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APPROXIMATION OF HOMOGENEOUS MEASURES IN THE 2-WASSERSTEIN METRIC

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Abstract. It is shown that, on certain weighted spaces of vector fields on $\mathbb{R}^3$, any homogeneous measure of finite energy density and dissipation can be approximated in the second Wasserstein distance by homogeneous measures supported by finite trigonometric polynomials of increasing period and degree. In particular, the periodic correlation functions of the approximation converge uniformly on compact sets of $\mathbb{R}^3$ to the correlation function of the given measure.

1. Introduction

Measures on function spaces invariant under shift of the argument are called homogeneous. Such measures are important in Kolmogorov's theory of turbulence. Properties of their correlation functions test the validity of the theory when compared to real data.

The first construction of homogeneous measures supported by weak solutions of the Navier-Stokes equations, called statistical solutions, is [VF1]. It relies on approximating an initial homogeneous measure μ by explicitly constructed homogeneous measures μ_l supported by trigonometric polynomials of degree l and period $2l$, see section 4.1 below. The approximation there is in characteristic: The characteristic functions of the μ_l's converge to the characteristic function of μ, see section 2.7. Throughout this note, such an approximation will be referred to as an **l-approximation** of the given homogeneous μ.

These statistical solutions are a weak limit P of push-forward measures $P_l := (S_l)_{\#}\mu_l$, for S_l the solution operator of the Galerkin approximation defined from such trigonometric polynomials, see section 2.2 for definitions. Convergence in characteristic suffices to show that the restriction of P at time 0 is the initial measure μ. The details of this construction are in [VF], with Appendix II there containing the details of the approximation $\mu_l \to \mu$. A similar construction yielding homogeneous and isotropic solutions, [DFK], relies on this convergence in characteristic.

On the other hand, several properties of the spatial correlation functions R_{ij} of homogeneous and isotropic fluid flows are taken for granted, see [D], for example. Often, arguments for the validity of these properties express the correlations as Fourier transforms. The existence of the Fourier transform of the correlation tensor has not been shown for homogeneous solutions, even when the Fourier transform of the correlation of the initial measure exists. What does hold is that the correlation tensor of any homogeneous measure is the Fourier-Stieltjes transform of a (possibly) non-differentiable function. This is thoroughly explained in [K].

The μ_l homogeneous measures above have, of course, periodic correlations, and hence Fourier series expansions. One can then try to show that an l-approximation also yields an approximation of the corresponding correlations to the correlation of μ, that this is also true for (almost) all times for statistical solutions, and then use Galerkin correlations to get information for the correlations of the solution itself.[1] This note substantiates the first of these steps:

Main Theorem. *Let μ be a homogeneous measure on the separable Hilbert space $\mathcal{H}^0(r)$ of vector fields on $\mathbb{R}^3$, as in Definition 2.1 below, and let μ_l be an l-approximation of μ, as in section 4.1 below. Then the μ_l's converge to μ weakly, up to subsequence, and the correlation functions of this subsequence converge to the correlation functions of μ pointwise.*

The precise statement of this theorem is Theorem 4.5. The proof of this main theorem is perhaps interesting in itself: It first improves the convergence of μ_l to μ from characteristic to weak and then shows that second moments converge:

$$\int \|u\|^2_{\mathcal{H}^0(r)}\mu_l(du) \to \int \|u\|^2_{\mathcal{H}^0(r)}\mu(du). \tag{1}$$

Recall here a standard result in the theory of optimal transport: Weak convergence and the convergence (1) is equivalent to convergence in the space $W_2(\mathcal{H}^0(r))$ of probability measures on $\mathcal{H}^0(r)$, with finite second moment, equipped with the second Wasserstein metric W_2, see section (2.4) for definitions. In this way, this note is a first step in revisiting the constructions [VF] and [DFK] in terms of the geometry of the Wasserstein space. (For example, the restriction at time t of the measures P_l describe absolutely continuous curves in $W_2(\mathcal{H}^0(r))$.)

Kuksin and Shirikian use the first Wasserstein metric W_1 in their study of stochastic (as opposed to statistical) solutions of the 2-dimensional Navier-Stokes equations on periodic domains, see [Ku], [KS].

Section 2 gathers the necessary definitions and some lemmas to be used later. Although the main result here concerns l-approximations, section 3 gives a general statement for the Wasserstein convergence of homogeneous measures, with an eye to proving convergence of correlations for statistical solutions, for all times, in forthcoming work. Section 4 then shows that the conditions of the general theorem hold for l-approximations. The final section remarks on how the main result still holds for homogeneous AND isotropic measures and alternative approximations.

[1] For example, in dim=1, convergence of correlations of an l-approximation of Burgers statistical solutions for almost all times implies immediately that the integral of the correlation function is constant in time, as already anticipated by Burgers, [Bu]. In fact, the main motivation behind this note is to develop tools for examining the spatial decay of correlation functions of Navier-Stokes statistical solutions in dimension 3, cf. [L].

2. Spaces and measures

2.1. **Spaces.** Non-trivial measures invariant under shifts exist on weighted Sobolev spaces of vector fields, but not on L^p spaces, the weight in the norm ensuring that balls in the function space are not be invariant under shifts, [VF], p. 208:

Definition 2.1. *For $r < -3/2$, define $\mathcal{H}^0(r)$ to be the space of measurable, solenoidal vector fields u on $\mathbb{R}^3$,*

$$\int u(x) \cdot \nabla \phi(x)\ dx = 0 \quad \text{for} \quad \text{all } \phi \in C_0^\infty(\mathbb{R}^3), \tag{2}$$

with finite $(0, r)$-norm:

$$\|u\|_{0,r}^2 = \int_{\mathbb{R}^3} \left(1 + |x|^2\right)^r |u(x)|^2\ dx. \tag{3}$$

Similarly, define $\mathcal{H}^1(r)$ to be the space of vector fields u on $\mathbb{R}^3$ satisfying (2) and with finite norm

$$\|u\|_{1,r}^2 = \int_{\mathbb{R}^3} \left(1 + |x|^2\right)^r \left(|u(x)|^2 + |\nabla u(x)|^2\right)\ dx. \tag{4}$$

Observe that the restriction on r implies that constant and periodic vector fields belong to the above spaces.

Lemma 2.2. *$\mathcal{H}^1(r')$ compactly embeds into $\mathcal{H}^0(r)$ for $r < r'$.*

Proof. For any u in the ball of radius M in $\mathcal{H}^1(r')$

$$\int_{\mathbb{R}^3} (1 + |x|^2)^{r'}\ |u(x)|^2\ dx < M. \tag{5}$$

Then for any $\epsilon > 0$ there exists an R_ϵ such that for all $R > R_\epsilon$,

$$\begin{aligned} \int_{\mathbb{R}^3 \setminus B_R} (1 + |x|^2)^r\ |u_n(x)|^2\ dx &= \int_{\mathbb{R}^3 \setminus B_R} (1 + |x|^2)^{r - r'} (1 + |x|^2)^{r'}\ |u_n(x)|^2\ dx \\ &\leq (1 + R^2)^{r - r'} M < \frac{\epsilon}{2}, \end{aligned} \tag{6}$$

for all such u.

At the same time, the restrictions $u|_{B_R}$ form a bounded set in $W^{1,2}(B_R)$, and therefore a precompact set in $L^2(B_R)$, by standard Sobolev embedding. In particular, [B], p. 239, there exist w_i in $L^2(B_R)$, $i = 1, 2, ..., N(\epsilon)$, such that for any u as above there exists an i with

$$\int_{B_R} (1 + |x|^2)^r\ |u(x) - w_i(x)|^2\ dx\ \leq\ \|u - w_i\|_{L^2(B_R)}\ <\ \frac{\epsilon}{2}. \tag{7}$$

Now extend each w_i trivially by setting it zero outside of B_R. Then

(8)
$$\int_{\mathbb{R}^n} (1+|x|^2)^r \ |u(x)-w_i(x)|^2 \ dx$$
$$= \int_{B_R} (1+|x|^2)^r \ |u(x)-w_i(x)|^2 \ dx + \int_{B_R'} (1+|x|^2)^r \ |u(x)|^2 \ dx < \epsilon.$$

Therefore the u's form a precompact set in $\mathcal{H}^0(r)$. □

2.2. **Shifts and push-forwards.** For u in these spaces let T_h be the translation operation defined, weakly, by

$$T_h u(x) = u(x+h). \tag{9}$$

For M a metric space denote by $\mathcal{B}(M)$ the σ-algebra of Borel sets of M. Let M_1, M_2 be metric spaces, and $\Psi : M_1 \to M_2$ be measurable.

For every Borel measure ν on M_1 define a new measure $\Psi_\# \nu$ on M_2:

$$\Psi_\# \nu(B) = \nu(\Psi^{-1} B) \quad \forall \ B \in \mathcal{B}(M_2). \tag{10}$$

The measure $\Psi_\# \nu$ is the **push forward of the measure ν under the map Ψ**. (10) is equivalent to

$$\int f(u) \ \Psi_\# \nu(du) = \int f(\Psi(v)) \ \nu(dv), \tag{11}$$

for any $\Psi_\# \nu$-integrable $f : M_2 \to \mathbb{R}$.

Definition 2.3. *A measure μ defined on $\mathcal{B}(\mathcal{H}^0(r))$ is called* **homogeneous** *if it is translation invariant:*

$$(T_h)_\# \mu = \mu \Leftrightarrow \int_{\mathcal{H}} F(T_h u) \ \mu(du) = \int_{\mathcal{H}} F(u) \ \mu(du), \tag{12}$$

for any μ-integrable F, for all h in $\mathbb{R}^3$.

2.3. **Point-wise averages and densities.** The homogeneity of a measure μ implies that the functionals on $L^1(\mathbb{R}^3)$

$$\begin{aligned} \phi &\mapsto \int\int |u(x)|^2 \phi(x) \ dx \ \mu(du), \\ \phi &\mapsto \int\int |\nabla u(x)|^2 \phi(x) \ dx \ \mu(du), \end{aligned} \tag{13}$$

are invariant under translation of ϕ, therefore the point-wise averages

$$\int |u(x)|^2 \ \mu(du), \quad \int |\nabla u(x)|^2 \ \mu(du), \tag{14}$$

can be defined by

$$\begin{aligned}\int\int |u(x)|^2\phi(x)\ dx\ \mu(du) &= \int |u(x)|^2\ \mu(du)\int \phi(x)\ dx,\\ \int\int |\nabla u(x)|^2\phi(x)\ dx\ \mu(du) &= \int |\nabla u(x)|^2\ \mu(du)\int \phi(x)\ dx,\end{aligned} \tag{15}$$

for any $\phi \in L_1(\mathbb{R}^3)$, and they are independent of $x \in \mathbb{R}^3$, see Chapter VII, section 1 of [VF]. The first average in (14) will be called **the energy density** and the second one **the density of the energy dissipation**.[2]

2.4. **Wasserstein convergence.** On $\mathcal{P}_p(X)$, the space of probability measures on a separable Hilbert space X with finite p-moments, consider the p-Wasserstein metric:

$$W_p(\mu_1,\mu_2) = \left(\inf_{\pi\in\Gamma(\mu_1,\mu_2)}\int_{X\times X} \|u-v\|_X^p \pi(du,dv)\right)^{1/p}, \tag{16}$$

with

$$\Gamma(\mu_1,\mu_2) = \{\pi \in \mathcal{P}(X\times X) : (pr_1)_\#\pi = \mu_1, (pr_2)_\#\pi = \mu_2\}. \tag{17}$$

(For an equivalent description of Γ in terms of couplings of random variables see [Ku], p. 41.)

The **p-Wasserstein space** is the metric space $W_p(X) = (\mathcal{P}_p(X), W_p)$. It is complete, separable, [AGS], p. 154, not locally compact, [AGS], p. 156, and the following holds as $n\to\infty$:

$$W_p(\mu_n,\mu)\to 0 \Leftrightarrow \left\{\begin{array}{l}\mu_n\to\mu,\ \text{weakly}\\ \int_X \|u\|_X^p\ \mu_n(du) \to \int_X \|u\|_X^p\ \mu(du)\end{array}\right\}, \tag{18}$$

see [AGS], p. 154, or [V], p. 212.

2.5. **Homogeneity in the Wasserstein space.** Of concern will be measures on $\mathcal{H}^0(r)$ of finite energy density. Taking ϕ in (15) to be the integrable weight of the $\mathcal{H}^0(r)$-norm, such measures satisfy

$$\int_{\mathcal{H}^0(r)} \|u\|^2_{\mathcal{H}^0(r)}\mu(du) \le +\infty, \tag{19}$$

i.e. they have finite second moment.

[2] A terminology justified by the identity $\frac{1}{2}\frac{d}{dt}\overline{u^2} = -\nu\overline{|\nabla u|^2}$, which formally follows after integrating by parts the Navier-Stokes equation of viscosity ν.

For X separable Hilbert, the space $\mathrm{Tan}_\mu W^2(X)$, tangent to $W_2(X)$ at a measure μ, is identified in [AGS] in terms of vector fields on X itself. It consists of vector fields perpendicular to those $v : X \to X$ satisfying

$$\int < \nabla F(u), v(u) >_X \ \mu(du) = 0, \tag{20}$$

for any F cylindrical function on X.

In this way, for $X = \mathcal{H}^0(r)$, the homogeneity of a measure μ with

$$\int_{\mathcal{H}^0(r)} \|u\|^2_{\mathcal{H}^1(r)} \ \mu(du) < +\infty \tag{21}$$

implies that the vector field

$$u \mapsto \nabla u \cdot h \tag{22}$$

is in $\mathrm{Tan}^\perp_\mu W_2(X)$, for all $h \in \mathbb{R}^3$. (Differentiate $\epsilon \mapsto \int_X F(T_{\epsilon h} u) \ \mu(du)$ at $\epsilon = 0$.) This restricts the tangent space at a homogeneous μ. That this vector field is in $\mathrm{Tan}^\perp_\mu W_2(X)$ and [AGS], Proposition 8.3.3 also give an approximation of μ other than an l-approximation (via absolutely continuous measures on finite dimensional subspaces), see section 5.2.

2.6. **Correlations.** Use the homogeneity of the measure as in (15) for any $h \in \mathbb{R}^3$ and any $\phi \in L^1(\mathbb{R}^3)$ to see that there is $R_{ij}(h)$ such that

$$\int_{\mathcal{H}^0(r)} \int_{\mathbb{R}^3} u_i(x) u_j(x+h) \phi(x) \ dx \ \mu(du) = R_{ij}(h) \int_{\mathbb{R}^3} \phi(x) \ dx. \tag{23}$$

Call the function $h \mapsto R_{ij}(h)$ on $\mathbb{R}^3$ the (i,j)-th **correlation function** of μ.

Correlation functions are often defined as $\overline{u_i(x) u_j(x+h)}$, with the overline indicating some average. This corresponds here to the bilinear form

$$\int_{\mathcal{H}^0(r)} < u_i, \phi >< u_j, \psi > \ \mu(du), \tag{24}$$

for ϕ and ψ smooth, with compact supports concentrated around x and $x+h$ respectively, and for $<,>$ the L^2-inner product. By Hölder, this is continuous on $\mathcal{H}^0(-r) \times \mathcal{H}^0(-r)$, for r still smaller than $-3/2$, as in Definition 2.1.

The following relates correlations as defined in (23) to the bilinear form (24) and will be used later. It is an elementary instance of the Kernel Theorem, cf. [GV], pp 167-169:

Lemma 2.4. *For μ homogeneous on $\mathcal{H}^0(r)$ with finite second moment, the following holds:*

$$\int_{\mathcal{H}^0(r)} < u_i, \phi >< u_j, \psi > \ \mu(du) = \int_{\mathbb{R}^3} R_{ij}(h) \int_{\mathbb{R}^3} \phi(x)\psi(x+h)\ dx\ dh \tag{25}$$

for any $\phi, \psi \in C_0^\infty(\mathbb{R}^3)$.

Proof. A simple change of variables, Fubini's Theorem (valid by the second moment assumption), and the definition of the correlation functions give:

$$\begin{aligned}
&\int_{\mathcal{H}^0(r)} < u_i, \phi >< u_j, \psi > \ \mu(du) \\
&\quad = \int_{\mathcal{H}^0(r)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} u_i(x)\ \phi(x)\ u_j(y)\ \psi(y)\ dx\ dy\ \mu(du) \\
&\quad = \int_{\mathcal{H}^0(r)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} u_i(x)\ \phi(x)\ u_j(x+h)\ \psi(x+h)\ dx\ dh\ \mu(du) \\
&\quad = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathcal{H}^0(r)} u_i(x)\ u_j(x+h)\ \mu(du)\ \phi(x)\ \psi(x+h)\ dx\ dh, \\
&\quad = \int_{\mathbb{R}^3} R_{ij}(h) \int_{\mathbb{R}^3} \phi(x)\ \psi(x+h)\ dx\ dh.
\end{aligned}$$

□

Remark 2.5. *(Regular Kernel.) For μ homogeneous with finite energy density and finite dissipation rate (assumptions that will be used below), the correlation functions R_{ij} are bounded and in C^2 with bounded derivatives, see* [VF]*, Lemma VIII.7.3. When μ is the evaluation of a Navier-Stokes homogeneous statistical solution at some positive time decay at spatial infinity for the R_{ij}'s is expected, but not rigorously shown. The hydrodynamic pressure and the decay rate of the correlation of the initial measure are expected to determine the rate of decay for $t > 0$, cf.* [BP], [L], [S].

2.7. **Convergence in characteristic.** Recall the characteristic function of a measure μ on X

$$\chi_\mu(\phi) = \int_X e^{i<u,\phi>_X}\ \mu(du), \tag{26}$$

for ϕ test function. Also recall that $\mu_n \to \mu$ **in characteristic** if

$$\chi_{\mu_n}(\phi) \to \chi_\mu(\phi), \tag{27}$$

for any ϕ. Finally recall that given μ and ν probability measures of finite first moments such that $\chi_\mu(\phi) = \chi_\nu(\phi)$ for all ϕ in a dense set in X, then

$\chi_\mu(u) = \chi_\nu(u)$ for all u in X (and hence $\mu = \nu$): Indeed, given $u_0 \in X, \phi_n \to u$ in X,

$$\int \left(e^{i<\phi_n,u>} - e^{i<u_0,u>}\right) \mu(du) \le \|\phi_n - u_0\| \int \|u\| \ \mu(du). \tag{28}$$

3. General results

Theorem 3.1. *Let $\{\mu_l\}_{l>0}$ be a family of homogeneous measures on $\mathcal{H}^0(r), r < -3/2$, with*

$$\int \left(|u(x)|^2 + |\nabla u(x)|^2\right) \mu_l(du) \le C \tag{29}$$

for all l and C independent of l. Then there is subsequence $\{\mu_{l(i)}\}_{i\in\mathbb{N}}$ converging weakly to some (necessarily homogeneous) measure μ on $\mathcal{H}^0(r)$, and if

$$\int |u(x)|^2 \mu_{l(i)}(du) \le \int |u(x)|^2 \mu(du) < \infty, \ i \in \mathbb{N}, \tag{30}$$

then $\mu_{l(i)} \to \mu$ in $W_2(\mathcal{H}^0(r))$.

Proof. Step 1: For some subsequence $l(i), i \in \mathbb{N}$, $\mu_{l(i)} \to \mu$ weakly as measures on $\mathcal{H}^0(r)$: For the given r, pick any r' satisfying $r < r' < -\frac{3}{2}$. By (29) and the definition of pointwise averages (15),

$$\int \|u\|^2_{\mathcal{H}^1(r')} \mu_l(du) < +\infty, \tag{31}$$

for all l. Therefore all Borel subsets of $\mathcal{H}^0(r)$ with infinite $\mathcal{H}^1(r')$ norm have μ_l-measure zero, for any l, i.e. all μ_l's are supported on $\mathcal{H}^1(r')$, for any such r'.[3]

Given the compactness of the embedding $\mathcal{H}^1(r')$ into $\mathcal{H}^0(r)$ from Lemma 2.2, it suffices to show that for all l

$$\int \|u\|_{\mathcal{H}^1(r')} \ \mu_l < C, \tag{32}$$

for C independent of l, cf. Lemma II.3.1 in [VF], or Remark 5.1.5 in [AGS]. This follows from (29) and Hölder.

Now rename $l(i)$ to l.

Step 2: The following holds:

$$\int \|u\|^2_{\mathcal{H}^0(r)} \mu_l(du) \to \int \|u\|^2_{\mathcal{H}^0(r)} \mu(du), \ l \to \infty. \tag{33}$$

[3]Note that (29) implies that the homogeneous μ_l's are supported in $\mathcal{H}^1(r)$, for any $r < -3/2$.

Indeed, it is standard that the weak convergence of μ_l to μ as measures on $\mathcal{H}^0(r)$ implies that

$$\liminf_l \int \|u\|^2_{\mathcal{H}^0(r)} \mu_l(du) \geq \int \|u\|^2_{\mathcal{H}^0(r)} \mu(du). \tag{34}$$

Then (30) implies that in addition,

$$\limsup_l \int \|u\|^2_{\mathcal{H}^0(r)} \mu_l(du) \leq \int \|u\|^2_{\mathcal{H}^0(r)} \mu(du). \tag{35}$$

This in turn implies that the second moments

$$\int \|u\|^2_{\mathcal{H}^0(r)} \mu_l(du) \tag{36}$$

are uniformly integrable in l, i.e.

$$\lim_{R\to\infty} \int_{\{\|u\|>R\}} \|u\|^2_{\mathcal{H}^0(r)} \mu_l(du) \to 0, \tag{37}$$

uniformly in l, see Lemma 5.1.7, [AGS]. It is also standard that this uniform integrability implies (33), see same Lemma in [AGS]. □

Remark 3.2. *By Hölder and (16), $W_2(\mathcal{H}^0(r))$ convergence implies $W_1(\mathcal{H}^0(r))$ convergence, hence convergence of expectations.*

Now convergence in $W_2(X)$ implies convergence of integrals of continuous functions of 2-growth, i.e.

$$\int f(u)\ \mu_l(du) \to \int f(u)\ \mu(du) \tag{38}$$

for any f satisfying $|f(u)| \leq C(\|u\|^2_X + 1)$, see Proposition 7.1.5 and Lemma 5.1.7 of [AGS], or Theorem 7.12 of [V]. Since for each fixed test ϕ the function

$$u \to < u, \phi >^2 \tag{39}$$

is a continuous function of 2-growth,

$$\begin{aligned} \int < u, \phi + \psi >^2\ \mu_l(du) &\to \int < u, \phi + \psi >^2\ \mu(du), \\ \int < u, \phi >^2\ \mu_l(du) &\to \int < u, \phi >^2\ \mu(du), \\ \int < u, \psi >^2\ \mu_l(du) &\to \int < u, \psi >^2\ \mu(du), \end{aligned} \tag{40}$$

hence

$$\int < u, \phi >< u, \psi >\ \mu_l(du) \to \int < u, \phi >< u, \psi >\ \mu(du), \tag{41}$$

for any ϕ, ψ test functions.

Then for the particular case of homogeneous measures, where correlation functions are defined, Wasserstein convergence implies the following:

Theorem 3.3. *Let μ_l, μ be as in Theorem 3.1. Then for each (i,j) there exists subsequence of the correlation functions R^l_{ij} which converges to the correlation function R_{ij} pointwise, uniformly on compact subsets of $\mathbb{R}^3$.*

The proof of Theorem 3.3 uses the following:

Lemma 3.4. *Under the assumptions of Theorem 3.1, the correlation functions R^l_{ij} corresponding to the measures μ_l and their first derivatives are uniformly bounded in l.*

Proof. First recall that, as for any homogeneous measure,

$$\frac{\partial}{\partial h_k} R^l_{ij}(h) = \int \frac{\partial u_i}{\partial x_k}(x+h)\ u_j(x)\ \mu_l(du), \tag{42}$$

see Lemma VIII.7.2 of [VF]. Following the definition of pointwise averages, easily calculate

$$\begin{aligned} \left|\frac{\partial}{\partial h_k} R^l_{ij}(h)\right| &= \left|\int \frac{\partial u_i}{\partial x_k}(x+h)\ u_j(x)\ \mu_l(du)\right| \\ &\le \left[\int \left|\frac{\partial u_i}{\partial x_k}(x+h)\right|^2\ \mu_l(du)\right]^{\frac{1}{2}} \left[\int |u_j(x)|^2\ \mu_l(du)\right]^{\frac{1}{2}} \\ &= \left[\int \left|\frac{\partial u_i}{\partial x_k}(x)\right|^2\ \mu_l(du)\right]^{\frac{1}{2}} \left[\int |u_j(x)|^2\ \mu_l(du)\right]^{\frac{1}{2}}, \end{aligned} \tag{43}$$

which by (29) are bounded above uniformly in l by C. Similarly,

$$\left|R^l_{ij}(h)\right| \le C. \tag{44}$$

□

Proof of theorem 3.3. Since the ∇R^l_{ij}'s are uniformly bounded by the previous lemma, the R^l_{ij}'s are uniformly equicontinuous. Also by the previous lemma, the sequence is equibounded, therefore by Arzela-Ascoli there exists Q_{ij} on $\mathbb{R}^3$ such that

$$R^l_{ij} \to Q_{ij} \tag{45}$$

pointwise, uniformly on compact subsets of $\mathbb{R}^3$, up to subsequence. In particular, for this subsequence,

$$\int\int_B R^l_{ij}(y-x)\Phi(x,y)dxdy \to \int\int_B Q_{ij}(y-x)\Phi(x,y)dxdy \tag{46}$$

on any B compact in $\mathbb{R}^6$ and Φ smooth with compact support in B. On the other hand, since

$$\int < u, \phi > < u, \psi > \ \mu_l(du) \to \int < u, \phi > < u, \psi > \ \mu(du) \tag{47}$$

for ϕ and ψ in $C_0^\infty(\mathbb{R}^3)$ by (41),

$$\int\int R_{ij}^l(y-x)\ \phi(x)\ \psi(y)\ \ dxdy \to \int\int R_{ij}(y-x)\ \phi(x)\ \psi(y)\ dxdy, \tag{48}$$

by (25). Now linear combinations of products $\phi(x)\psi(y)$ are dense in $C_0^\infty(\mathbb{R}^6)$, (see for example [F], Theorem 4.3.1), therefore, as the R_{ij}^l's are bounded uniformly in l by Lemma 3.4,

$$\int\int R_{ij}^l(y-x)\ \Phi(x,y)\ \ dxdy \to \int\int R_{ij}(y-x)\ \Phi(x,y)dxdy, \tag{49}$$

on any B. Therefore $Q_{ij} = R_{ij}$. In particular, R_{ij}^l converge pointwise to R_{ij}, and uniformly so on compacts. □

4. Application: Homogeneous measures on trigonometric polynomials

4.1. **Overview of l-approximations.** The construction of homogeneous and isotropic statistical solutions of the Navier-Stokes equations is based on approximating ANY homogeneous μ on $\mathcal{H}^0(r)$ by homogeneous μ_l's supported on:

$$\mathcal{M}_l = \left\{ \sum_{\substack{k\in\frac{\pi}{l}\mathbb{Z}^3,\\ |k|\le l}} a_k e^{ik\cdot x} : a_k \cdot k = 0,\ a_k = \overline{a}_{-k}\ \forall\ k \right\}, \tag{50}$$

the finite-dimensional space of divergence-free, real, vector valued trigonometric polynomials of degree l and period $2l$. Note that $\mathcal{M}_l \subset \mathcal{H}^0(r)$ holds for all l. A concise description of the μ_l's follows, with full details available at Appendix II of [VF]. (The construction is not straightforward as one must obtain **divergence free** periodic vector fields.)

- Given l, fix cut-off function ψ_l with support well within $\mathbb{T}_l = [-l, l]^3$. This is used to cut in x-space.
- Also fix for the given l a cut-off ζ_l with support in a ball of radius decreasing in l. This is used to cut in frequency space.
- Given $u \in \mathcal{H}^0(r)$, define

$$w_l(x) = u(x) - \int u(y) \int e^{i(x-y)\xi}\zeta_l(\xi)\ d\xi\ dy. \tag{51}$$

- Define u_l^s to be the divergence free part of the projection on $\mathcal{M}_l$ of the periodization

$$u_l^T(x) = \sum_{j\in\mathbb{Z}^3} (\psi_l\ w_l)(x + 2lj) + C_u, \tag{52}$$

for C_u a constant that, as only derivatives will be of concern, does not need to be specified here. Then define

$$\begin{aligned} U_l^s : \mathcal{H}^0(r) &\to \mathcal{M}_l \\ u &\mapsto u_l^s. \end{aligned} \tag{53}$$

- Finally, define

$$\mu_l = (\alpha \circ (U_l^s \times Id))_\#(\mu \times \tau_l), \tag{54}$$

where τ_l is the normalized Lebesque measure on $\mathbb{T}_l$, Id the identity on $\mathbb{T}_l$, and $\alpha(u,h) = T_h u$.

Given μ homogeneous, an approximation μ_l of μ constructed according to (51)–(54) will be referred to as **an l-approximation of** μ.

Having averaged push forwards via l-periodics over $\mathbb{T}_l$, μ_l is homogeneous with respect to all shifts in $\mathbb{R}^3$, therefore μ_l-pointwise averages can be defined. The main result of Appendix II in [VF] then reads:

Theorem 4.1. *$\mu_l \to \mu$ in characteristic as $l \to \infty$, and*

$$\int |u(x)|^2\ \mu_l(du) \le \int |u(x)|^2\ \mu(du). \tag{55}$$

Remark 4.2. *Note that the correlations of the μ_l's are also $2l$-periodic, as for any test ϕ*

$$\int_{\mathbb{R}^3} u_i(x + 2l + h)u_j(x)\phi(x)\ dx = \int_{\mathbb{R}^3} u_i(x + h)u_j(x)\phi(x)\ dx, \tag{56}$$

for any u in the support of μ_l.

4.2. **An improved energy estimate.** The following extends part 3, Proposition 2.1, Appendix II, in [VF]:

Lemma 4.3. *For any homogeneous measure μ on $\mathcal{H}^0(r)$ with*

$$\int_{\mathcal{H}^0(r)} \|u\|^2_{\mathcal{H}^1(r)}\ \mu(du) < \infty, \tag{57}$$

there exist finite complex measures $\mathfrak{M}_{ij}, \mathfrak{N}_{ij}$ on $\mathbb{R}^3$ such that the following hold for averages of the distributional Fourier transforms of u's and any ψ of rapid decay:

$$\begin{aligned} \int_{\mathcal{H}^0(r)} < \widetilde{u_i}, \psi > \overline{< \widetilde{u_j}, \psi >}\ \mu(du) &= \int_{\mathbb{R}^3} |\psi(x)|^2\ \mathfrak{M}_{ij}(dx), \\ \int_{\mathcal{H}^0(r)} < \widetilde{\nabla u_i}, \psi > \overline{< \widetilde{\nabla u_j}, \psi >}\ \mu(du) &= \int_{\mathbb{R}^3} |\psi(x)|^2\ \mathfrak{N}_{ij}(dx), \end{aligned} \tag{58}$$

with Hermitian inner products in $\mathbb{C}$ and $\mathbb{C}^3$ used in the integrands of the left hand sides. In particular,

$$\begin{aligned} \sum_{i=1}^{3} \int \mathfrak{M}_{ii}(dx) &= \int |u(x)|^2\ \mu(du), \\ \sum_{i=1}^{3} \int \mathfrak{N}_{ii}(dx) &= \int |\nabla u(x)|^2\ \mu(du). \end{aligned} \tag{59}$$

Proof. For the second equality in (58):

$$\begin{aligned} &\int_{\mathcal{H}^0(r)} < \widetilde{\nabla u_i}, \psi > \overline{< \widetilde{\nabla u_j}, \psi >}\ \mu(du) \\ &= \int_{\mathbb{R}^3} \int_{\mathcal{H}^0(r)} \int_{\mathbb{R}^3} \nabla u_i(x+h)\ \nabla u_j(x) \tilde{\psi}(x+h) \tilde{\psi}(-x)\ dx\ \mu(du)\ dh, \end{aligned} \tag{60}$$

where Fubini is justified by (57). Then the definition of the correlation function and the identity

$$\begin{aligned} (\partial_n \partial_m R_{ij})(h) &= \int \partial_n u_i(x)\ \partial_m u_j(x+h)\ \mu(du) \\ &= \int \partial_m u_i(x)\ \partial_n u_j(x+h)\ \mu(du), \end{aligned} \tag{61}$$

(see Lemma VII.7.2 of [VF] for this), imply that

(62)
$$
\begin{aligned}
&\int_{\mathcal{H}^0(r)} < \widetilde{\nabla u_i}, \psi > \overline{< \widetilde{\nabla u_j}, \psi >}\ \mu(du) \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tilde{\psi}(x+h)\tilde{\psi}(-x)\ dx(\nabla^2 R_{ij})(h)\ dh \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \psi(y')\psi(y)\ e^{-ih\cdot y'}\ e^{-ix\cdot(y'-y)}\ dy'\ dy\ dx\ (\nabla^2 R_{ij})(h)\ dh \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \psi(y')\psi(y)\delta(y'-y)\ dy\ e^{-ih\cdot y'}\ dy'\ (\nabla^2 R_{ij})(h)\ dh \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\psi(y')|^2\ e^{-ih\cdot y'}\ dy'(\nabla^2 R_{ij})(h)\ dh \\
&= < \widetilde{\nabla^2 R_{ij}}, |\psi|^2 > .
\end{aligned}
$$

The first equality of (58) is proved by exactly the same argument, cf. [VF], p. 539. □

Lemma 4.4. *For μ homogeneous measure on $\mathcal{H}^0(r)$ supported on $\mathcal{H}^1(r)$ and μ_l an l-approximation of μ as a measure on $\mathcal{H}^0(r)$, the following holds:*

$$
\int_{\mathcal{H}^0(r)} |\nabla u(x)|^2\ \mu_l(du) \le C \int_{\mathcal{H}^0(r)} \left(|\nabla u(x)|^2 + |u(x)|^2\right)\ \mu(du). \tag{63}
$$

Proof. First note that by the definition of μ_l

$$
\begin{aligned}
\int_{\mathcal{H}^0(r)} |\nabla u(x)|^2\ \mu_l(du) &= \int_{\mathcal{H}^0(r)}\int_{\mathbb{T}_l} |\nabla u_l^s(x+h)|^2\ \tau_l(dh)\mu(du) \\
&= \int_{\mathcal{H}^0(r)} \left(\sum_{k\in\Gamma_l} |\widehat{\nabla u_l^s}(k)|^2\right) \mu(du) \\
&= C\sum_{k\in\Gamma_l} |k|^2 \int_{\mathcal{H}^0(r)} |\widehat{u_l^s}(k)|^2 \mu(du) \\
&\le C \sum_{k\in\Gamma_l, |k|\le l} |k|^2 \int_{\mathcal{H}^0(r)} |\widehat{u_l^T}(k)|^2 \mu(du),
\end{aligned} \tag{64}
$$

for u_l^T as in (52). Using (2.28′) of Appendix II of [VF], rewrite this as

(65)

$$\begin{aligned}
&\frac{C}{|\mathbb{T}_l|^2}\sum_{k\in\Gamma_l,|k|\le l}|k|^2\int_{\mathcal{H}^0(r)}\left|\int_{\mathbb{R}^3}(1-\zeta_l(\xi))\,\tilde{u}(\xi)\;\widetilde{\psi}_l(\xi-k)\;d\xi\right|^2\mu(du)\\
=&\frac{C}{|\mathbb{T}_l|^2}\sum_{k\in\Gamma_l,|k|\le l}\sum_i\int_{\mathcal{H}^0(r)}\left|\int_{\mathbb{R}^3}(1-\zeta_l(\xi))\,\tilde{u}_i(\xi)\;k\;\widetilde{\psi}_l(\xi-k)\;d\xi\right|^2\mu(du)\\
=&\frac{C}{|\mathbb{T}_l|^2}\sum_{k\in\Gamma_l,|k|\le l}\sum_i\int_{\mathcal{H}^0(r)}\left|\int_{\mathbb{R}^3}\Big\{(1-\zeta_l(\xi))\,\tilde{u}_i(\xi)\;(k-\xi)\;\widetilde{\psi}_l(\xi-k)\right.\\
&\qquad\left.+(1-\zeta_l(\xi))\,\xi\;\tilde{u}_i(\xi)\;\;\widetilde{\psi}_l(\xi-k)\Big\}\;d\xi\right|^2\mu(du)\\
=&\frac{C}{|\mathbb{T}_l|^2}\sum_{k\in\Gamma_l,|k|\le l}\sum_i\int_{\mathcal{H}^0(r)}\left|\int_{\mathbb{R}^3}\Big\{\tilde{u}_i(\xi)\;\;(1-\zeta_l(\xi))\;\;\widetilde{\nabla\psi_l}(\xi-k)\right.\\
&\qquad\left.+\;\widetilde{\nabla u_i}(\xi)\;\;(1-\zeta_l(\xi))\;\;\widetilde{\psi}_l(\xi-k)\Big\}\;d\xi\right|^2\mu(du)\\
\le&\frac{C}{|\mathbb{T}_l|^2}\sum_{k\in\Gamma_l,|k|\le l}\sum_i\int_{\mathcal{H}^0(r)}\Big\{|<\tilde{u}_i,(1-\zeta_l)\,\widetilde{\nabla\psi_l}(.-k)>|^2\\
&\qquad+|<\widetilde{\nabla u_i},(1-\zeta_l)\,\widetilde{\psi}_l(.-k)>|^2\Big\}\,\mu(du).
\end{aligned}$$

Now use $\mathfrak{M}=\sum_i\mathfrak{M}_{ii}$ and $\mathfrak{N}=\sum_i\mathfrak{N}_{ii}$ to rewrite this as:

(66)

$$\begin{aligned}
&C\left\{\int_{\mathbb{R}^3}(1-\zeta_l(\xi))^2\sum_{k\in\Gamma_l,|k|\le l}\frac{|\widetilde{\nabla\psi_l}(\xi-k)|^2}{|\mathbb{T}_l|^2}\mathfrak{M}(d\xi)\right.\\
&\qquad\left.+\int_{\mathbb{R}^3}(1-\zeta_l(\xi))^2\sum_{k\in\Gamma_l,|k|\le l}\frac{|\widetilde{\psi}_l(\xi-k)|^2}{|\mathbb{T}_l|^2}\mathfrak{N}(d\xi)\right\}\\
\le&C\left\{\int_{\mathbb{R}^3}\sum_{k\in\Gamma_l,|k|\le l}\frac{|\widetilde{\nabla\psi_l}(\xi-k)|^2}{|\mathbb{T}_l|^2}\mathfrak{M}(d\xi)+\int_{\mathbb{R}^3}\sum_{k\in\Gamma_l,|k|\le l}\frac{|\widetilde{\psi}_l(\xi-k)|^2}{|\mathbb{T}_l|^2}\mathfrak{N}(d\xi)\right\}.
\end{aligned}$$

Observe next that there is l_0 such that for $l\ge l_0$, using Parseval,

$$\begin{aligned}
\frac{1}{|\mathbb{T}_l|^2}\sum_k|\widetilde{\nabla\psi_l}(\xi-k)|^2&=\frac{1}{|\mathbb{T}_l|}\int_{\mathbb{T}_l}|\nabla\psi_l|^2\;dx,\\
&\le\frac{C}{|\mathbb{T}_l|\;l^{2\kappa}}\int_{\mathbb{T}_l}1\;dx,\quad\text{by (2.12) in Appendix II of [VF]},\\
&=\frac{C}{l^{2\kappa}}\le 1.
\end{aligned}\tag{67}$$

With this and the original estimate (2.44) from [VF], p. 547, the right hand side of (66) is smaller than

$$C\left\{\int_{\mathbb{R}^3}\mathfrak{M}(d\xi)+\int_{\mathbb{R}^3}\mathfrak{N}(d\xi)\right\}. \tag{68}$$

Using (59), obtain

$$\int_{\mathcal{H}^0(r)}|\nabla u(x)|^2\ \mu_l(du)\le C\int_{\mathcal{H}^0(r)}\left(|u(x)|^2+|\nabla u(x)|^2\right)\ \mu(du). \tag{69}$$ □

Theorem 4.5. *Given μ homogeneous on $H^0(r)$, let μ_l be an l-approximation defined by (51)–(54). Then, up to subsequence, $W_2(\mu_l,\mu)\to 0$ as $l\to\infty$ and the correlation functions R^l_{ij} of μ_l's converge to the correlation functions R_{ij} of μ uniformly on compact subsets of $\mathbb{R}^3$.*

Proof. Lemma 4.4 shows that (29) holds. Hence, the μ_l's converge weakly to a homogeneous measure μ on $\mathcal{H}^0(r)$ by Theorem 3.1. At the same time, Theorem 4.1 gives μ as limit of the μ_l's in characteristic. It is standard that these two limits must be equal, cf. [GS], p.370.

It follows from (55) that (30) is also satisfied. Therefore, from Theorem 3.1 it follows that $\mu_l\to\mu$ in $W_2(\mathcal{H}^0(r))$. And from Theorem 3.3 finally follows that the correlation functions R^l_{ij} converge to the correlation functions R_{ij} uniformly on compact subsets of $\mathbb{R}^3$. □

5. Final remarks

5.1. **Homogeneous and isotropic solutions.** On $\mathbb{R}^3$, of interest in statistical hydrodynamics are homogeneous measures μ that are also isotropic, i.e. invariant also under rotations:

$$(R_\omega)_\#\mu=\mu, \tag{70}$$

with $R_\omega u(x)=u(\omega^{-1}x)$, for all $\omega\in O(3)$. The results of section 4 hold true for the analogue of the l-approximation of such measures by homogeneous and isotropic μ_l's, now supported on $\cup_\omega R_\omega\mathcal{M}_l$. This follows from the fact that the improved energy estimate of Lemma (4.4) holds. Its proof of remains the same word for word, after integrations with respect to $\mu(du)$ are replaced by integrations with respect to $\mu(du)d\omega$. The details are in [Ka].

The following two subsections contain remarks that will be expanded on future work.

5.2. **An alternative approximation in $W_2(\mathcal{H}^0(r))$.** It follows from (22) above and [AGS], Proposition 8.3.3, that a homogeneous μ with finite energy density and finite density of energy dissipation can be approximated, also in $W_2(\mathcal{H}^0(r))$, by measures μ_n, $n \in \mathbb{N}$, such that for each n: μ_n is supported on some n-dimensional subspace of $\mathcal{H}^0(r)$, is absolutely continuous with respect to the n-Lebesque measure, and satisfies

$$\int < \nabla\Phi(u), u_n >_{\mathcal{H}^0(r)} \mu_n(du) = 0, \tag{71}$$

for u_n smoothings of

$$\int_{\{\mathrm{pr}_n(v)=u\}} \mathrm{pr}_n(\nabla v \cdot h)\ \mu_u(dv), \tag{72}$$

for μ_u the disintegration of μ with respect to $(\mathrm{pr}_n)_{\#}\mu$.

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