OPERATORS WITH SINGULAR CONTINUOUS SPECTRUM: I. GENERAL OPERATORS

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§0. Introduction

The Baire category theorem implies that the family, $\mathcal{F}$, of dense sets G_δ in a fixed metric space, X, is a candidate for generic sets since it is closed under countable intersections; and if X is perfect (has no isolated point), then $A \in \mathcal{F}$ has uncountable intersections with any open ball in X.

There is a long tradition of soft arguments to prove that certain surprising sets are generic. For example in $\mathcal{C}[0,1]$, a generic function is nowhere differentiable. Closer to our concern here, Zamfirescu [20] has proven that a generic monotone function has purely singular continuous derivative, and Halmos [7]-Rohlin [14] have proven that a generic ergodic process is weak mixing but not mixing. We will say a set $S \subset X$ is Baire typical if it is a dense G_δ and a set $S \subset X$ is Baire null if its complement is Baire typical.

Our goal is to look at generic sets of self-adjoint operators and show that their spectrum is quite often purely singular continuous. Here are three of our results that give the flavor of what we will prove in §3 and §4.

Consider the sequence space, $[-a,a]^{\mathbb{Z}}$, of sequences v_n with $|v_n| \leq a$. Given any such v, we can define a Jacobi matrix $J(v)$ as the tridiagonal matrix with $J_{n,n\pm 1} = 1$ and $J_{n,n} = v_n$. View J as a self-adjoint operator on $\ell^2(\mathbb{Z})$. It is known (e.g. [4,17,16]) that if one puts a product of normalized Lebesgue measures on $[-a,a]^{\mathbb{Z}}$ (i.e., the v_n are independent random variables each uniformly distributed in $[-a,a]$) then, $J(v)$ is a.e. an operator with spectrum $[a-2,a+2]$ and the spectrum there is pure point. So our first result is somewhat surprising.

Theorem 1. *View $[-a,a]^{\mathbb{Z}}$ in the product topology. Then* $\{v \mid J(v)$ has spectrum $[-a-2,a+2]$ and the spectrum is purely singular continuous$\}$ *is Baire typical.*

We also have some results if $\mathbb{Z}$ is replaced by $\mathbb{Z}^\nu$ and the Jacobi matrix by the multidimensional discrete Schrödinger operator. One might think that the weakness of the topology and the one dimension are critical. They are not, as our second result shows.

For $V \in \mathcal{C}(\mathbb{R}^\nu)$, let $S(V)$ be the Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R}^\nu)$.

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Theorem 2. *Let $\mathcal{C}_\infty(\mathbb{R}^\nu)$ be the continuous functions vanishing at infinity in $\|\cdot\|_\infty$. Then*

$$\{V \mid S(V) \text{ has purely singular continuous spectrum on } (0,\infty)\}$$

is Baire typical.

Note that for $V \in \mathcal{C}_\infty(\mathbb{R})$, the essential spectrum, $\mathrm{spec}_{\mathrm{ess}}(S(V)) = [0,\infty)$, so Theorem 2 says that generically, the singular continuous spectrum, $\mathrm{spec}_{\mathrm{sc}}(S(V)) = [0,\infty)$, the absolutely continuous spectrum, $\mathrm{spec}_{\mathrm{ac}} = \emptyset$, and the pure point spectrum, $\mathrm{spec}_{\mathrm{pp}}(S(V)) \subset (-\infty,0]$. For the discrete one-dimensional (Jacobi matrix) case, we will be able to say something about decay. For example when $\nu = 1$, a generic $v \in \ell^p$ $(2 < p < \infty)$ has a $J(v)$ with purely singular continuous spectrum in $[-2,2]$. For $p = 1$, we know $\mathrm{spec}_{\mathrm{ac}}(J(v)) = [-2,2]$ so the singular spectrum result doesn't extend to all p. $1 < p \leq 2$ is open.

Our third example is related to the celebrated theorem of Weyl-von Neumann [18,19,8] that given any self-adjoint A and any ϵ, there exists a Hilbert-Schmidt operator B with $\|B\|_2 < \epsilon$ (where $\|C\|_2 = \mathrm{tr}(C^*C)^{1/2}$) so that $A+B$ has only point spectrum. That is not the generic situation.

Definition. A self-adjoint operator, C, is called *usual* if and only if $\{\psi \mid C\psi = \lambda\psi$ and $\lambda \in \mathrm{spec}_{\mathrm{disc}}(C)$, the discrete spectrum of $C\} \cup \{\psi \mid d\mu_\psi^C(\lambda)$ is purely singular continuous$\}$ span the space $\mathcal{H}$. Here $d\mu_\psi^C$ is the spectral measure for (C,ψ), that is

$$\int e^{i\lambda t} d\mu_\psi^C(\lambda) = (\psi, e^{i\lambda C}\psi). \tag{0.1}$$

I_2 is the Hilbert-Schmidt operators in $\|\cdot\|_2$ norm.

Theorem 3. *Let A be a fixed self-adjoint operator. Then $\{B \in I_2 \mid A+B$ is usual$\}$ is Baire typical.*

For example, if $\mathrm{spec}(A) = [-1,1]$, generically $A+B$ has purely singular continuous spectrum in $(-1,1)$.

In §1 we prove two results asserting that certain families of operators are always sets G_δ. We will use that to prove criteria for generic singular spectrum in §2. We then study general operators in §3 and Schrödinger/Jacobi operators in §4.

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§1. Soft Stuff

A metric space, X, of (perhaps unbounded) self-adjoint operators on a separable Hilbert space, $\mathcal{H}$, will be called *regular* if and only if:

(1) X is complete.
(2) If $A_n \to A$ in the metric topology, then $A_n \to A$ in the strong resolvent sense.

Our main technical results are three:

Theorem 1.1. *Fix $C \subset \mathbb{R}$ closed and X a regular metric space of operators. Then*

$$\{A \mid A \text{ has no eigenvalues in } C\}$$

is a G_δ in X.

Theorem 1.2. *Fix $U \subset \mathbb{R}$ open and X a regular metric space of operators. Then*

$$\{A \mid \text{ For any spectral measure for } A,\ (\mu_\psi^A)_{\rm ac}[U] = 0\}$$

is a G_δ in X.

Remarks. 1. Note the word "dense" does not appear before G_δ. That will hold sometimes, as we will analyze.

2. μ_ψ^A is defined in (0.1). $(\nu)_{\rm ac}$ means the absolutely continuous component of ν.

Theorem 1.3. *Fix $K \subset \mathbb{R}$ closed and X a regular metric space of operators. Then*

$$\{A \mid K \subset \operatorname{spec}(A)\}$$

is a G_δ.

Lemma 1.4. *Let A_n be a sequence of self-adjoint operators on $\mathcal{H}$ so that $A_n \to A$ in strong resolvent sense for some self-adjoint A. Let K be a compact subset of $\mathbb{R}$; φ, a fixed vector in $\mathcal{H}$, and $\epsilon > 0$. Suppose there exist eigenvectors η_n of A_n:*

$$A_n \eta_n = \lambda_n \eta_n$$

with $\|\eta_n\| = 1$, $\lambda_n \in K$ and $|\langle \eta_n, \varphi\rangle| \geq \epsilon$. Then A has an eigenvector η with

$$A\eta = \lambda\eta$$

with $\lambda \in K$, $\|\eta\| = 1$ and $|\langle \eta, \varphi\rangle) \geq \epsilon$.

Proof. K is compact and $\{\psi \in \mathcal{H} \mid \|\psi\| \leq 1\}$ is compact in the weak topology. So we can pass to a subsequence and suppose $\eta_n \to \eta_\infty$ weakly and $\lambda_n \to \infty$. We will show that $\eta_\infty \in D(A)$ and $A\eta_\infty = \lambda\eta_\infty$. Since $|(\eta_\infty, \varphi)| \geq \epsilon$, we have $\eta_\infty \neq 0$ and so $\eta = \eta_\infty / \|\eta_\infty\|$ is the required vector.

Let $\psi \in \mathcal{H}$ be arbitrary. Then

$$\begin{aligned} \left((A+i)^{-1}\eta_\infty, \psi\right) &= \left(\eta_\infty, (A-i)^{-1}\psi\right) \\ &= \lim_n \left(\eta_n, (A_n - i)^{-1}\psi\right) \\ &= \lim_n \left((A_n + i)^{-1}\eta_n, \psi\right) \\ &= \lim_n \left((\lambda_n + i)^{-1}\eta_n, \psi\right) \\ &= \left((\lambda + i)^{-1}\eta_\infty, \psi\right). \end{aligned} \tag{1.1}$$

It follows that $\eta_\infty = (\lambda + i)$, $(A+i)^{-1}\eta_\infty \in D(A_\infty)$ and $A\eta_\infty = \lambda\eta_\infty$. (1.1) holds because $(A_n - i)^{-1}\psi$ converges to $(A-i)^{-1}\psi$ in norm and $\eta_n \to \eta_\infty$ weakly with $\|\eta_n\| \leq 1$. $\square$

Proof of Theorem 1.1. Fix $K \subset \mathbb{R}$ compact, $\epsilon > 0$ and $\varphi \in \mathcal{H}$. Then Lemma 1.4 implies that

$$\begin{aligned} &Q(K, \varphi, \epsilon) = \\ &\qquad \{A \in X \mid \exists \eta \in D(A) \text{ with } \|\eta\| = 1, |\langle \varphi, \eta\rangle| \geq \epsilon,\ A\eta = \lambda\eta \text{ for some } \lambda \in K\} \end{aligned}$$

is a closed subset of X.

Fix $\{\varphi_l\}_{l=1}^{\infty}$ an orthonormal basis of $\mathcal{H}$. For $n, l, m \in \mathbb{Z}_+$, let

$$Q_{n,l,m} = Q(C \cap [-n, n], \varphi_l, m^{-1}).$$

Then

$$\cup Q_{n,l,m} = \{A \mid A \text{ has an eigenvalue in } C\}$$

is an F_σ, so its complement is a G_δ as claimed. $\square$

Lemma 1.5. *Let (a, b) be a fixed open interval in $\mathbb{R}^n$ and let $d\mu$ be a measure on $\mathbb{R}$. Then μ is purely singular on (a, b) if and only if for each $n > 2$, there exists $\epsilon_n > 0$ and f_n obeying*

1. $0 \le f_n \le 1$,
2. *f_n is supported in $(a - \epsilon_n, b + \epsilon_n)$,*
3. $\int_{-\infty}^{\infty} f_n(s)ds < 2^{-n}$,
4. $\mu\left(\chi_{[a-\epsilon_n, b+\epsilon_n]} - f_n\right) < 2^{-n}$,
5. $\epsilon_n < 2^{-n}$.

Proof. Suppose such ϵ_n and f_n exist. Let

$$C_n = \left\{x \mid f_n(x) > \frac{1}{2}\right\}.$$

Then (with $|\cdot|$ = Lebesgue measure):

$$|C_n| < 2^{-n+1}$$
$$\mu([a - \epsilon_n, b + \epsilon_n] \backslash C_n) < 2^{-n+1}$$

and

$$C_n \subset [a - \epsilon_n, b + \epsilon_n].$$

It follows that

$$C = \bigcap_m \bigcup_{n=m}^{\infty} C_n$$

obeys $|C| = 0$ and $\mu([a, b] \backslash C) = 0$.

Conversely, suppose that μ is purely singular continuous on (a, b). Find C in (a, b) so $|C| = 0$ and $\mu((a, b) \backslash C) = 0$. By adding a and/or b to C, we can suppose $C \subset [a, b]$ and $\mu([a, b] \backslash C) = 0$. Since $\lim_{\epsilon \downarrow 0} \mu([a - \epsilon, a)) = 0$ and $\lim_{\epsilon \downarrow 0} \mu((b, b + \epsilon]) = 0$, we can choose $\epsilon_n < 2^{-n}$ so that

$$\mu([a - \epsilon_n, a)) + \mu((b, b + \epsilon_n]) < 2^{-n-1}.$$

By regularity of measures, we can find $K_n \subset C \subset U_n \subset (a - \epsilon_n, b + \epsilon_n)$ so that $|U_n| < 2^{-n}, \mu([a, b] \backslash K_n) < 2^{-n-1}$. By Urysohn's lemma, find f continuous with $0 \le f \le 1$, $f \equiv 1$ on K_n and $\operatorname{supp} f \subset U_n$. Then

$$\int_{-\infty}^{\infty} f_n(s)ds \le |U_n| < 2^{-n}$$

while

$$\mu\left(\chi_{[a-\epsilon_n,b+\epsilon_n]} - f_n\right) \le \mu([a-\epsilon_n,a)) + \mu([a,b]\backslash K_n) + \mu((b,b+\epsilon_n]) < 2^{-n}$$

as required. $\square$

Proof of Theorem 1.2. Let $\varphi \in \mathcal{H}$, $a, b \in \mathbb{R}$ and

$$Q(\varphi,a,b) = \{A \mid d\mu_\varphi^A \text{ is purely singular on } (a,b)\}.$$

By Lemma 1.5

$$Q(\varphi,a,b) = \bigcap_{n=2}^{\infty} \bigcup_{(f,\epsilon)\in B_n} \mathcal{A}_n(f,\epsilon;\varphi)$$

where B_n is the set of pairs (f,ϵ) obeying (1–3; 5) of Lemma 1.5 and

$$\mathcal{A}_n(f,\epsilon;\varphi) = \{A \mid (\varphi, [\chi_{[a-\epsilon,b+\epsilon]}(A) - f(A)]\varphi) < 2^{-n}\}.$$

We claim each A is open, equivalently that

$$\mathcal{A}_n^c(f,\epsilon;\varphi) = \{A \mid (\varphi, \chi_{[a-\epsilon,b+\epsilon]}(A) - f(A)\varphi) \ge 2^{-n}\}$$

is closed. For let $A_l \in \mathcal{A}_n^c$ converge to A in strong resolvent sense. Then $\lim(\varphi, f(A_l)\varphi) = (\varphi, f(A)\varphi)$ (see, e.g., [12]). Let h_m be continuous functions with $h_m \downarrow \chi_{[a-\epsilon,b+\epsilon]}$ monotonically. Then $h_m(A) \to h_m(A)$ strongly, so

$$\begin{aligned}(\varphi, \chi_{[a-\epsilon,b+\epsilon]}(A)\varphi) &= \inf_m(\varphi, h_m(A)\varphi)\\ &= \inf_m[\lim_n(\varphi, h_m(_l)\varphi)]\\ &\ge \overline{\lim_n}(\varphi, \chi_{[a-\epsilon,b+\epsilon]}(A_l)\varphi)\end{aligned}$$

so the claim is proven.

Any open set U is a countable union of open intervals $I_n = (a_n, b_n)$. Let φ_l be an orthonormal basis for $\mathcal{H}$. Then the set that the theorem asserts is a G_δ is just

$$\bigcap_{n=1}^{\infty} \bigcap_{l=1}^{\infty} Q(\varphi_l, a_n, b_n)$$

which is indeed therefore a G_δ. $\square$

The following is an expression of the well-known fact of lower semicontinuity of the spectrum under strong limits.

Lemma 1.6. *If $A_n \to A$ in strong resolvent sense and $(a,b) \cap \operatorname{spec}(A_n) = \emptyset$, then $(a,b) \cap \operatorname{spec}(A) = \emptyset$.*

Proof. Let f be the function $f(x) = \operatorname{dist}(x, \mathbb{R}\backslash(a,b))$. Then $(a,b)\cap\operatorname{spec}(B) = \emptyset$ if and only if $f(B) = 0$. By the continuity of the functional calculus of $A_n \to A$ in strong resolvent sense, then $f(A) = \text{s-lim}\, f(A_n) = 0$ if $(a,b)\cap\operatorname{spec}(A_n) = \emptyset$. $\square$

Proof of Theorem 1.3. Let λ_n be a countable dense set in K. Then

$$\{A \mid K \subset \operatorname{spec}(A)\} = \bigcap_n \{A \mid \lambda_n \in \operatorname{spec}(A)\}$$

so we need only consider the cases where $K = \{\lambda\}$. But

$$\{A \mid \lambda \notin \operatorname{spec}(A)\} = \bigcup_{n=1}^{\infty} \left\{A \mid \left(\lambda - \frac{1}{n}, \lambda + \frac{1}{n}\right) \cap \operatorname{spec}(A) = \emptyset\right\}$$

is an F_σ by Lemma 1.6. Thus, its complement is a G_δ. $\square$

§2. Welcome to Wonderland

The main point in the way to generate generic singular spectrum is

Theorem 2.1. *Let X be a regular metric space of self-adjoint operators. Suppose that for some interval (a,b), we have that*

(i) $\{A \mid A$ has purely continuous spectrum on $(a,b)\}$ *is dense in X.*
(ii) $\{A \mid A$ has purely singular spectrum on $(a,b)\}$ *is dense in X.*
(iii) $\{A \mid A$ has (a,b) in its spectrum$\}$ *is dense in X.*

Then $\{A \mid (a,b) \subset \operatorname{spec}_{\mathrm{sc}}(A), (a,b) \cap \operatorname{spec}_{\mathrm{pp}}(A) = \emptyset, (a,b) \cap \operatorname{spec}_{\mathrm{ac}}(A) = \emptyset\}$ *is a dense G_δ.*

Proof. Because (a,b) is an F_δ, each of the sets in (i)–(iii) is a G_δ by Theorems 1.1–3. (For example, the set in (i) is the intersection of the same sets for $[a+\frac{1}{n}, b-\frac{1}{n}]$.) Thus, by hypothesis they are dense G_δ's. By the Baire category theorem, their intersection is a dense G_δ. □

Remarks. 1. We pick an interval for definiteness. In many cases, one can say things about other sets.

2. We pick the same set (a,b) for convenience. In some examples later, we will take $(a,b)=\mathbb{R}$ in (ii), but replace (a,b) by a closed set in (i).

Here is a spectacular corollary, which we call the Wonderland Theorem:

The Wonderland Theorem. *Let X be a regular metric space of operators. Suppose*

(a) $\{A \mid A$ has purely absolutely continuous spectrum$\}$ *is dense in X;*
(b) $\{A \mid A$ has purely point spectrum$\}$ *is dense in X.*

Then Baire typically, A has only singular continuous spectrum.

Proof. Strictly speaking, this is not a corollary of the theorem but of its proof, since we do not specify the spectrum. By Theorem 1.1 and (a)

$$\{A \mid A \text{ has purely continuous spectrum}\}$$

is Baire typical. Similarly, by Theorem 1.2 and (b)

$$\{A \mid A \text{ has purely singular spectrum}\}$$

is Baire typical. So their intersection is Baire typical. □

§3. General Operators

We apply the theory to general self-adjoint operators first. Throughout, $\mathcal{H}$ is a fixed separable Hilbert space.

Theorem 3.1. *Fix $a>0$. Let $X = \{A \mid A$ is self-adjoint, $\|A\| \le a\}$ which is a complete metrizable space in the strong topology. Then*

$$\{A \mid \operatorname{spec}(A) = [-a,a]; A \text{ has purely singular continuous spectrum}\}$$

is Baire typical.

Remark. For example, if φ_n is an orthonormal basis

$$\rho(A, A') = \sum_{n=1}^{\infty} \min(2^{-n}, \|(A - A')\varphi_n\|)$$

is a metric.

Proof. This will use the Wonderland Theorem. By the Weyl-von Neumann theorem, the operators with point spectrum are norm dense, but there is a simpler argument since we only need strong density. Since the same argument is needed for dense absolutely continuous spectrum, we give it.

Pick an orthonormal basis $\{\varphi_n\}_{n=-\infty}^{\infty}$ (this way of counting will be convenient) and let P_N be the projection onto $\{\varphi_n\}_{|n|\le N}$ so that $P_N \to 1$ strongly. Let α_n be a counting of the rationals in $[-a, a]$ and let B be the diagonal operator $B\varphi_n = \alpha_n\varphi_n$. Then

$$P_N A P_N + (1 - P_N)B(1 - P_N) \xrightarrow{s} A.$$

The operator on the left has spectrum $[-a, a]$ and it is pure point. So we have two of the three hypotheses of the Wonderland Theorem.

To prove that absolutely continuous spectrum operators are dense, we need only prove that an operator A with point spectrum and $\|A\| \le a - \epsilon$ can be approximated since we have just proven such operators are dense. Let $\{\varphi_n\}$ be the eigenvectors of A (say, $A\varphi_n = \alpha_n\varphi_n$) and let $A_N = P_N A P_N$. Fix a sequence δ_N with $0 < \delta_N < \frac{\epsilon}{2}$ and $\delta_N \to 0$. Let B_N be defined by

$$B_N\varphi_n = \delta_N(\varphi_{n+(2N+1)} + \varphi_{n-(2N+1)}) + \beta_n\varphi_n$$

where $\beta_n = \alpha_j$ for the unique j with $n \equiv j \mod (2N - 1)$. Then $\|B_N\| \le a$ since $\delta_N \le \frac{\epsilon}{2}$ and $B_N \to A$ strongly as $N \to \infty$. Each B_N is a direct sum of $N + 1$ operators of the form

$$\alpha_n\mathbb{I} + \delta_N J$$

where J is the tridiagonal operator with zeros on diagonal and 1 on the two principal off diagonals. J has absolutely continuous spectrum and thus so does $\alpha_n\mathbb{I} + \delta_N J$ and B_N. □

Surprisingly, the strong topology is only relevant to be sure that the spectrum is $[-a, a]$:

Theorem 3.2. *Fix $a < b$. Let $X = \{A \mid A$ is self-adjoint and $\operatorname{spec}(A) = [a, b]\}$ in the operator norm topology. (X is closed in $\mathcal{L}(\mathcal{H})$, so complete.) Then*

$$\{A \mid A \text{ has purely singular continuous spectrum}\}$$

is Baire typical.

Proof. We will use the Wonderland Theorem. By the Weyl-von Neumann theorem, given $A \in X$ and ϵ, we can find B_1 so $\|B_1\| < \frac{\epsilon}{2}$ and $C_1 \equiv A + B_1$ has pure point spectrum. C_1 may have eigenvalues in $(a - \frac{\epsilon}{2}, a) \cup (b, b + \frac{\epsilon}{2})$ and so not be in X, but we can change those eigenvalues to a or b with an operator B_2 of norm at most $\frac{\epsilon}{2}$. Then, $C_2 = A + B_1 + B_2 \in X$ has pure point spectrum and $\|C_2 - A\| < \epsilon$.

By the above, we need only show operators in X with pure point spectrum can be approximated by operators with purely absolutely continuous spectrum. So, suppose $A \in X$ has pure point spectrum.

Let $c = b - a$. Given n, let

$$I_1 = \left[a, a + \frac{c}{2^n}\right), I_2 = \left[a + \frac{c}{2^n}, a + \frac{2c}{2^n}\right), \dots, I_{2^n} = \left[b - \frac{c}{2^n}, b\right].$$

Let α_j be the midpoint of I_j. Suppose

$$A\varphi_k = \lambda_k \varphi_k$$

is the orthonormal family of eigenvalues for A. Define B_n by

$$B_n \varphi_k = \alpha_j \varphi_k \quad \text{if } \lambda_k \in I_j$$

so $\|B_n - A\| \leq \frac{c}{2^{n+1}}$ and B_n is a direct sum of $\alpha_1 \mathbb{I} \oplus \cdots \oplus \alpha_{2^n} \mathbb{I}$ with each $\mathbb{I}$ an infinite dimensional identity. Let D be a self-adjoint operator with purely absolutely continuous spectrum on $[-1, 1]$ (e.g., the matrix with 0 on diagonal and $\frac{1}{2}$ on the two principal off diagonals). Let

$$C_n = (\alpha_1 \mathbb{I} + \frac{c}{2^{n+1}} D) \oplus \cdots \oplus (\alpha_{2^n} \mathbb{I} + \frac{c}{2^{n+1}} D).$$

Then, $C \in X$, C has purely absolutely continuous spectrum and $\|A - C_n\| < \frac{c}{2^n}$. $\square$

Theorem 3.3. *Let A be a fixed self-adjoint operator. Let I_2 be the Hilbert-Schmidt operators. Then for a dense G_δ of B in I_2:*

(1) $\mathrm{spec}_{\mathrm{ac}}(A + B)$ *is empty.*
(2) $A + B$ *has no eigenvalues on* $\mathrm{spec}_{\mathrm{ess}}(A + B) = \mathrm{spec}_{\mathrm{ess}}(A)$.

Remarks. 1. This is equivalent to Theorem 3 of the introduction.

2. Given Kuroda's extension of the Weyl-von Neumann theorem [11], this theorem extends to I_p with $p > 1$. If A has no absolutely continuous spectrum, one can take $p = 1$.

Proof. By the Baire category theorem, it suffices to prove the set with (i), (ii) separately are given by dense G_δ's. By Theorem 1.2, the set of operators B with $\mathrm{spec}_{\mathrm{ac}}(A + B)$ empty is a G_δ, and by the Weyl-von Neumann theorem, it is dense so (i) yields a dense G_δ.

By Weyl-von Neumann and a simple additional argument, given ϵ, we can find B_0 with $\|B_0\|_2 < \frac{\epsilon}{2}$ so $A_0 \equiv A + B_0$ has simple pure point spectrum. Let φ be a cyclic vector for A_0 and let P_0 be the projection onto $\{\alpha\varphi \mid \alpha \in \mathbb{C}\}$. By a theorem of [3], $A_0 + \lambda P_0$ has no eigenvalues in $\mathrm{spec}(A_0)$ for Baire typical λ so we can find $|\lambda_0| < \frac{\epsilon}{2}$ so that $A_0 + \lambda_0 P_0$ has no eigenvalues on $\mathrm{spec}_{\mathrm{ess}}(A)$. Take $B = B_0 + \lambda_0 P_0$ so $\|B\|_2 < \epsilon$. This proves the density of the set in (ii). It is a G_δ by Theorem 1.1. $\square$

§4. Jacobi Matrices and Schrödinger Operators

We will begin with the Jacobi matrix case and prove Theorem 1 of the introduction.

Theorem 4.1. *Fix $a > 0$. Let X be the set of Jacobi matrices on $\ell^2(\mathbb{Z})$:*

$$Au_n = u_{n+1} + u_{n-1} + x_n u_n$$

where x_n is an arbitrary sequence with $|x_n| \le a$. Put the topology of pointwise convergence on $\{x_n\}$ (so X is a compact metrizable space). Then

$$\{A \in X \mid \operatorname{spec}(A) = [-a-2, a+2], \operatorname{spec}(A) \text{ is purely singular continuous}\}$$

is Baire typical.

Proof. We use the Wonderland Theorem. Let $d\mu$ be the product of Lebesgue measures $(2a)^{-1} dx_n$ so $\operatorname{supp}(d\mu) = [-a, a]^{\mathbb{Z}}$. Let $D = \{A \in X \mid \operatorname{spec}(A) = [-a-2, a+2], \operatorname{spec}(A)$ is pure point$\}$. Then $\mu(X \backslash D) = 0$ by Anderson localization (see, e.g., [14]). D is dense by the support result.

Given any x_n, let

$$\begin{aligned} x_n^j &= x_n \quad |n| \le j \\ &= \text{chosen to be periodic of period } 2j+1 \quad \text{ if } n > |j|. \end{aligned}$$

Thus, $x^{(j)} \to x$ and the Jacobi matrix associated to $x^{(j)}$ has purely absolutely continuous spectrum. □

Remark. We do not need the full proof of Anderson localization; it suffices that the Jacobi matrices associated to Lebesgue typical sequences have no a.c. spectrum and this is easier to prove.

For random Jacobi matrices in higher dimension, it is believed that there is sometimes a.c. spectrum, but that is not so for the generic matrix. Let $\mathbb{Z}^\nu$ have the norms $|n| = \sum_{j=1}^{\nu} |n_j|$ and $\|n\| = \sup_j |n_j|$.

Theorem 4.2. *Fix $a > 0$. Let X be the set of Jacobi matrices on $\ell^2(\mathbb{Z}^\nu)$*

$$Au_n = \sum_{|j|=1} u_{n+j} + x_n u_n$$

where x is an arbitrary multisequence with $|x_n| \le n$. Put the topology of pointwise convergence on $\{x_n\}$. Then

$$\{A \in X \mid \operatorname{spec}(A) = [-a-2\nu, a+2\nu]; \operatorname{spec}(A) \text{ is purely singular continuous}\}$$

is Baire typical.

We need a lemma which shows how "loose" generic really is:

Lemma 4.3. *In the setup of Theorem* 4.2, *suppose that there is a single operator* $A_0 \in X$, $\mathrm{spec}_{\mathrm{ac}}(A_0) = \emptyset$. *Then,* $\mathrm{spec}_{\mathrm{ac}}(A) = \emptyset$ *for a dense set of* A *in* X.

Proof. Let $x_n^{(0)}$ be the multisequence defining A_0. Given $B \in X$ with multisequence x_n, define A_j by the multisequence $x_n^{(j)}$ where

$$\begin{aligned} x_n^{(j)} &= x_n \qquad |n| \leq j \\ &= x_n^{(0)} \quad\ |n| > j. \end{aligned}$$

Since $x_n^{(j)} \to x_n$ pointwise, $A_j \to B$. But $A_j \to A_0$ is finite rank, so $\mathrm{spec}_{\mathrm{ac}}(A_j) = \mathrm{spec}_{\mathrm{ac}}(A_0) = \emptyset$. $\square$

Proof of Theorem 4.2. We use the Wonderland Theorem. For any rational $q \in [-a, a]$, the set of potentials x_n equal to q if $|n| \geq j$ for some j is dense. Such a potential yields an operator A with $[q - 2\nu, q + 2\nu] \in \mathrm{spec}(A)$, so generically $\bigcup_q [q - 2\nu, q + 2\nu]$ is in] $\mathrm{spec}(A)$.

As in the proof of Theorem 4.1, the periodic multisequences are dense and each yields an operator A with no point spectrum, so the operators with no point spectrum are dense.

By the lemma, we need only find the operator A in our space with no a.c. spectrum. Let $\{y_i\}_{i\in\mathbb{Z}}$ be a specific sequence in $[\frac{-a}{\nu}, \frac{a}{\nu}]^{\mathbb{Z}}$ whose one-dimensional Jacobi matrix J_0 has only dense point spectrum in $[-2 - \frac{a}{\nu}, 2 + \frac{a}{\nu}]$. Let

$$x_n = y_{n_1} + \cdots + y_{n_\nu}$$

so the corresponding A has the form

$$J \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes J \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes J$$

in $\ell^2(\mathbb{Z}^\nu) = \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}) \otimes \cdots \otimes \ell^2(\mathbb{Z})$. Then $\mathrm{spec}(A)$ is also pure point. $\square$

Theorem 4.4. *Let* c_0 *be the sequences* $\{x_n\}_{n\in\mathbb{Z}}$ *with* $|x_n| \to 0$. *For* $x \in \ell^p$ *or in* c_0, *let* $J(x)$ *be the corresponding Jacobi matrix on* $\ell^2(\mathbb{Z})$. *Then* $\mathrm{spec}_{\mathrm{ess}}(J(x)) = [-2, 2]$ *and*

$$\{x \mid (x) \text{ has purely singular continuous spectrum on } [-2, 2]\}$$

is Baire typical in c_0 *and in each* ℓ^p $(p > 2)$ *when these spaces are given the norm topology.*

Proof. Since $x_n \to 0$ at $\pm\infty$, the diagonal matrix is compact and $\mathrm{spec}_{\mathrm{ess}}(J(x)) = \mathrm{spec}_{\mathrm{ess}}(J(x = 0)) = [-2, 2]$. Thus, it suffices to find dense sets with no point spectrum in $[-2, 2]$ and with no a.c. spectrum in $[-2, 2]$. If x has finite support, then any solution of $J(x)u = \lambda u$ with $\lambda \in (-2, 2)$ must be a plane wave outside a finite set and so is not in ℓ^2. Since the sequences, x, of compact support are dense, we have the required density of operators without point spectrum.

As in the proof of the last theorem, we need only find one x in our space with no a.c. spectrum. In [15], Simon showed that if a_n is a typical random sequence, independent and uniformly distributed in $[-1, 1]$, then $x_n = (|n| + 1)^{-\beta} a_n$ yields a

$J(x)$ with pure point spectrum so long as $\beta < \frac{1}{2}$. This yields the required examples in ℓ^p or c_0. $\square$

Remarks. 1. One could instead look at sequences x_n with $\sup \left|(1+|n|)^\beta x_n\right| < \infty$ in the obvious norm and get the result so long as $\beta < \frac{1}{2}$.

2. For $p = 1$, or $\beta > 1$ (in the language of Remark 1), $J(x)$ has lots of a.c. spectrum, so the result requires some slow falloff hypothesis. It is likely the result remains true for $1 < \beta \leq \frac{1}{2}$ and $1 < p \leq 2$ but it is open.

3. We are unable to extend this result to the higher dimensional $(\mathbb{Z}^\nu)$ case because neither the method used in Theorem 4.2 (taking $x_n = y_{n_1} + \cdots + y_{n_\nu}$) or Theorem 4.5 (spherical symmetry) works.

We turn next to Schrödinger operators. We will begin with the case where $V \to 0$ at infinity.

Theorem 4.5. *Let $\mathcal{C}_\infty(\mathbb{R}^\nu)$ be the continuous function of $\mathbb{R}^\nu$ which vanish at infinity in the uniform norm. Then for a Baire typical set of V, $-\Delta + V$ has purely singular continuous spectrum on all of $(0,\infty)$.*

Proof. By general principles, (see, e.g., [13]), $\text{spec}_{\text{ess}}(-\Delta + V) = [0,\infty)$ so we need only show that for a dense set $\text{spec}_{\text{ac}}(-\Delta + V) = \emptyset$ and for another dense set, $\text{spec}_{\text{pp}}(-\Delta + V) \subset (-\infty, 0]$.

If V has compact support, it is well known [13] that $\text{spec}_{\text{pp}}(-\Delta + V) \subset (-\infty, 0]$, so we have that required dense set.

Suppose we find one $V \in \mathcal{C}_\infty(\mathbb{R}^\nu)$ with $\text{spec}_{\text{ac}}(-\Delta + V) = \emptyset$. Suppose W is another potential with $W(x) = V(x)$ for $|x| > R$ for some R. Then $\text{spec}_{\text{ac}}(-\Delta + W) = \emptyset$ by using Dirichlet decoupling as in Deift-Simon [2]. Any $W_0 \in \mathcal{C}_\infty(\mathbb{R}^\nu)$ is a limit of functions equal to V outside of some ball, so we get the required density. Thus we need only find one V.

To find the required V, we choose V spherically symmetric and given by a typical potential in the analysis of Kotani-Ushiroya [10]. These go to zero at infinity and are known to have $\text{spec}(-\frac{d^2}{dx^2} + V(r))$ pure point. Each partial wave Hamiltonian $-\frac{d^2}{dr^2} + \frac{c}{r^2} + V(r)$ also has no a.c. spectrum by trace class theory, so $-\Delta + V$ has no a.c. spectrum. $\square$

Remark. By looking carefully at [10], the result extends to $L^p(\mathbb{R}^\nu)$, $p > 2n$.

Here is a typical example for random Schrödinger operators.

Theorem 4.6. *For $v \in [-a,a]^{\mathbb{Z}^\nu}$, define V on $\mathbb{R}^\nu$ by*

$$V(x) = v(i(x))$$

where $i(x)$ is defined by

$$i(x) = j \quad \text{if } j_\alpha \leq x_\alpha < j_\alpha + 1.$$

Then for a Baire typical v, $-\Delta + V$ has spectrum $[-a,\infty)$ and is purely singular continuous there.

Proof. By using periodic v, we see that Baire typically V has no point spectrum. As in the last theorem, we need only find a single v with no a.c. spectrum. Take

$v(i) = \tilde{v}_{i_1} + \cdots + \tilde{v}_{i_n}$ for a one-dimensional $\tilde{v}$. If $-\frac{d^2}{dx^2} + \tilde{V}$ has point spectrum, so does $-\Delta + V$. Thus localization in the one-dimensional case [5,9] completes the proof. $\square$

Finally, we want to say something about the almost periodic case with a series of remarks.

1. Consider the almost Mathieu equation, the Jacobi matrix with $v, \lambda\cos(\pi\alpha n + \theta)$ for λ, θ fixed. For α rational, the potential is periodic and there is no point spectrum. It follows that for Baire typical α, there is no point spectrum either. This is a soft version of Gordon's theorem (Gordon [6], Avron-Simon [1]).

2. Fix λ, α in the almost Mathieu equation with α irrational. Suppose that there is a single θ_0 leading to purely s.c. spectrum. Then its translates are dense and so Baire typically, there will be only s.c. spectrum. It may well happen that for α with good Diophantine properties and $\lambda > 2$, we have pure point spectrum for Lebesgue typical θ and purely s.c. spectrum for Baire typical θ.

3. The argument in Remark 1 applies to generic potentials, v, in spaces of limit periodic potentials.

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