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Classical Limit of the Matrix Elements on Quantized Lobachevskii Plane

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Abstract

It is proved that the matrix elements $\widehat{F}_{n,n+k}$ between harmonic oscillator eigenvectors of any smooth observable in the quantized Lobachevskii plane converge to the Fourier coefficients F_k of the corresponding classical observable $F(A,\phi)$ at the classical limit $n \to \infty, \hbar \to 0, n\hbar \to A$, k fixed, where A, ϕ are the oscillator action-angle variables. The Wigner functions are then defined and, as a consequence of the above result, their convergence to $\delta(A - A_0)e^{-ik\phi}$ at the classical limit is proved when computed on the harmonic oscillator eigenstates n and $n + k$.

1 Introduction

Let $\widehat{H}$ be the Schrödinger operator representing the quantization of a completely integrable classical system in d-degrees of freedom. Let $\widehat{F}$ be any quantum observable, and $\widehat{F}_{mn}$ its matrix elements among eigenstates of $\widehat{H}$. It is well known (see e.g. [LL], §48) that at the classical limit the matrix elements $\widehat{F}_{mn}$ are expected to converge to the Fourier coefficients F_{n-m} of F. Here F is the corresponding classical observable written in the action-angle variable of the Hamiltonian H.
This means that if

$$F(A,\phi) = \sum_{k=-\infty}^{+\infty} F_k(A) e^{ik\cdot\phi}$$

is the Fourier series expansion of the classical observable F in the action-angle variables A, ϕ then, as k is fixed,

$$\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} \langle \widehat{F} f_n, f_{n+k}\rangle = F_k(A). \tag{1.1}$$

In particular, 1.1 entails that the diagonal matrix elements of $\widehat{F}$ tend to the mean value of the classical observable F on the d-torus labeled by A, i.e.

$$\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} \langle \widehat{F} f_n, f_n\rangle = F_0(A) = \frac{1}{(2\pi)^d}\int_0^{2\pi} F(A,\phi)d\phi. \tag{1.2}$$

Formula (1.2) has been proved in [C].

It is known that either, in Schrödinger representation and in the Bargmann one as well, when the correspondence $F \to \widehat{F}$ is the Weyl quantization, 1.1 is equivalent to the assertion that the Wigner functions $W_{n,n+k}(A,\phi)$ tend to $\delta(A-A_0)e^{-ik\cdot\phi}$ in the sense of distributions, i.e.

$$\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} W_{n,n+k}(A,\phi) = \delta(A-A_0)e^{-ik\cdot\phi}. \tag{1.3}$$

Formula 1.3, for $k=0$, has been proved in great generality by stationary phase arguments in [By] and [BB]; however the complete mathematical details for the general assertion 1.1 have been so far obtained (see [R]) only when H is the harmonic oscillator, by proving directly 1.3 in the Bargmann representation.
The proof has been subsequently simplified in [DBR] using the anti-Wick quantization, always in the Bargmann representation.

In this paper we deal with the quantization on the Lobachevskii plane (see [Ber1]) that, according to the Poincaré model of hyperbolic geometry, is identified with the complex unit circle D. It represents the fundamental example of quantization out of the Heisenberg group representations. In this model the definition of the Wigner function is not a priori clear; here we find its expression through the analogy between unitary representations of the Heisenberg group on the Hilbert space of holomorphic functions on $\mathbb{C}$ introduced by Bargmann [Bar] on one side and the $SU(1,1)$ unitary representation on the Hilbert space of the holomorphic functions on the unit disc in $\mathbb{C}$ with the Poincaré metric on the other side.

1.1 holds in each one of the three possible quantization procedures: covariant, contravariant and Weyl. The result in Weyl case implies in particular 1.3.

2 Classical mechanics

The complex unit circle $D = \{z \in \mathbb{C}; |z| < 1\}$ is a homogeneous space for the group $SU(1,1)$ of motions of the Lobachevskii plane. $SU(1,1)$ acts on D as follows

$$z \in D \longrightarrow g \cdot z = \frac{az+b}{\bar{b}z+\bar{a}}$$

where

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU(1,1); \qquad a, b \in \mathbb{C}; \quad |a|^2 - |b|^2 = 1.$$

D admits a Kähler structure. The 2-form invariant under motions is

$$\Omega = i(1 - z\bar{z})^{-2} dz \wedge d\bar{z}. \tag{2.1}$$

The 2-form Ω defines a symplectic structure on D. In particular the Poisson bracket is

$$\{f, g\} = i(1 - z\bar{z})^2 \left(\frac{\partial f}{\partial \bar{z}} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}} \right).$$

If we set

$$u = \frac{\bar{z}}{1 - z\bar{z}},$$

then z and u form a pair of complex canonical variables: $\{u, z\} = i$.

The harmonic oscillator in D is described by the Hamiltonian

$$H(z,\bar{z}) = \frac{z\bar{z}}{1 - z\bar{z}} = zu. \tag{2.2}$$

The corresponding equations of motion are

$$\dot{z} = iz$$
$$\dot{\bar{z}} = -i\bar{z}$$

and the phase trajectories are circle centered at the origin.

The action variable is given by

$$A = \frac{1}{2\pi}\int_{|z|<|z_0|}\Omega = \frac{z_0\bar{z}_0}{1 - z_0\bar{z}_0} = H(z_0) = E \tag{2.3}$$

where E is the energy computed on the trajectory $|z| = |z_0|$, and

$$z\bar{z} = \frac{A}{1+A}. \tag{2.4}$$

The angle variable ϕ canonically conjugate to A is solution of the equation

$$\{A,\phi\} = i\left(\bar{z}\frac{\partial\phi}{\partial\bar{z}} - z\frac{\partial\phi}{\partial z}\right) = 1$$

and can be put equal to

$$\phi = \mathrm{Arg}z. \tag{2.5}$$

In these coordinates the hamiltonian takes the form

$$H(A,\phi) = A$$

and the equations of motion become

$$A = const.$$
$$\phi = t + \phi_0.$$

3 Quantization

The Hilbert space of quantum states for the Lobachevskii plane (see e.g. [BER1], [BER2]) is defined by

$$\mathcal{F}_\hbar = \{f;\ f \in L^2(D, d\alpha);\ f \text{ holomorfic in } D\}$$

where

$$d\alpha(z,\bar{z}) = (1 - z\bar{z})^{1/\hbar} d\mu(z,\bar{z}), \qquad d\mu(z,\bar{z}) = \left(\frac{1}{\hbar} - 1\right)\frac{\Omega}{2\pi} = \left(\frac{1}{\hbar} - 1\right)\frac{1}{\pi}\frac{dzd\bar{z}}{(1 - z\bar{z})^2}.$$

Here $1 < 1/\hbar \in \mathbb{N}$ (see [BER3]).

The scalar product in $\mathcal{F}_\hbar$ is given by

$$\langle f, g\rangle_{\mathcal{B}_\hbar} = \left(\frac{1}{\hbar} - 1\right)\frac{1}{\pi}\int_D f(z)\overline{g(z)}(1 - z\bar{z})^{1/\hbar - 2} dzd\bar{z}.$$

A complete orthonormal system in $\mathcal{F}_\hbar$ is defined by the sequence of functions

$$f_n(z) = \left(\frac{\Gamma(1/\hbar + n)}{n!\,\Gamma(1/\hbar)}\right)^{1/2} z^n = \left(\frac{(1/\hbar)_n}{n!}\right)^{1/2} z^n, \qquad n = 0, 1, \ldots \tag{3.1}$$

where $(a)_n := \Gamma(a+n)/\Gamma(a)$.

A standard computation by means of the orthonormal system proves that the family of elements

$$e_v(z) = (1 - z\bar{v})^{-1/\hbar} \tag{3.2}$$

called *coherent states*, represent a supercomplete system of vectors in $\mathcal{F}_\hbar$. The Parseval identity

$$\langle f, g\rangle = \int_D \langle f, e_v\rangle\langle e_v, g\rangle d\alpha(v, \bar{v})$$

and the self-reproducing property

$$\langle f, e_v\rangle = \int_D f(z)\overline{e_v(z)} d\alpha(z,\bar{z}) = f(v) \qquad \forall f \in \mathcal{F}_\hbar,\ \ \forall v \in D$$

hold as in the standard Bargmann space [BAR].

In 1974, F.A.Berezin (see [Ber1]) introduced the covariant and contravariant symbols as holomorphic functions of $(z, \bar{z})$ on D and their quantization as operators in $\mathcal{F}_\hbar$. Since the variables $(z, \bar{z})$ are not canonically conjugated, the Weyl symbol cannot be the symmetrization of the covariant and contravariant ones. The Weyl quantization is here defined by "reflection" operation ([Ber1], recalled in §3 below): this reduces to the standard Weyl quantization in the case $(\mathbb{R}^{2n}; \Omega = \sum dp_i \wedge dq_i)$.

After recalling the three types of symbol-operator correspondences, we will compute the classical limit 1.1 for all three cases.

1. Covariant symbol. Let $\widehat{F}$ be a bounded linear operator in $\mathcal{F}_\hbar$. Its *covariant* symbol, denoted $F(z, \bar{z})$ is defined (see [Ber1]) in terms of the operator $\widehat{F}$ by

$$F(z, \bar{z}) = \frac{\langle \widehat{F}\, e_z, e_z \rangle}{\langle e_z, e_z \rangle}. \tag{3.3}$$

where e_z; $z \in D$ is given by 3.2.

This definition shows that the covariant symbol is uniquely defined for any operator $\widehat{F}$. $F(z, \bar{z})$ admits an analytic continuation:

$$F(z, \bar{v}) = \frac{\langle \widehat{F}\, e_v, e_z \rangle}{\langle e_v, e_z \rangle} = (1 - z\bar{v})^{1/\hbar} \langle \widehat{F}\, e_v, e_z \rangle, \qquad (z, v) \in D \times D. \tag{3.4}$$

The action of the operator $\widehat{F}$ is given in terms of its symbol by the formula

$$(\widehat{F} f)(z) = \int_D f(v) F(z, \bar{v}) \langle e_v, e_z \rangle d\alpha(v, \bar{v})$$

$$= \int_D f(v) F(z, \bar{v}) \left(\frac{1 - v\bar{v}}{1 - z\bar{v}} \right)^{1/\hbar} d\mu(v, \bar{v}).$$

We denote $\widehat{F}_W$ the quantized observable whose covariant symbol is the classical observable F and we denote the correspondence

$$F \to \widehat{F}_W$$

covariant correspondence.

It is proved in [GR] that, in the covariant correspondence, the operator corresponding to the symbol

$$F(z, \bar{z}) = \sum_{m,l,p} c_{mlp} \frac{z^m \bar{z}^l}{(1 - z\bar{z})^p} \tag{3.5}$$

is

$$\widehat{F}_W = \sum_{m,l,p} z^m \left(\prod_{q=0}^{p-1} \frac{\hbar z \partial_z + 1 + q\hbar}{1 + q\hbar} \right) \left[(\hbar z \partial_z + 1)^{-1} \hbar \partial_z \right]^l . \tag{3.6}$$

In particular this implies the following (formal) quantization rules

$$\sum_{n,m} c_{nm} z^n \bar{z}^m \quad \longleftrightarrow \quad \sum_{n,m} c_{nm} z^n \left[(\hbar z \partial_z + 1)^{-1} \hbar \partial_z \right]^m$$

$$u = \frac{\bar{z}}{1 - z\bar{z}} \quad \longleftrightarrow \quad \widehat{u} = \hbar \partial_z .$$

2.2 entails that the quantized hamiltonian of the harmonic oscillator in this correspondence is

$$\widehat{H} = \hbar z \partial_z$$

Then the functions f_n, defined by 3.1, are eigenfunctions of the operator $\widehat{H}$ with eigenvalues $\hbar n$ which become the action A at the classical limit.

We prove now that for a classical observable F of the form 3.5, the matrix elements of the corresponding operator $\widehat{F}_W$, given by 3.6, tend, at the classical limit, to the Fourier coefficients of F written in the action-angle variables. By linearity it is sufficient to prove the following

Proposition 3.1 *Let*

$$F(z, \bar{z}) = \frac{z^m \bar{z}^l}{(1 - z\bar{z})^p}$$

and the corresponding operator $\widehat{F}_W$ given by 3.6. Then

$$\lim_{\substack{n \to \infty \\ \hbar \to 0 \\ n\hbar \to A}} \langle \widehat{F}_W f_n, f_{n+k} \rangle = \frac{1}{2\pi} \int_0^{2\pi} F(A, \phi) e^{-ik\phi} d\phi, \qquad k \ fixed \tag{3.7}$$

where, by a standard abuse of notation,

$$F(A, \phi) \equiv F \left(\sqrt{\frac{A}{A+1}} e^{i\phi}, \sqrt{\frac{A}{A+1}} e^{-i\phi} \right)$$

is the function $F(z, \bar{z})$ re-expressed in the action-angle variables (A, ϕ) through 2.3 and 2.5.

Remark. By 2.4, the r.h.s of 3.7 can be written as

$$\frac{1}{2\pi}\left(\frac{A}{A+1}\right)^{(m+l)/2}(A+1)^p\int_0^{2\pi}e^{i(m-l-k)\phi}d\phi. \tag{3.8}$$

Proof. Acting with the operator $\widehat{F}_W$, defined by 3.6, on the eigenfunctions f_n we easily obtain

$$\widehat{F}_W f_n=\left(\frac{\Gamma(n+1/\hbar)}{n!\Gamma(1/\hbar)}\right)^{1/2}\left(\prod_{q=0}^{p-1}\frac{\hbar(n-l)+1+q\hbar}{1+q\hbar}\right)\left(\prod_{q=0}^{l-1}\frac{\hbar(n-q)}{\hbar(n-q)+1}\right)z^{m+n-l} \tag{3.9}$$

because $\left[(\hbar z\partial_z+1)^{-1}\hbar\partial_z\right]z^n=\dfrac{\hbar n}{\hbar(n-1)+1}z^{n-1}$.

As k is fixed we have

$$\lim_{\substack{n\to\infty\\ \hbar\to 0\\ n\hbar\to A}}\prod_{q=0}^{p-1}\frac{\hbar(n-l)+1+q\hbar}{1+q\hbar}=(A+1)^p \qquad\text{and}\qquad \lim_{\substack{n\to\infty\\ \hbar\to 0\\ n\hbar\to A}}\prod_{q=0}^{l-1}\frac{\hbar(n-q)}{\hbar(n-q)+1}=\left(\frac{A}{A+1}\right)^l$$

After substitution in 3.9 we obtain

$$\lim_{\substack{n\to\infty\\ \hbar\to 0\\ n\hbar\to A}}\langle\widehat{F}_W f_n,f_{n+k}\rangle=\lim_{\substack{n\to\infty\\ \hbar\to 0\\ n\hbar\to A}}\left(\frac{\Gamma(1/\hbar+n)\Gamma(1/\hbar+n+k)}{n!\,\Gamma(1/\hbar)(n+k)!\,\Gamma(1/\hbar)}\right)^{1/2}A^l(A+1)^{p-l}\langle z^{m+n-l},z^{n+k}\rangle.$$

By Stirling's formula (see [MO]) we get, at the classical limit

$$\left(\frac{\Gamma(1/\hbar+n)\Gamma(1/\hbar+n+k)}{n!\,\Gamma(1/\hbar)(n+k)!\,\Gamma(1/\hbar)}\right)^{1/2}=\frac{1}{\sqrt{2\pi}}\hbar^{1/2}\frac{(A+1)^{n+1/\hbar+(k-1)/2}}{A^{n+(k+1)/2}}\Big(1+O(\hbar)\Big). \tag{3.10}$$

Hence

$$\lim_{\substack{n\to\infty\\ \hbar\to 0\\ n\hbar\to A}}\langle\widehat{F}_W f_n,f_{n+k}\rangle=\lim_{\substack{n\to\infty\\ \hbar\to 0\\ n\hbar\to A}}\frac{1}{\sqrt{2\pi}}\hbar^{1/2}\frac{(A+1)^{n+p-l+1/\hbar+(k-1)/2}}{A^{n-l+(k+1)/2}}\langle z^{m+n-l},z^{n+k}\rangle. \tag{3.11}$$

Evaluating the scalar product $\langle z^{m+n-l},z^{n+k}\rangle$ in polar coordinates we obtain

$$\langle z^{m+n-l},z^{n+k}\rangle=\left(\frac{1}{\hbar}-1\right)\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 t^{n+(m+k-l)/2}(1-t)^{1/\hbar-2}e^{i(m-l-k)\phi}dtd\phi.$$

Remembering (see [MO]) that

$$\int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \Re(x)>0 \ \Re(y)>0 \tag{3.12}$$

the last integral becomes

$$\langle z^{m+n-l}, z^{n+k}\rangle = \left(\frac{1}{\hbar}-1\right)\frac{1}{2\pi}\frac{\Gamma\Big(n+1+(m+k-l)/2\Big)\Gamma(1/\hbar-1)}{\Gamma(n+1/\hbar+(m+k-l)/2)}\int_0^{2\pi} e^{i(m-l-k)\phi}d\phi.$$

By Stirling's formula, $\forall a\in\mathbb{R}$, at the classical limit

$$\frac{\Gamma\Big(n+1+a\Big)\Gamma(1/\hbar-1)}{\Gamma\Big(n+1/\hbar+a\Big)} = \sqrt{2\pi}\hbar^{1/2}\frac{A^{n+a+1/2}}{(A+1)^{n+1/\hbar+a-1/2}}\Big(1+O(\hbar)\Big). \tag{3.13}$$

Hence

$$\lim_{\substack{n\to\infty\\ \hbar\to 0\\ n\hbar\to A}}\langle z^{m+n-l}, z^{n+k}\rangle = \lim_{\substack{n\to\infty\\ \hbar\to 0\\ n\hbar\to A}}\frac{1}{\sqrt{2\pi}}\frac{1}{\hbar^{1/2}}\frac{A^{n+(m-l+k+1)/2}}{(A+1)^{n+1/\hbar+(m-l+k-1)/2}}\int_0^{2\pi} e^{i(m-l-k)\phi}d\phi.$$

To conclude the proof we substitute the last expression in to 3.11. This yields 3.8 and hence 3.7. ■

2. *Contravariant symbol.* We recall (see [Ber1]) that a function $\overset{o}{F}(v,\bar v)$ such that

$$\widehat{F} = \int_D \overset{o}{F}(v,\bar v)\widehat{P}_v\langle e_v, e_v\rangle d\alpha(v,\bar v) \tag{3.14}$$

where $\widehat{P}_v$ is an orthogonal projector on e_v, i.e.

$$\widehat{P}_v f = \frac{\langle f, e_v\rangle e_v}{\langle e_v, e_v\rangle}$$

is called a *contravariant symbol* of the operator $\widehat{F}$. One has

$$(\widehat{F}f)(z) = \int_D \overset{o}{F}(v,\bar v)\langle f, e_v\rangle e_v(z)d\alpha(v,\bar v)$$

$$= \left(\frac{1}{\hbar}-1\right)\int_D \overset{o}{F}(v,\bar v)f(v)\left(\frac{1-v\bar v}{1-z\bar v}\right)^{1/\hbar}d\mu(v,\bar v).$$

Unlike the covariant symbol, the contravariant one is not always defined for every bounded operator, and when defined need not be unique (see [BER1]).

The relationship between the covariant and contravariant symbols $F(z,\bar{z})$ and $\overset{o}{F}(z,\bar{z})$ of the operator $\widehat{F}$ is expressed by the formula

$$F(z,\bar{z}) = (\sigma_\hbar \overset{o}{F})(z,\bar{z}) \tag{3.15}$$

where

$$(\sigma_\hbar \overset{o}{F})(z,\bar{z}) = \int_D \overset{o}{F}(v,\bar{v}) \frac{\langle e_z, e_v\rangle\langle e_v, e_z\rangle}{\langle e_z, e_z\rangle} d\alpha(v,\bar{v}) \tag{3.16}$$

$$= \int_D \overset{o}{F}(v,\bar{v}) \left[\frac{(1-z\bar{z})(1-v\bar{v})}{(1-z\bar{v})(1-v\bar{z})}\right]^{1/\hbar} d\mu(v,\bar{v}). \tag{3.17}$$

For differentiable functions f, the operator $\sigma_\hbar$ has the following asymptotic expansion (see [BER1] for details)

$$(\sigma_\hbar f)(z,\bar{z}) = f(z,\bar{z}) + \hbar\Delta f(z,\bar{z}) + o(\hbar) \tag{3.18}$$

where Δ is the Laplace-Beltrami operator on the Lobachevskii plane formally defined as

$$\Delta = (1-z\bar{z})^2 \frac{\partial^2}{\partial z \partial \bar{z}}. \tag{3.19}$$

Moreover $\sigma_\hbar$ can be written in terms of the Δ operator through the formula

$$\sigma_\hbar = \prod_{k=0}^{\infty} \left[1 - \hbar^2 \frac{\Delta}{(1+k\hbar)(1+(k-1)\hbar)}\right]^{-1}. \tag{3.20}$$

Given a classical observable $F(z,\bar{z})$, we denote with $\widehat{F}_{AW}$ the operator obtained by the contravariant correspondence. Then by 3.15 and 3.18 we deduce that if $F(v,\bar{v})$ is a classical observable, the covariant symbol of $\widehat{F}_{AW}$ tends to $F(v,\bar{v})$ as $\hbar \to 0$.

Proposition 3.2 *Let $F(z,\bar{z})$ be a smooth classical observable (say in C_0^∞) and $\widehat{F}_{AW}$ the corresponding operator obtained by the contravariant correspondence. Then*

$$\lim_{\substack{n\to\infty,\hbar\to 0 \\ n\hbar \to A}} \langle \widehat{F}_{AW} f_n, f_{n+k}\rangle = \frac{1}{2\pi}\int_0^{2\pi} F(A,\phi) e^{-ik\phi} d\phi$$

where by a standard abuse of notation

$$F(A,\phi) = F\left(\sqrt{\frac{A}{A+1}}e^{i\phi}, \sqrt{\frac{A}{A+1}}e^{-i\phi}\right).$$

Proof. 3.14 yields

$$\langle \widehat{F}_{AW} f_n, f_{n+k}\rangle = \langle \int_D F(v,\bar{v})\langle f_n, e_v\rangle e_v(z) d\alpha(v,\bar{v}), f_{n+k}\rangle$$

$$= \int_D F(v,\bar{v})\langle f_n, e_v\rangle\langle e_v, f_{n+k}\rangle d\alpha(v,\bar{v})$$

$$= \left(\frac{1}{\hbar}-1\right)\frac{1}{\pi}\int_D F(v,\bar{v})\left(\frac{(1/\hbar)_n(1/\hbar)_{n+k}}{n!(n+k)!}\right)^{1/2} v^n\bar{v}^{n+k}(1-v\bar{v})^{1/\hbar-2} dv d\bar{v}.$$

Transforming into the action-angle variables, i.e. setting $v = \sqrt{\frac{A}{A+1}}\rho e^{i\phi}$ we obtain

$$\langle \widehat{F}_{AW} f_n, f_{n+k}\rangle = \frac{1}{2\pi}\int_0^{\frac{A+1}{A}}\int_0^{2\pi} C(n,\hbar,\rho)F(\rho A,\phi)e^{-ik\phi} d\rho d\phi \tag{3.21}$$

where

$$C(n,\hbar,\rho) = \left(\frac{1}{\hbar}-1\right)\left(\frac{(1/\hbar)_n(1/\hbar)_{n+k}}{n!(n+k)!}\right)^{1/2}\left(1-\frac{A}{A+1}\rho\right)^{1/\hbar-2}\left(\frac{A}{A+1}\right)^{n+1+k/2}\rho^{n+k/2}$$

and by the standard abuse of notation

$$F(\rho A,\phi) = F\left(\sqrt{\frac{A}{A+1}}e^{i\phi}, \sqrt{\frac{A}{A+1}}e^{-i\phi}\right).$$

Since $F \in C_0^\infty(D)$, to conclude the proof, by 3.21 it is enough to prove that, at the classical limit ($n \to \infty$, $\hbar \to 0$, $n\hbar \to A$) the function $C(n,\hbar,\rho)$ tends, in the sense of distributions, to the Dirac measure supported on 1; that is

$$\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} C(n,\hbar,\rho) = \delta(\rho-1). \tag{3.22}$$

We must therefore prove:

1) $\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} C(n,\hbar,\rho)=\infty \quad \text{for} \quad \rho=1.$

2) $\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} C(n,\hbar,\rho)=0 \quad \text{for} \quad \rho\neq 1.$

3) $\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} \int_0^{\frac{A+1}{A}} C(n,\hbar,\rho)d\rho=1.$

By 3.10 it follows that

$$\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} C(n,\hbar,\rho)=\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} \frac{1}{\sqrt{2\pi}}\frac{A^{1/2}(A+1)^{1/2}}{(A+1-A\rho)^2}\rho^{k/2}\hbar^{-1/2}\Big[(A+1-A\rho)\rho^A\Big]^{1/\hbar}.$$

If $\rho=1$ then $\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} C(n,\hbar,\rho)=\lim_{\hbar\to 0}\hbar^{1/2}=\infty$ and this proves the first assertion.

Since the function $(A+1-A\rho)\rho^A$ is less than 1 as $\rho\neq 1$ and it is equal to 1 as $\rho=1$, then for $\rho\neq 1$

$$\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} C(n,\hbar,\rho)=\lim_{\hbar\to 0}\left[(A+1-A\rho)\rho^A\right]^{1/\hbar}=0.$$

Concerning the third assertion we have, by 3.10 and performing the change of variable $t=\dfrac{A}{A+1}\rho$,

$$\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} \int_0^{\frac{A+1}{A}} C(n,\hbar,\rho)d\rho=\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} \frac{\hbar^{-1/2}}{\sqrt{2\pi}}A^{-1}\left(\frac{A}{A+1}\right)^{1/2-n-k/2}\int_0^1(1-t)^{1/\hbar-2}t^{n+k/2}dt.$$

Then by the integral representation 3.12:

$$\begin{aligned}&\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} \int_0^{\frac{A+1}{A}} C(n,\hbar,\rho)d\rho\\ &=\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} \frac{\hbar^{-1/2}}{\sqrt{2\pi}}A^{-1}\left(\frac{A}{A+1}\right)^{1/2-n-k/2}\frac{\Gamma(1/\hbar-1)\Gamma(n+k/2+1)}{\Gamma(n+1/\hbar+k/2)}.\end{aligned}$$

The third assertion follows now by the general formula 3.13 and this concludes the proof. ∎

3. *Quantization by means of reflection.* If a group of motions G acts on a phase space $\mathcal{M}$ then the transformation in the algebra of the classical observables $C^\infty(\mathcal{M})$ (a Lie algebra with respect to the Poisson bracket)

$$(\tau_g F)(z, \bar{z}) = F(g \cdot z, \overline{g \cdot z}) \qquad g \in G \tag{3.23}$$

is an automorphism of the algebra. As is known, all automorphisms of the algebra of bounded operators in a Hilbert space are internal. Therefore, the bounded operator which generates the automorphism 3.23 exists in $\mathcal{F}_\hbar$,

$$\frac{\langle \widehat{\mathcal{U}}_g^{-1} \widehat{A}\, \widehat{\mathcal{U}}_g e_z, e_z \rangle}{\langle e_z, e_z \rangle} = \frac{\langle \widehat{A} e_{g\cdot z}, e_{g\cdot z}\rangle}{\langle e_{g\cdot z}, e_{g\cdot z}\rangle} = A(g \cdot z, \overline{g \cdot z}) = (\tau_g A)(z, \bar{z}).$$

The operator $\widehat{\mathcal{U}}_g$ is defined up to an arbitrary complex multiplier. If the indefinite multiplier equals 1 in modulus, $g \to \widehat{\mathcal{U}}_g$ is a unitary, irreducible representation of G. In our case the operator $\widehat{\mathcal{U}}_g$ has the following explicit form

$$(\widehat{\mathcal{U}}_g f)(z) = \phi(-\bar{b}z + a)^{-1/\hbar} f\left(\frac{\bar{a}z - b}{-\bar{b}z + a}\right), \qquad |\phi| = 1.$$

If we denote $\mathcal{U}_g(z, \bar{z})$ the covariant symbol of the operator $\widehat{\mathcal{U}}_g$ we obtain

$$\mathcal{U}_g(z, \bar{z}) = \frac{\langle \widehat{\mathcal{U}}_g e_z, e_z\rangle}{\langle e_z, e_z\rangle} = \phi\left(\frac{1 - z\bar{z}}{a - \bar{a}z\bar{z} + b\bar{z} - \bar{b}z}\right)^{1/\hbar}. \tag{3.24}$$

Consider in D the reflection at the point v. The generating element $g \in SU(1,1)$ is

$$g(v, \bar{v}) = \frac{1}{(1 - v\bar{v})}\begin{pmatrix} i(1 + v\bar{v}) & -2iv \\ 2i\bar{v} & -i(1 + v\bar{v}) \end{pmatrix}.$$

The transformation

$$\tau_v(z) = g(v, \bar{v}) \cdot z = \frac{-(1 + v\bar{v})z + 2v}{-2\bar{v}z + 1 + v\bar{v}}$$

is indeed involutive and admits v as fixed point.

The operator $\widehat{\mathcal{U}}_g$ for $g = g(v, \bar{v})$ will be denoted $\mathcal{U}_{v,\bar{v}}$. According to the general formula 3.24 the covariant symbol of the operator $\mathcal{U}_{v,\bar{v}}$, up to the multiplier ϕ, $|\phi| = 1$, has the form

$$\mathcal{U}_\hbar(v, \bar{v}|z, \bar{z}) = \left[\frac{(1 - z\bar{z})(1 - v\bar{v})}{(1 - z\bar{v})(1 - v\bar{z})}\right]^{1/\hbar} \frac{1}{\left[1 + \dfrac{z - v}{1 - \bar{z}v}\dfrac{\bar{z} - \bar{v}}{1 - z\bar{v}}\right]^{1/\hbar}}. \tag{3.25}$$

We fix the multiplier so that the covariant symbol of the operator $\widehat{\mathcal{U}}_{v,\bar{v}}$ has exactly the form 3.25, i.e.

$$(\widehat{\mathcal{U}}_{v,\bar{v}}f)(z) = \left(\frac{1+v\bar{v}-2\bar{v}z}{1-v\bar{v}}\right)^{-1/\hbar} f\left(\frac{-(1+v\bar{v})z+2v}{2\bar{v}z+(1+v\bar{v})}\right).$$

The function

$$\mathcal{F}(z,\bar{z}) = 2Sp(\widehat{F}\widehat{U}_{z,\bar{z}}) \tag{3.26}$$

is called the *covariant Weyl symbol* of the operator $\widehat{F}$ and the function $\overset{o}{\mathcal{F}}$ such that

$$\widehat{F} = 2\int \overset{o}{\mathcal{F}}(v,\bar{v})\widehat{\mathcal{U}}_{v,\bar{v}}d\mu(v,\bar{v}) \tag{3.27}$$

is called the *contravariant Weyl symbol* of the operator.

Recall that here $\mathcal{F} \neq \overset{o}{\mathcal{F}}$ because the Lobachevskii plane is a realization of a surface of non zero (negative and constant) curvature; in the flat case, the standard Bargmann space, $\mathcal{F} = \overset{o}{\mathcal{F}}$.

The connection between the contravariant symbol, $\overset{o}{F}$ and the covariant Weyl symbol, $\mathcal{F}$ and the connection between the contravariant Weyl symbol, $\overset{o}{\mathcal{F}}$ and the covariant symbol, F are given (see [Ber1]) by the formulas

$$\mathcal{F}(z,\bar{z}) = (T_\hbar \overset{o}{F})(z,\bar{z}) \tag{3.28}$$

$$F(z,\bar{z}) = (T_\hbar \overset{o}{\mathcal{F}})(z,\bar{z}) \tag{3.29}$$

where

$$(T_\hbar f)(z,\bar{z}) = 2\int_D f(v,\bar{v})\mathcal{U}_\hbar(v,\bar{v}|z,\bar{z})d\mu(v,\bar{v}). \tag{3.30}$$

The $T_\hbar$ operator can be explicitly written in the form

$$(T_\hbar f)(z,\bar{z}) = \frac{2}{\pi}\left(\frac{1}{\hbar}-1\right)\int_D f(v,\bar{v})\left[\frac{(1-z\bar{z})(1-v\bar{v})}{(1-z\bar{v})(1-v\bar{z})}\right]^{1/\hbar} \frac{1}{\left[1+\frac{z-v}{1-\bar{z}v}\frac{\bar{z}-\bar{v}}{1-z\bar{v}}\right]^{1/\hbar}}\frac{dvd\bar{v}}{(1-v\bar{v})^2}.$$

$T_\hbar$ maps the constant function equal to one in to itself and, when acting on smooth functions on D, has the following asymptotic expansion as $\hbar \to 0$

$$(T_\hbar f)(z,\bar{z}) = f(z,\bar{z}) + \frac{\hbar}{2}\Delta f(z,\bar{z}) + o(\hbar), \tag{3.31}$$

where Δ is the Laplace-Beltrami operator. Since $T_\hbar$ commutes with all transformations $f(z,\bar{z}) \to f(g\cdot z, \overline{g\cdot z})$ where $g\cdot z$ is a motion on the Lobachevskii plane then it can be written through the Laplace-Beltrami operator Δ (see [BER1]) by the formula

$$T_\hbar = \prod_{k=0}^{\infty}\left[1-\hbar^2\frac{\Delta}{(1+2k\hbar)(1+(2k-1)\hbar)}\right]^{-1}. \tag{3.32}$$

Comparing 3.20 with 3.32 we can deduce that

$$\sigma_\hbar = T_\hbar T'_\hbar \tag{3.33}$$

where

$$T'_\hbar = \prod_{k=0}^{\infty}\left[1-\hbar^2\frac{\Delta}{(1+(2k+1)\hbar)(1+(2k)\hbar)}\right]^{-1}.$$

The covariant Weyl symbol $\mathcal{F}$ and the contravariant Weyl symbols $\overset{o}{\mathcal{F}}$ are related by the formula

$$\overset{o}{\mathcal{F}} = T'_\hbar \overset{o}{F} = T'_\hbar (T_\hbar)^{-1}\mathcal{F}. \tag{3.34}$$

The *Wigner function* for $F, G \in \mathcal{F}_\hbar$ is defined as

$$W(F,G)(z,\bar{z}) = 2\langle \widehat{\mathcal{U}}_{v,\bar{v}}F, G\rangle. \tag{3.35}$$

The Wigner function on the harmonic oscillator eigenfunctions can be used through 3.27 to express the matrix elements of an operator $\widehat{F}$ through its contravariant Weyl symbol, i.e.

$$\langle \widehat{F}f_n, f_m\rangle = \int_D \overset{o}{\mathcal{F}}(v,\bar{v})W(f_n,f_m)(v,\bar{v})d\mu(v,\bar{v}). \tag{3.36}$$

Formula 3.36 is the analog of the representation of matrix elements of the observables through the Wigner functions valid in the canonical quantization case. The only difference is that here the properly 3.35, which can be proved in the canonical case, is taken as definition.

Proposition 3.3 *The Wigner function admits the representation:*

$$W(f_n,f_m)(v,\bar{v}) = 2\left(\frac{\Gamma(n+1/\hbar)m!}{\Gamma(m+1/\hbar)n!}\right)^{1/2}\frac{(-1)^m}{(n-m)!}\left(\frac{1-v\bar{v}}{1+v\bar{v}}\right)^{1/\hbar}\left(\frac{1+v\bar{v}}{2\bar{v}}\right)^{n-m} \tag{3.37}$$

$$\times {}_2F_1\left(-n, \frac{1}{\hbar}+m; m-n+1; \frac{4v\bar{v}}{(1+v\bar{v})^2}\right)$$

if $m > n$ *and*

$$W(f_n, f_m)(v,\bar v) = 2\left(\frac{\Gamma(m+1/\hbar)n!}{\Gamma(n+1/\hbar)m!}\right)^{1/2} \frac{(-1)^n}{(m-n)!}\left(\frac{1-v\bar v}{1+v\bar v}\right)^{1/\hbar}\left(\frac{1+v\bar v}{2v}\right)^{m-n} \tag{3.38}$$

$$\times {}_2F_1\left(-m, \frac{1}{\hbar}+n; n-m+1; \frac{4v\bar v}{(1+v\bar v)^2}\right)$$

if $m < n$*, where* ${}_2F_1$ *is the hypergeometric function (see [MO]).*

Proof. We have

$$W(f_n, f_m)(v,\bar v) = 2\langle \widehat{\mathcal{U}}_{v,\bar v} f_n, f_m\rangle = \frac{2}{\pi}\left(\frac{1}{\hbar}-1\right)\left(\frac{(1/\hbar)_n}{n!}\frac{(1/\hbar)_m}{m!}\right)^{1/2}$$

$$\times \int_D \left(\frac{1+v\bar v - 2\bar v z}{1-v\bar v}\right)^{-1/\hbar}\left(\frac{2v-(1+v\bar v)z}{1+v\bar v+2\bar v z}\right)^n \bar z^m (1-z\bar z)^{1/\hbar-2} dz d\bar z$$

where in general $(a)_n = \Gamma(a+n)/\Gamma(a)$.

Performing the integral in polar coordinates we obtain

$$W(f_n, f_m)(v,\bar v) = \frac{2}{\pi}\left(\frac{1}{\hbar}-1\right)\left(\frac{(1/\hbar)_n}{n!}\frac{(1/\hbar)_m}{m!}\right)^{1/2}(1-v\bar v)^{1/\hbar} I, \tag{3.39}$$

where

$$I = \int_0^{2\pi}\int_0^1 \left(1+v\bar v - 2\bar v\rho e^{i\phi}\right)^{-1/\hbar-n}\left(2v-(1+v\bar v)\rho e^{i\phi}\right)^n \rho^{m+1} e^{-im\phi}(1-\rho^2)^{1/\hbar-2} d\rho d\phi.$$

Since

$$\left(1+v\bar v - 2\bar v\rho e^{i\phi}\right)^{-1/\hbar-n} = (1+v\bar v)^{1/\hbar-n}\sum_{k=0}^{\infty}\frac{\Gamma(-1/\hbar-n+1)}{k!\Gamma(-1/\hbar-n-k+1)}\left(-\frac{2\bar v}{1+v\bar v}\rho e^{i\phi}\right)^k$$

and

$$\left(2v-(1+v\bar v)\rho e^{i\phi}\right)^n = \sum_{l=0}^{n}\begin{pmatrix} n \\ l \end{pmatrix}(-1)^l(1+v\bar v)^l \rho^l e^{il\phi}(2v)^{n-l}$$

we have

$$I = \int_0^{2\pi}\int_0^1 \sum_{k=0}^{\infty}\sum_{l=0}^{n}\frac{\Gamma(-1/\hbar-n+1)}{k!\Gamma(-1/\hbar-n-k+1)}\begin{pmatrix} n \\ l \end{pmatrix}(-1)^{l+k}(1+v\bar v)^{-1/\hbar-n-k+l}(2\bar v)^k(2v)^{n-l}$$

$$\times \rho^{m+k+l+1}(1-\rho^2)^{1/\hbar-2} e^{i(l+k-m)\phi} d\rho d\phi.$$

The integral is non zero only for $l = m - k$ and $k \in [\max\{0, m-n\}, m]$. Hence

$$I = 2\pi \sum_{k=\max\{0,m-n\}}^{m} \frac{\Gamma(-1/\hbar - n + 1)}{k!\,\Gamma(-1/\hbar - n - k + 1)} \left(\begin{array}{c} n \\ m-k \end{array} \right) (-1)^m (1+v\bar{v})^{-1/\hbar - n - 2k + m}$$

$$\times (2\bar{v})^k (2v)^{n-m+k} \int_0^1 \rho^{2m}(1-\rho^2)^{1/\hbar-2} d\rho$$

$$= \pi \sum_{k=\max\{0,m-n\}}^{m} \frac{m!\,\Gamma(1/\hbar - 1)\Gamma(-1/\hbar - n + 1)}{k!\,\Gamma(m + 1/\hbar)\Gamma(-1/\hbar - n - k + 1)} \left(\begin{array}{c} n \\ m-k \end{array} \right) (-1)^m \tag{3.40}$$

$$\times (1+v\bar{v})^{-1/\hbar - n - 2k + m} (2\bar{v})^k (2v)^{n-m+k}.$$

If $m > n$ with the substitution $k \to k + m - n$ we obtain

$$I = \pi (-1)^m (1+v\bar{v})^{-1/\hbar + n - m} (2\bar{v})^{m-n} \frac{m!\,\Gamma(1/\hbar - 1)}{\Gamma(m + 1/\hbar)}$$

$$\times \sum_{k=0}^{n} \frac{\Gamma(-1/\hbar - n + 1)\Gamma(n+1)}{k!\Gamma(-1/\hbar - m - k + 1)\Gamma(k + m - n + 1)\Gamma(n - k + 1)} \left(\frac{4v\bar{v}}{(1+v\bar{v})^2} \right)^k .$$

Since from Gamma function properties (see [MO])

$$\Gamma(-1/\hbar - n + 1) = (-1)^n \frac{\Gamma(-1/\hbar + 1)\Gamma(-1/\hbar)}{\Gamma(1/\hbar + n)},$$

$$\Gamma(-1/\hbar - m - k + 1) = (-1)^{m+k} \frac{\Gamma(-1/\hbar + 1)\Gamma(-1/\hbar)}{\Gamma(1/\hbar + m + k)}$$

and

$$\frac{\Gamma(n+1)}{\Gamma(n-k+1)} = (-1)^k \frac{\Gamma(-n+k)}{\Gamma(-n)}.$$

substituting I in 3.39 and remembering the definition of the Hypergeometric function we obtain 3.37.

3.38 follows from 3.40 by analogous computation. ∎

Remark. In the particular case $n = m$ we have

$$W(f_n, f_n)(v, \bar{v}) = 2(-1)^n \left(\frac{1 - v\bar{v}}{1 + v\bar{v}} \right)^{1/\hbar} \sum_{k=0}^{n} \frac{(-n)_k (1/\hbar + n)_k}{k!k!} \frac{(4v\bar{v})^k}{(1+v\bar{v})^{2k}}$$

$$= 2(-1)^n \left(\frac{1 - v\bar{v}}{1 + v\bar{v}}\right)^{1/\hbar} {}_2F_1\left(-n, \frac{1}{\hbar} + n; 1; \frac{4v\bar{v}}{(1 + v\bar{v})^2}\right).$$

Proposition 3.4 *If $F \in C_0^\infty(D)$ then*

$$\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} \int_D W(f_n, f_{n+k})(v,\bar{v})F(v,\bar{v})d\mu(v,\bar{v}) = \frac{1}{2\pi}\int_0^{2\pi} F(A,\phi)e^{-ik\phi}d\phi.$$

Proof. Let $F \in C_0^\infty(D)$ and let $\widehat{F}_{WY}$ be the corresponding operator obtained through contravariant Weyl quantization given by 3.27. Let $\overset{o}{F}_\hbar$ be the contravariant symbol of the operator $\widehat{F}_{WY}$. It follows by 3.34

$$\overset{o}{F}_\hbar = (T'_\hbar)^{-1}\overset{o}{\mathcal{F}} \overset{\hbar\to 0}{\longrightarrow} \overset{o}{\mathcal{F}}$$

and by 3.36

$$\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} \int_D W(f_n, f_{n+k})(v,\bar{v})F(v,\bar{v})d\mu(v,\bar{v}) = \lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} \langle \widehat{F}_{AW} f_n, f_{n+k}\rangle$$

then by Proposition 3.2

$$\lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} \int_D W(f_n, f_{n+k})(v,\bar{v})F(v,\bar{v})d\mu(v,\bar{v}) = \lim_{\substack{n\to\infty,\hbar\to 0\\ n\hbar\to A}} \frac{1}{2\pi}\int_0^{2\pi} \overset{o}{F}_\hbar(A,\phi)e^{-ik\phi}d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \overset{o}{\mathcal{F}}(A,\phi)e^{-ik\phi}d\phi.$$

This concludes the proof. ∎

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