simplicity of presentation. The basic relation here is that angular momentum, which is the quantity remaining meaningful in the classical limit equals $\hbar s$, where s is the usual integer or half-integer spin parameter labelling the irreducible representations of SU(2). Hence in this context $\hbar \to 0$ simply means $s \to \infty$. The nonstandard approach to this limit is to take an an irreducible representation U of ${}^*\mathrm{SU}(2)$ with infinite spin $s \in \frac{1}{2}{}^*\mathbb{N}$, where ${}^*\mathbb{N}$ denotes the nonstandard version of the natural numbers. The nonstandard notions of "representation" and "irreducibility" are again defined by the Transfer Principle. Since standard rotations $R \in \mathrm{SU}(2)$ are contained in ${}^*\mathrm{SU}(2)$ in the form ${}^*R \in {}^*\mathrm{SU}(2)$, and since the nonstandard representation space is also a complex vector space with scalar multiplication restricted to standard numbers, SU(2) is also represented by the infinite spin representation, but, of course, this representation is highly reducible as a standard representation.

The analogue of the Theorem in the previous section is proved precisely along the same lines, with the Weyl operators replaced by the representing unitaries U_R, and the coherent state Ω replaced by the eigenvector of the rotations around the 3-axis with maximal eigenvalue (namely s). The analogue of the phase space is the orbit of the one-dimensional projection $|\Omega\rangle\langle\Omega|$ under rotations, which is isomorphic in an obvious way to the (nonstandard) sphere. The measure "$d\xi$" is thus replaced by the integration over the sphere. The operators $S^{\uparrow}, S^{\downarrow}$ are also defined analogously, and that $S^{\downarrow}$ maps the constant function into the identity operator is obvious from the irreducibility of U.

Only one thing really has to be verified, namely that the kernel describing the operator $S^{\downarrow}S^{\uparrow}$ is infinitesimal outside the so-called monad $\{\phi : \phi \approx 0\}$ of the north pole, i.e. if

$$R = \begin{pmatrix} \cos(\phi/2) & \sin(\phi/2) \\ -\sin(\phi/2) & \cos(\phi/2) \end{pmatrix} \tag{15}$$

is the SU(2)-rotation taking the north pole to latitude $\phi \leq 2\pi$,

$$|\langle \Omega, U_R \Omega \rangle|^2 \approx 0 \quad , \quad \text{unless} \quad \phi \approx 0 \quad . \tag{16}$$

This can be seen without computation by realizing the spin-s representation as the subrepresentation of a tensor product of $2s$ copies of the defining (spin-1/2) representation, namely the representation on the completely symmetric (Bose-) subspace. In this subspace Ω is identified with the product of $2s$ "spin up" vectors. The matrix element we have to compute is thus equal to $(\cos(\phi/2))^{2s}$, and the estimate (16) follows.

Scaled commutators of the form $is[A, B]$ become Poisson brackets as before. The "phase space" on which this bracket lives is the sphere, which is slightly unusual in that it does not contain unbounded momenta. The Poisson bracket is uniquely determined by the condition that the three coordinate functions on the sphere (which are the limits of the angular momentum components divided by s) satisfy angular momentum "commutation" relations. In differential geometric terms this is saying that the symplectic form is the surface 2-form of the sphere. Moreover, the dynamics converges in the same sense as before.

For the proof of these statements it is helpful once again to look at the spin-s representation as a representation on the permutation symmetric subspace of $2s$ spins-1/2. One can then appeal to the theory of mean-field lattice systems in which permutation symmetry is the key ingredient, and in which these results already exist. We refer the reader to [RW, DW] for the standard version of this theory, and to [Wo, WH] for its nonstandard form.

Generalizations to compact Lie groups other than SU(2), or to several interacting spins are straightforward using the appropriate coherent states and symbol maps [Sim]. For a discussion of phase measurements in the classical limit, also using nonstandard methods, see [Oza].

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