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INVARIANT MANIFOLDS ASSOCIATED TO INVARIANT SUBSPACES WITHOUT INVARIANT COMPLEMENTS: A GRAPH TRANSFORM APPROACH

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Abstract. We use the graph transform method to prove existence of invariant manifolds near fixed points of maps tangent to invariant subspaces of the linearization.

In contrast to the best known of such theorems, we do not assume that the corresponding space for the linear map is a spectral subspace. Indeed, we allow that the spaces invariant under the linearization do not have an invariant complement.

We also do not need that the spectrum of the operator restricted to the spaces satisfies the usual dominance conditions.

We prove some uniqueness theorems and show how this can be used to prove results for flows.

More general theorems have been proved in [CFdlL03a] by another method.

1. Introduction

If X is a Banach space and $F : X \to X$ $F(0) = 0$ is a local diffeomorphism there are many theorems in the literature establishing the existence of manifolds invariant under F which are tangent to invariant spaces of the linearization of F at 0.

These classical theorems usually assume that there is a decomposition

$$X = E_1 \oplus E_2 \tag{1}$$

which is invariant under $DF(0)$ (often the spaces $E_{1,2}$ are spectral subspaces for $DF(0)$). That is:

$$DF(0)E_1 = E_1, \quad DF(0)E_2 = E_2, \tag{2}$$

and such that they satisfy the domination condition:

$$||DF(0)|_{E_1}|| \cdot ||DF^{-1}(0)|_{E_2}|| < 1 \ . \tag{3}$$

The conclusions of the classical invariant manifold theorems (See [Rue89] for a comparison of different invariant manifold theorems) are that one can find a manifold $\mathcal{W}$ invariant under F, tangent to E_1 at the origin, so that one can think of $\mathcal{W}$ as a non-linear analogue of E_1.

The goal of this paper is to weaken somewhat the conditions (2) and (3) in the stable and pseudo-stable manifold theorems. The main result, Theorem 3.1 only needs as hypothesis weaker versions of (2) and (3). The Theorem 3.1 contains results for stable and pseudo-stable manifolds.

More explicitly, we will assume that the space E_1 is invariant under the map, but we will not assume that the decomposition (1) is invariant under $DF(0)$. (in particular, we will not assume that the decomposition corresponds to spectral subspaces). Also, we will not assume that domination (3) happens. For most of the results on stable manifolds presented here, it will suffice to assume that

$$||A_2^{-1}||||A_1||^2 < 1. \tag{4}$$

(where $A_{1,2}$ are the restriction of A to the spaces $E_{1,2}$ respectively, see (9) for a more precise definition).

Invariant spaces for a linear without an invariant complement occur naturally as the spaces associated to the eigenvalues of a nontrivial Jordan block. For a Jordan block , even if the space generated by the eigenvalue – or the generalized eigenvalues of smallest index – is invariant, there is no invariant complementary space. See Example 3.7.

Some situations where invariant subspaces without invariant complements appear in a natural way are: skew product systems, bifurcations of systems with nilpotent part and – an application that has motivated us significantly – the flows of resonant systems.

As a motivation to weakening the assumption (3), we note again that the situation of a non-trivial Jordan block happens in resonant systems. We may want to associate invariant manifolds to the spaces generated by the eigenvalues. We also note that the convergence to the origin in a linear contraction is dominated by the eigenvalues closest to the unit circle, not the eigenvalues closest to the origin (In the case of a non-trivial Jordan block, the convergence to the equilibrium is also tangent the space corresponding to the eigenvalue).

Hence, in the study of asymptotic convergence of non-linear systems, it is interesting to give meaning to manifolds associated to the eigenvalues closest to the unit circle – the so called *slow manifolds*. In [dlL97] and [CFdlL03c] one can find a more comprehensive discussion of slow manifolds and their role in applications.

The goal of this paper is to present proofs of results on existence of invariant manifolds of the above results that are based in the graph transform method. The results we present will cover the usual strong stable manifold theorem and the pseudo-stable ones. The proofs presented here are based in rather well known variants of the graph transform approach.

We will present the results in the generality of Banach spaces, since assuming finite dimensional spaces does not simplify the proofs. One motivation for our working in Banach spaces is that, in this way, the results lift immediately to invariant foliation theorems using a well known device [HP70]. (As it turns out, in some of the results proved here, when doing the lifting in [HP70], one does not obtain foliations as was observed in [JPdlL95].

The paper [CFdlL03a] contains results on existence of invariant manifolds associated to very general invariant spaces, which, in particular are not required to have a complement. The paper [CFdlL03b] studies the dependence on parameters of these manifolds. The paper [CFdlL03c] is concerned with simple proofs, applications and examples.

The results presented here (except for the results on pseudo-stable manifolds and the technical use of fractional regularities) are less general than those in [CFdlL03a] and the method of [CFdlL03a] has several advantages – see the discussion in those papers –. Nevertheless, we hope that the present note could be useful for people more familiar with the graph transform method than with with parameterization method of [CFdlL03a].

One tool that we will use is *"conical norms"* or weighted norms which also plays a role in [CFdlL03a]. As we will see this is the key technical tool that allows to weaken (3). See Remark 4.5 for a discussion of alternatives and the reasons why one can improve (3) with them.

We postpone the statement of the main results till Section 3 so that we can introduce some notation. The proofs of the statements will follow in subsequent sections.

2. Notation

We consider the Banach spaces $C^{k+\alpha}$, $k \in \mathbb{N}$, $\alpha \in [0, \mathrm{Lip}]$ endowed with the usual uniform norms as well as the spaces C^∞ with its usual Frechet topology and C^ω endowed with a Banach topology based on taking sups of a complex extension. When the size of the extension is important, we will denote it as a subindex, for example inC^ω_ρ.

That is, we denote, when $k \in \mathbb{N}$,

$$||\Phi||_{C^k(U,X)} = \max_{i=0,\ldots,k} (\sup_{x\in U} |D^i\Phi|).$$

When $r = k + \alpha$, $k \in \mathbb{N}$, $\alpha \in (0, \mathrm{Lip}]$

$$||\Phi||_{C^r(U,X)} = \max_{i=0,\ldots,k} (||\Phi||_{C^k(U,X)}, H_\alpha(D^k\Phi))$$

where H_α is the seminorm

$$H_\alpha(\Phi) = \sup_{x,\tilde{x}\in U, x\neq\tilde{x}} |\Phi(x) - \Phi(y)| \cdot |x - y|^{-\alpha}. \tag{5}$$

We note that, when $\alpha = 0$, for functions normalized to $\Phi(0) = 0$ we have

$$||\Phi||_{C^0} \leq H_0(\Phi) \leq 2||\Phi||_{C^0}$$

We adopt the convention that $\mathrm{Lip} > \beta$ for any $\beta \in [0, 1)$, nevertheless, $k + \mathrm{Lip} < k + 1$. In the arithmetic expressions such as (5),(7), the symbol Lip takes the value 1.

Of course, the reader may choose to ignore the borderline cases where Lip enters. Indeed, we note that the results and their proofs are significantly easier for $r \geq 2$. Hence, in a first reading it could be worth to concentrate in the case $r > 2, r \notin \mathbb{N}$.

Given a regularity index s as above, we will denote

$$\tilde{s} = \begin{cases} s; & s \notin \mathbb{N} \\ s - 1 + \mathrm{Lip}; & s \in \mathbb{N} \end{cases} \tag{6}$$

(In the notation above, we note that when $s = \infty, \omega$, then $\tilde{s} = \infty, \omega$ respectively.

Two useful and elementary inequalities are (since they are so well known, we omit a precise formulation including assumptions on domains and the existence of products)

$$(7)\qquad \begin{aligned} H_\alpha(\Psi\cdot\Phi) &\le H_\alpha(\Psi)\|\Phi\|_{C^0} + \|\Psi\|_{C^0} H_\alpha(\Phi) \\ H_\alpha(\Psi\circ\Phi) &\le H_\alpha(\Psi)(\mathrm{Lip}(\Phi))^\alpha. \end{aligned}$$

We recall that a cutoff function is a function Ψ such that it is identically equal to 1 in the ball of radius 1 and zero outside a ball of radius 2.

In finite dimensional spaces, the existence of such a cut-off function is obvious. Similarly, Hilbert spaces – or Banach spaces whose norms which are smooth functions out of the origin – do have cut-off functions. Nevertheless, there are infinite dimensional Banach spaces for which no such smooth cut-off exists. For example, the usual C^0 space of a closed interval does not have smooth cut-off functions (see [DGZ93]).

3. Statement of results

The main result of this paper is the following Theorem 3.1. We note that Theorem 3.1 incorporates results about stable and pseudostable cases. The hypothesis and conclusions on regularity etc. are somewhat different in both cases since the manifolds in the stable cases are – roughly – as regular as the map, but in the pseudo-stable cases the regularity is also limited by ratios of norms.

Theorem 3.1. *Let X be a Banach space. Let $F : X \to X$ be C^r, $r \in \mathbb{N} + [0, \mathrm{Lip}] \cup \{\infty, \omega\}$, $r \ge 1$ be such that $F(0) = 0$. Denote $DF(0) = A$.*

Assume that

A) *There exists a decomposition $X = X_1 \oplus X_2$ into closed subspaces such that the space X_1 is invariant under A. That is,*

$$(8)\qquad A(X_1) \subset X_1\ .$$

Note that we do not assume that X_2 is invariant under A.

Denote by Π_1, Π_2 the projections over X_1, X_2. Denote also

$$(9)\qquad A_1 = \Pi_1 A \Pi_1, \quad A_2 = \Pi_2 A \Pi_2.$$

Assume furthermore that we are in one of the following two cases:

Stable case

B.1.1

$$\|A_1\| < 1.$$

B.1.2 *Let $s = \min(2, r)$,*

Assume that we have the following weak dominance condition

$$(10)\qquad \|A_2^{-1}\|\|A_1\|^s < 1\ .$$

In particular, if $r \geq 2$ we just assume

$$\|A_2^{-1}\| \cdot \|A_1\|^2 < 1. \tag{11}$$

Pseudo-stable case

B.2.1

$$\|A_2^{-1}\| < 1.$$

B.2.1 *The space X_1 admits smooth cut-off functions.*

B.2.2 *For some real number $1 \leq s$, we have*

$$\|A_2^{-1}\|\|A_1\|^s < 1 \ . \tag{12}$$

Then, there exists U a neighborhood of 0 and a map $\Phi : U \subset X_1 \to X_2$ such that

i) *Φ is Lipschitz in U and differentiable at the origin.*

ii) *$\Phi(0) = 0$, $D\Phi(0) = 0$.*

iii) *The graph of Φ is locally invariant under F. (In the stable case, the graph of Φ is invariant.*

iv.1) *In the stable case we have:*

$$\Phi \in C^{\tilde{r}}$$

where $\tilde{r}$ is defined in (6).

iv.2) *In the pseudo-stable case we have:*

$$\Phi \in C^{\tilde{s}}$$

(where $\tilde{s}$ is defined in (6)) for any $s \leq r$ for which (12) holds.

Remark 3.2. Instead of conditions on the norms of the operators in (10) and (12), we could have used bounds on the spectral radius by redefining the norms using adapted norms. By choosing an adapted norm, we can arrange at the same time that the norm of the operator, its diagonal blocks and the inverses, are as close as desired to the spectral radii of these operators.

We refer to the appendix of [CFdlL03a] for a discussion of adapted norms in upper triangular operators. Of course, in finite dimensional spaces, the construction of these adapted norms are elementary.

In many ways, the formulation in terms of spectral properties is more intrinsic since the assumptions are independent of the norm used. Nevertheless, in this paper we will just use the norm formulation of the hypothesis.

Since we will not use a spectral formulation of the hypothesis, the statements work just as well for real Banach spaces and for complex Banach spaces. □

Remark 3.3. The method of proof will also provide with some uniqueness statements. We will describe them in Remark 4.11 and Remark 4.18. We anticipate, however that the conditions that give uniqueness are very different in the stable case and in the pseudo-stable case. In the stable case, we obtain uniqueness under the assumption of regularity of the invariant manifold in a neighborhood of the origin. In the pseudo-stable case, we obtain uniqueness by imposing conditions on the behavior at infinity.

This leads to some apparently paradoxical situations. For example, there are some cases where one space invariant for $DF(0)$ satisfies the hypothesis of the stable part of Theorem 3.1 for the map F and the hypothesis of the pseudo-stable part for the map F^{-1}. Nevertheless, it could happen – indeed it happens generically – that the manifolds obtained applying the two results are different.

Furthermore, we note that for the pseudostable results in Theorem 3.1, we only obtain the uniqueness results when we perform some preliminary preparations – described in Section 4.1 for the map which include arbitrary choices. Even if we obtain a unique manifold for each choice of the preparation, it could happen – see Example 5.4 – that the manifold produced depends on the preparation. Hence, the invariant manifolds claimed in the pseudo-stable case of Theorem 3.1 are very far from unique.

□

Remark 3.4. We will obtain some results for flows just by applying the results for maps to the time t map of the flow. Given the uniqueness results alluded to above, it is not hard to show that one obtains manifolds that are indeed invariant for the flow. We have developed these rather standard arguments in Section 6.1. Note also that it is quite possible to write a direct proof of the results for flows using an strategy very similar to that used here. We will indicate how this is possible in Section 6.2. □

Remark 3.5. Note that in the pseudo-stable case, we do not make any assumptions on what is $||A_1||$ other than (12). This allows to consider maps where

$$||A_1|| \geq 1. \tag{13}$$

Of course, due to (10), we need that the possible expansion of A_1 is dominated by the contraction of A_2^{-1} (notice that in (12) we are requiring that $s \geq 1$).

Even if it is not strictly logically necessary, in the study of pseudo-stable manifolds one can assume without loss of generality (13) since the case where (13) fails can be study more efficiently by the methods discussed in the stable case.

Pseudo-stable manifolds have been established by the graph transform in [HPS77] and by other methods in [Irw80], [dlLW95]. Indeed the treatment in this paper is quite similar to that of [HPS77], the only novelty is the removal of the assumption of invariance of the complement, but the proof we present here in the pseudo-stable case is essentially the same as in [HPS77]. □

Remark 3.6. The use of $\tilde{r}$ and $\tilde{s}$ in the conclusions does not belong. This is quite common in invariant manifold theory. Nevertheless, to improve the regularity conclusions on the manifold from $r - 1 + \mathrm{Lip}$ to r seems to require a separate argument. Such arguments are very standard in the literature. In particular, one often uses the tangent functor trick.

One possibility for such an argument is to derive a functional equation satisfied by the first derivative – it is a linear equation on the derivative. Then show that the solutions of this equation are C^{r-1}. We refer to [CFdlL03a] for arguments in a similar situation. Other arguments which also give sharp regularity occur in [ElB01]. We will not pursue this improvement here. □

A simple example where Theorem 3.1 applies and the classical theorems do not – of course the results in [CFdlL03a] do apply – is the following.

Example 3.7. *Consider* $X = \mathbb{R}^5$ *and a* C^2 *map* F*, leaving the origin fixed and such that the linearization at the origin is:*

$$A = \begin{pmatrix} 1/2 & 1 & & & \\ & 1/2 & 1 & & \\ & & 1/2 & & \\ & & & 2/5 & \\ & & & & 1/3 \end{pmatrix}.$$

Note that $1/2 > 2/5 > 1/3 > (1/2)^2$.

Therefore, Theorem 3.1 applies to either of the following splittings (as well as others based on the same ideas)

$$X_1^a = \{(x, 0, 0, 0, 0)|x \in \mathbb{R}\} \quad X_2^a = \{(0, y, z, t, u)|y, z, t, u \in \mathbb{R}\}$$
$$X_1^b = \{(x, y, 0, 0, 0)|x, y \in \mathbb{R}\} \quad X_2^b = \{(0, 0, z, t, u)|z, t, u \in \mathbb{R}\}$$
$$X_1^c = \{(x, 0, 0, t, 0)|x, t \in \mathbb{R}\} \quad X_2^c = \{(0, y, z, 0, u)|y, u \in \mathbb{R}\}$$
$$X_1^d = \{(x, 0, 0, 0, u)|x, u \in \mathbb{R}\} \quad X_2^d = \{(0, y, z, t, 0)|y, z, t \in \mathbb{R}\}.$$

It suffices to take a space invariant under the block of $1/2$ *and adjoin it or not the eigenspace corresponding to* $1/3$ *or the eigenspace corresponding to* $2/5$. *Of course, more possibilities appear if the eigenvalues* $2/5$, $1/3$ *would have had a non-trivial Jordan block.*

Systems such as those considered in Example 3.7 appear naturally as the time one maps of systems with resonances. The time one map of a resonant system will often have a nontrivial Jordan block. Understanding well the geometric properties at resonances seems to be an step toward providing explanations of many empirically known, but not yet rigorously analyzed phenomena.

Upper triangular couplings are called in the literature *master-slave* systems. When the master and the slave are identical systems – which often happens in electronics – the linearization has Jordan blocks.

Remark 3.8. We note that the results of [dlL97], [ElB01], [CFdlL03a] apply also to situations in which rather than a gap condition we have non-resonance conditions. In particular, we could have applied the results of those papers to Example 3.7 with $1/3$ replaced by $1/7$. Nevertheless, the results of this paper do not go through with such a change. □

Remark 3.9. Note that the equations (10) and (12) have the same form.

Roughly speaking the regularity obtained for the manifold is the minimum of the regularity of the map and the set of numbers s that satisfy the condition in (10) or (12).

In the stable case, we have that the larger that we take s, the easier is to satisfy (10). Hence the limitations for the regularity of the conclusions come only for the regularity of the map.

In the pseudo-stable case, (12) is false for large enough s. Hence, the limitation for regularity given by (12) is a genuine limitation which may be stronger than the limitation due to the regularity of the map.

Another consideration is that, in order that the proof goes through, we need to assume certain limitations on s. In the stable case, the proof we present goes through with $s = 2$. This is weaker than the usual domination condition $s = 1$. The reason why we can take $s = 2$ is related to the fact that the nonlinear part of the map vanishes to order s. If we had $D^i N(0) = 0$ for $i \leq k$, it would suffice to take $s \geq k$. This observation is further exploited in [dlL97] which, under extra non-resonance conditions shows that one can make transformations which make N to be in a normal form so that it vanished to higher order. We will not discuss these improvements in detail.

In the pseudo-stable case, the order of vanishing of N does not help and we just need to use the classical condition $s \geq 1$. □

4. Proof of Theorem 3.1

In this section we will present the proof of Theorem 3.1. We will present first the proof for the stable case and then, the proof for the pseudo-stable case. Both proofs are based in the same functional equation and have some common preparatory work, nevertheless, the final analysis will be rather different.

4.1. **Preliminaries.** In this section, we carry out some preliminary preparations of the problem that can be performed without loss of generality. They simplify subsequent analysis.

We write

$$\begin{aligned} DF(0) &= A \\ F(x) &= Ax + N(x). \end{aligned}$$

Clearly, $N(0) = 0$, $DN(0) = 0$.

With respect to the decomposition $X = X_1 \oplus X_2$, we can write

$$A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}. \tag{14}$$

We will assume without loss of generality that

$$\|x\|_X = \max\left(\|\Pi_1 x\|_{X_1}, \|\Pi_2 x\|_{X_2}\right)$$

Furthermore, by changing $\|\cdot\|_{X_2}$ to $\|\cdot\|_{\widetilde{X}_2} = \lambda \|\cdot\|_{X_2}$ with λ sufficiently large, we can assume that

$$\|B\|_{X_2 \to X_1} \leq \varepsilon \tag{15}$$

where ε is arbitrarily small. Later in the proof we will impose a finite number of conditions that ε has to satisfy.

As a consequence of (15) we have:

$$\|A\| \leq \max\left(\|A_1\|, \|A_2\|\right) + \varepsilon$$

As standard in invariant manifold theory, we observe that if we introduce a scaling $F_\lambda(x) = \lambda F(\lambda^{-1}x)$, we have that $DF_\lambda(0) = DF(0)$, hence none of the previous properties of the linear map are altered. At the same time can arrange by taking λ big enough that

$$\|N\|_{C^r(B_1)} \le \varepsilon \,. \tag{16}$$

Hence, we will assume, without loss of generality that we have (15) and (16).

4.1.1. *Preparations for the pseudo-stable case.* In the pseudo-stable case, we need a further reduction that allows us to assume that N is C^r small in $X_1 \times B_1(X_2)$ where $B_1(X_2)$ is the unit ball centered at the origin in X_2. Given a cutoff function Ψ on X_1, it suffices to consider the mapping

$$\tilde{F}(x) = Ax + \Psi(\Pi_1 x)N(x)$$

Since we have arranged by scalings that N is C^r small in the ball of radius 2, the Leibniz formula for the derivatives of products and the formula for the products of Hölder functions show that $\tilde{F}$ is small in the ball of radius 2.

Note that the map $\tilde{F}$ agrees with the map F in a neighborhood of the origin. Hence a manifold which is invariant for $\tilde{F}$, will be locally invariant for F.

As we will see later, the uniqueness results established for the pseudo-stable case, will be uniqueness results for the manifolds invariant under $\tilde{F}$ and which satisfy some conditions on the behavior at ∞. Since the construction of $\tilde{F}$ out of F involves the choice of the cutoff function Ψ, it is quite possible that different choices of Ψ will lead to different invariant manifolds for different $\tilde{F}$ and, hence, different locally invariant manifolds for F.

4.2. **A functional equation for the invariance.** We follow a rather standard variation of the graph transform method.

If $x = (y, \Phi(y))$ is a point in the graph of Φ we have

$$F(x) = \big(A_1 y + B\Phi(y) + N_1(y, \Phi(y)), A_2\Phi(y) + N_2(y, \Phi(y))\big)$$

where N_1, N_2 are shorthands for $\Pi_1 N$, $\Pi_2 N$ respectively.

The condition that $F(x)$ is also in the graph of Φ reads – ignoring for the moment questions of domains on where the composition is defined–

$$\Phi\big(A_1 y + B\Phi(y) + N_1(y, \Phi(y))\big) = A_2\Phi(y) + N_2(y, \Phi(y)). \tag{17}$$

The equation (17) is, furthermore formally equivalent – again ignoring questions of domains of definitions of the functions – to

$$\Phi(y) = A_2^{-1}\big[\Phi(A_1 y + B\Phi(y) + N_1(y, \Phi(y))\,) - N_2(y, \Phi(y))\big]. \tag{18}$$

Remark 4.1. The issue of equivalence between (17) and (18) are certainly not trivial. As we will see when we discuss uniqueness, different equations which are formally equivalent, may have different solutions. As argued in [dlLW95], [JPdlL95] this is one of the reasons why different notions of slow invariant manifolds lead to different objects. See also Example 5.4 where we exemplify that the same formal equation may have very different solutions depending on the precise requirements on domains. □

We will study (18) as a fixed point problem.

We denote by $\mathcal{T}$ the operator which to a function Φ associates the R.H.S. of (18). That is

$$\mathcal{T}[\Phi](y) = A_2^{-1}\big[\Phi(A_1 y + B\Phi(y) + N_1(y, \Phi(y))) - N_2(y, \Phi(y))\big] \tag{19}$$

This operator is not exactly the operator associated to the graph transform, but it is closely related to it – the fixed points we will produce for $\mathcal{T}$ will be also fixed points of the graph transform operator – and slightly simpler. This is the operator that is considered in many classical proofs, e.g. that of [LI83].

The proof we present consists in showing that the operator $\mathcal{T}$ is well defined on a space of functions, that it has a fixed point and that the fixed point is such that it allows us to reverse the formal derivation, (that is, we will show that the solution of (18) we produce is also a solution of (17) since we will show that the domains and range match so that we can reverse the derivation of (17)).

The proofs of these results will be different in the stable case and in the pseudo-stable case. Both of them follow the classical proofs of invariant manifold theorems following the method of the graph transform. We identify some space of functions which is mapped into itself by $\mathcal{T}$ and on which $\mathcal{T}$ is a contraction. We will present first the proof in the stable case and later the proof for the pseudo-stable case. The cases of C^∞ and C^ω regularity for the stable case will be done separately in Section 4.4.5.

4.3. **Formulas for derivatives.**

A result that we will use both in the stable and in the pseudo-stable cases is the following purely formal Lemma 4.2.

Lemma 4.2. *Assume that for an open set of y we can define $\mathcal{T}[\Phi]$ as in* (19).

If Φ is C^i, $i \le r$, then $\mathcal{T}[\Phi]$ is C^i and, moreover, we have

$$
\begin{aligned}
D^i\mathcal{T}[\Phi] =& A_2^{-1} D^i\Phi(A_1 + B\Phi + N_1(\cdot,\Phi))\cdot \\
&\cdot \left[(A_1 + BD\Phi + D_1N_1(\cdot,\Phi) + D_2N(\cdot,\Phi)D\Phi)\right]^{\otimes i} \\
&+ A_2^{-1} D\Phi(A_1 + B\Phi + N_1(\cdot,\Phi))D_2N_1(\cdot,\Phi)D^i\Phi \\
&+ A_2^{-1} D_2N_2(\cdot,\Phi)D^i\Phi \\
&+ R_i(D\Phi,\ldots,D\Phi^{i-1})
\end{aligned}
\tag{20}
$$

where R_i is a polynomial in the derivatives of Φ whose coefficients are polynomial expressions involving the derivatives of F – up to order i – evaluated at Φ

Proof. The formula (20) can be obtained from Faa Di Bruno formula, but is significantly easier.

The main point of (20) is that we can identify the only term in $D^i\mathcal{T}[\Phi]$ which contains as a factor $D^i\Phi$.

The formula (20) is easily established by induction starting form the obvious case $i = 1$. Assuming that (20) is true, we compute $D^{i+1}\mathcal{T}[\Phi]$ by taking one more derivative on both sides.

We note that taking the derivative of R_i we do not obtain derivatives of Φ of order higher than i. To establish that R is a polynomial we compute the derivative of the terms by using the product rule. When we take a derivative of a factor $D^j\Phi$, we obtain $D^{j+1}\Phi$ and when we take the derivative of $D^jF \circ \Phi$ we obtain $D^{j+1}F \circ \Phi D\Phi$. Both factors are of the desired form.

To establish the claims about the terms with higher derivatives, it suffices to observe that the only way that we get derivatives of order $i+1$ when we take derivatives of the expression is that, when we apply the product rule, take the derivatives on the factor $D^i\Phi$. If we take derivatives on the other factors, we obtain terms which are polynomials in derivatives of order lower that $i+1$ so that can consider them as part of R_{i+1}. □

4.4. **Proof of Theorem 3.1 in the stable case.** We now start the proof of Theorem 3.1 when $r < \infty$. The cases $r = \infty, \omega$ will be postponed till Section 4.4.5

4.4.1. *Some spaces of functions.* In this section, we introduce some spaces and norms that we will use later.

The main novelty with respect to most of the standard proofs of invariant manifold theorems in the literature is that we take advantage of the fact that the functions we are seeking vanish at the origin to second order (slightly less in the less regular cases). Hence we can use weighted norms (22) or norm based on derivatives which lead to stronger contraction properties for the operator considered (see Proposition 4.3). This improved contraction property is what allows us to weaken the dominance condition.

We consider $r = k + \alpha$ $k \in \mathbb{N}$, $\alpha \in [0, \mathrm{Lip}]$ and define the spaces

$$\chi_{\delta_0,\cdots,\delta_k;\delta^\alpha} = \Big\{\Phi : B_1 \subset X_1 \to X_2, \Phi \in C^r,$$

$$(21) \qquad \begin{aligned} &\|D^i\Phi\|_{C^0(B_1)} \le \delta_i, i = 0, \dots, k, \\ &H_\alpha(D^k\Phi) \le \delta^\alpha, \\ &\Phi(0) = 0, D\Phi(0) = 0, \\ &\sup_{y \in B_1 - \{0\}} |D\Phi(y)||y|^{-s+1} < \infty\Big\} \end{aligned}$$

where s is the same as that entering in (10), namely, $s = \min(2, r)$. Hence, when $r \ge 2$, the exponent of $|y|$ in the last condition for the derivative in (21) is -1. In case $r = 1 + \alpha$, the exponent $-s + 1$ is just $-\alpha$.

When $\alpha = 0$, the parameter in the definition of χ does not play any role since δ_k and δ^0 would control the C^0 norm of $D^k\Phi$. Hence, we will just suppress δ^0 Hence, when dealing with integer regularity, we will use the notation $\chi_{\delta_0,\dots,\delta_k}$.

We will assume that $\delta_0 \le 1$ so as to make sure that $N(y, \Phi(y))$ is always well defined.

We will endow $\chi_{\delta_0,\dots,\delta_k;\delta^\alpha}$ with the topology induced by

$$(22) \qquad \|\!|\Phi|\!\| = \sup_{y \in B_1 - \{0\}} |D\Phi(y)|/|y|^s$$

where, we recall $s = 2$ whenever $r \ge 2$, $s = \alpha$ when $r = 1 + \alpha$, $\alpha \in (0, \mathrm{Lip}]$ and $s = 0$ when $r = 1$.

It is not hard to check that (22) is a norm in the space of functions which satisfy the normalization $\Phi(0) = 0$.

Note that the topology induced by (22) is finer than the topology induced by the C^0 norm.

An important result [LI73] Lemma 2-5 is that when $r = k + \alpha$, the closure of χ under C^0 – a fortiori under the weaker topology we consider – is contained in the set of functions which are $C^{\tilde{r}}$ and, which, moreover satisfy

$$\begin{aligned} &\Phi(0) = 0, \quad D\Phi(0) = 0, \\ &\|D^i\Phi\|_{C^0(B_1)} \le \delta_i, \quad 0 \le i \le \tilde{k}, \\ &H_{\alpha,B_1}(D^{r-1}\Phi) \le \delta^\alpha. \end{aligned}$$

We also note that when $r \geq 2$ and the space X_1 is separable, a variant of the Ascoli-Arzelá argument concludes that the space χ is precompact with the topology given.

The following two propositions study the behavior of $\|\!| \cdot |\!\|$ under operations that appear frequently in the graph transform approach.

The first property of the norm (22) that we will use is that it behaves very well under composition with contractions.

Proposition 4.3. *Let $\Gamma : B_1 \to B_1$ be a C^1 function such that*

$$\Gamma(0) = 0, \qquad ||D\Gamma||_{C^0(B_1)} \leq 1.$$

Assume that $\|\!|\Phi|\!\| < \infty$.

Then,

$$\|\!|\Phi \circ \Gamma|\!\| \leq \|\!|\Phi|\!\| ||D\Gamma||^s_{C^0(B_1)} \tag{23}$$

The main point of the Proposition is that we obtain the exponent s in the bound in (23). In the most typical case $r \geq 2$, then the exponent $s = 2$. Since $||D\Gamma||_{C^0(B_1)}$ is smaller than 1, this is quite worthwhile. Indeed, this is the reason why we can improve (3) to (11) when $r \geq 2$ or to (10) for low regularities.

Proof. Clearly, the function $\Phi \circ \Gamma$ is C^1 and it satisfies $D(\Phi \circ \Gamma)(0) = 0$.

We estimate for $y \neq 0$

$$\begin{aligned} |D\Phi \circ \Gamma(y)|/|y|^{s-1} &\leq \frac{|(D\Phi) \circ \Gamma(y)||\Gamma(y)|^{s-1}|D\Gamma(y)|}{|y|^{s-1}|\Gamma(y)|^{s-1}} \\ &\leq \sup_{y \in B_1 - \{0\}} (|D\Phi y|/|y|^{s-1}) \left[\sup_{y \in B_1 - \{0\}} (|\Gamma(y)|/|y|) \right]^{s-1} \\ &\quad \cdot \sup_{y \in B_1 - \{0\}} (|D\Gamma(y)|) \end{aligned}$$

from which it clearly follows that $\|\!|\Phi \circ \Gamma|\!\|$ is finite and that it satisfies the estimates in (23). □

Proposition 4.4. *Let $N_2 : X_1 \oplus X_2 \to X_2$ satisfy $DN_2(0,0) = 0$. Assume that $N_2 \in C^{1+\alpha}$, $0 \leq \alpha \leq \mathrm{Lip}$. Let Φ be a $C^{1+\alpha}$ function, $\Phi(0) = 0$, $D\Phi(0) = 0$.*

Then, the function $\eta(y) = N_2(y, \Phi(y))$ satisfies

$$\|\!|\eta|\!\| \leq ||N||_{C^{1+\alpha}} + ||N||_{C^1} \|\!|\Phi|\!\|$$

where, as before

$$\|\!|\eta(y)|\!\| = \sup_{y \in B_1 - \{0\}} |D\eta(y)|/|y|^{\alpha}.$$

Proof. We have

$$\begin{aligned}|DN(y,\Phi(y))|/|y|^\alpha \leq & |(D_1N)(y,\Phi(y))|/|y|^\alpha \\ & + |(D_2N)(y,\Phi(y))||D\Phi(y)|/|y|^\alpha \\ \leq & ||N||_{C^{1+\alpha}} + ||N||_{C^1} \|\Phi\| \end{aligned}$$

□

Remark 4.5. We note that the introduction of the *conical norm* $\|\cdot\|$ is mainly useful for the cases $r \in 1 + [0, \mathrm{Lip}]$. In the cases $r \geq 2$, we could just use the topology induced by $||D^2\Phi||_{C^0(B_1)}$.

Since we are considering spaces of C^2 functions which satisfy the normalizations $\Phi(0) = 0$, $D\Phi(0) = 0$, we see that $||D^2\Phi||_{C^0}$ is indeed a norm.

For our purposes, the main property that we need is that the norm considered behaves well under composition with a contraction and that we have improved contraction properties analogous to (23).

For $||D^2\Phi||_{C^0(B_1)}$ we have

$$\begin{aligned}||D^2(\Phi\circ\Gamma)||_{C^0(B_1)} \leq & ||D^2\Phi||_{C^0(B_1)}||D\Gamma||^2_{C^0(B_1)} \\ & + ||D\Phi||_{C^0(B_1)}||D^2\Gamma||_{C^0(B_1)} \\ \leq & ||D^2\Phi||_{C^0(B_1)}\big(||D\Gamma||^2_{C^0(B_1)} + ||D^2\Gamma||_{C^0(B_1)}\big)\end{aligned}$$

which can be used in a similar way as (23) provided that we can make $D^2\Gamma$ small.

Hence, we could use $||D^2\Phi||_{C^0}$ in the subsequent arguments rather than the conical norm. The conical norm $\|\Phi\|$ turns out to be somewhat simpler to estimate and, since $\|\Phi\| \leq ||D^2\Phi||_{C^0}$ the uniqueness statements in the conical norm are slightly more general (the spaces in which the conical norm is defined include functions that are not C^2).

In case that we consider functions Φ which vanish to order k – which is possible if N vanishes to order k, perhaps after some preliminary transformations which are possible under finitely many non-resonance conditions –, it is possible to use the norms $||D^k\Phi||_{C^0}$, The paper [dlL97] includes a general discussion of these norms and shows. In [CFdlL03a], these cases are studied with wighted norms with higher powers. Other norms which also lead to improved contraction properties occur in [ElB01]. □

4.4.2. *The operator $\mathcal{T}$ is well defined in the χ spaces.* We will first check that the RHS of (18) indeed defines an operator on χ.

We first note that using the conventions arranged in Section 4.1 we have, for $|y| \le 1$, $\Phi \in \chi_{\delta_0,\dots,\delta_k;\delta^\alpha}$

$$\begin{aligned} |A_1 y + B\Phi(y) + N_1(y,\Phi(y))| &\le \|A_1\| + \varepsilon\delta_0 + \|N_1\|_{C_0} \\ &\le \|A_1\| + 2\varepsilon \end{aligned} \tag{24}$$

If we impose the condition that ε is small enough, we can ensure that the RHS of (24) is smaller than 1.

Once we have that the function $\mathcal{T}[\Phi]$ is well defined in the indicated domain, the chain rule tells us that $\mathcal{T}[\Phi]$ is C^r.

Hence, the RHS of (18) can be defined for all the $\Phi \in \chi_{\delta_0,\dots,\delta_k;\delta^\alpha}$.

4.4.3. *The range of operator $\mathcal{T}$ on the spaces χ.* In this section, we show that $\mathcal{T}[\chi]$ is contained in another set also of the form χ but with different parameters. In particular, we will show that it is possible to arrange with the prenormalizations introduced in Section 4.1 that one can find domains χ that get mapped into themselves.

Lemma 4.6. *In the conditions of Theorem 3.1 after making the adjustments in Section 4.1 so that $\|B\|$, $\|N\|_{C^s(B_1)}$ are small enough, $s = \min(r,2)$.*

Then, it is possible to find $\delta_0, \cdots, \delta_k, \delta^\alpha$, satisfying $\delta_0 = \delta_1 = 1$, $\delta_i > 0$ as well as:

$$\mathcal{T}(\chi_{\delta_0=1,\delta_1=1\cdots\delta_k;\delta^\alpha}) \subset \chi_{\delta_0=1,\delta_1=1,\cdots,\delta_k;\delta^\alpha}.$$

Proof. First, it is clear by the chain rule that if $\Phi \in C^r$, then $\mathcal{T}[\Phi] \in C^r$.

The fact that $\mathcal{T}[\Phi](0) = 0$ and $D\mathcal{T}[\Phi](0) = 0$ are just an easy calculation.

We denote

$$\Gamma[\Phi](y) = A_1 y + B\Phi(y) + N_1(x,\Phi(y)). \tag{25}$$

Therefore:

$$\begin{aligned} D\Gamma[\Phi](y) = A_1 &+ BD\Phi(y) + D_1N_1(x,\Phi(y)) \\ &+ D_2N_2(y,\Phi(y))D\Phi(y). \end{aligned} \tag{26}$$

We estimate

$$\mathrm{Lip}(\Gamma[\Phi]) \le \|D\Gamma[\Phi]\|_{C^0(B_1)} \le \|A_1\| + \varepsilon \ . \tag{27}$$

The fact that $|||\mathcal{T}[\Phi]|||$ is finite is a very easy consequence of Propositions 4.3 and 4.4. (The operator $\mathcal{T}$ just differs from the case considered in Proposition 4.3 by multiplication in the left by a linear operator.)

The heart of the matter is to obtain estimates for the derivatives of $\mathcal{T}[\Phi]$ and for $|||\mathcal{T}[\Phi]|||$ in terms of those of Φ.

From (20), using the triangle inequality, the Banach algebra properties of multiplication, the formula (27) and that for functions $\Phi \in$

$\chi_{\delta_0,\dots,\delta_k;\delta^\alpha}$, we have bounds on the derivatives, we obtain that, when $\Phi \in \Phi \in \chi_{\delta_0,\dots,\delta_k;\delta^\alpha}$, we have:

$$
\begin{aligned}
\|D^i\mathcal{T}[\Phi]\|_{C^0(B_1)} \le & \|D\Phi^i\|_{C^0(B_1)}\big(\|A_1\|+3\varepsilon\big)^i\|A_2^{-1}\| \\
& + P_i(\delta_0,\dots,\delta_{i-1})
\end{aligned} \tag{28}
$$

where P_i is a real polynomial with positive coefficients.

The coefficients of P are obtained by estimating the derivatives of N. Even if we will not use it in this paper – except for $i = 1, 2$ – we note that the polynomial P_i can be assumed to be arbitrarily small by assuming that $\|N\|_{C^i}$ is sufficiently small.

Similarly, using (7) we estimate the Holder seminorm in the unit ball as:

$$
\begin{aligned}
& H_{\alpha,B_1}(D^i\mathcal{T}[\Phi]) \\
& \quad \le H_{\alpha,B_1}(D\Phi^i)\big(\|A_1\|+3\varepsilon\big)^{i+\alpha}\|A_2^{-1}\| + P_i(\delta_1,\dots,\delta_{i-1}).
\end{aligned} \tag{29}
$$

The rest of the proof of Lemma 4.6 will be different according to whether $r \ge 2$, $r \in 1 + (0, \mathrm{Lip}]$ or $r = 1$.

For $r \ge 2$, which is the main case, we note that, because of the assumption (11), we can arrange as in Section 4.1, that

$$
\begin{aligned}
& \|A_2^{-1}\|\|D\Gamma\|^2_{C^0(B_1)} \equiv \gamma < 1 \\
& \|D\Gamma\|_{C^0(B_1)} < 1.
\end{aligned}
$$

We choose $\delta_0 = \delta_1 = 1, \delta_2 = 1$. Because of (28) we get

$$
\|D^2\mathcal{T}[\Phi]\|_{C^0} \le \gamma\delta_2 + P_2(1,1)
$$

Recalling that we can make the coefficients or P_2 as small as desired by arranging that $\|N\|_{C^2}$, $\|B\|$ are sufficiently small, we can arrange that

$$
\|D^2\mathcal{T}[\Phi]\|_{C^0} \le 1. \tag{30}
$$

Using that we have the normalizations $\mathcal{T}[\Phi] = 0$, $D\mathcal{T}[\Phi] = 0$, we can use the mean value theorem to obtain from (30)

$$
\|D\Phi\|_{C^0} \le 1, \qquad \|\Phi\|_{C_0} \le 1/2.
$$

This establishes the desired result for $r = 2$.

In case that $r = k + \alpha > 2$, we proceed to choose the δ_i, δ^α so that the desired conclusions hold. It is important to emphasize that the smallness conditions that we will be imposing on $\|N\|_{C^2}$, $\|B\|$ will be independent of k. This will be the basis of the study of the C^∞ case.

We observe that, we have for $k \ge i > 2$

$$
\|A_2^{-1}\|\|D\Gamma\|^i_{C^0(B_1)} \equiv \gamma_k < 1.
$$

Using (28) we have

$$\|D^i\mathcal{T}[\Phi]\|_{C^0} \leq \gamma_i\delta_i + P_i(1,\delta_2,\ldots,\delta_{i-1})$$

Hence, we can choose recursively δ_i so that $\|D^i\mathcal{T}[\Phi]\|_{C^0} \leq \delta_i$.

Similarly, taking into account (29) we obtain

$$H_\alpha[D^k\mathcal{T}[\Phi]] \leq \gamma_{k,\alpha}\delta^\alpha + P_{k,\alpha}(1,\delta_2,\ldots,\delta_k)$$

where $\gamma_{k,\alpha} \equiv \|A_2^{-1}\|\|D\Gamma\|^{k+\alpha}_{C^0(B_1)} < 1$. Hence, it is possible to choose also the δ^α.

This finishes the proof of Lemma 4.6 in the case $r \geq 2$.

Remark 4.7. We call attention that in this case, we have only used the assumption (11). This is somewhat weaker than the usual dominance condition (3).

As it often happens, when considering the composition with contractions, derivatives of higher order have better estimates than derivatives of lower order.

In our case, we deduced estimates for the lower derivatives from those for the higher derivatives because we have the normalization that Φ has a second order tangency at zero, so that we could estimate the first derivative and the function by the second derivative.

This is, of course related to the fact that, by the definition of derivative, the nonlinear part has a tangency of order 2 with the linear part.

The same type of argument can be carried out using derivatives or order higher than 2 if we can ensure that N has a tangency of high enough order. Indeed in [dlL97] it is shown that if A satisfies certain non-resonance conditions, it is possible to make changes of variables that reduce N so that high order tangencies are preserved. In such a case, it is possible to use similar arguments with higher derivatives and obtain conditions weaker than (11) because the exponent of $\|A_1\|$ is bigger than 2.

Arguments with a similar flavor but applied to somewhat different operators happen in [CFdlL03a].

We also refer to Example 5.1 to show that some of these conditions are necessary. □

Now, we consider the range of the operator $\mathcal{T}$ in the low regularity cases for which the use of the second derivative is not possible.

For the case $r = 1$, we observe that the estimate (28) we have $\|D\mathcal{T}[\Phi]\|_{C^0} \leq \gamma_1\delta_1 + \varepsilon$ where $\gamma_1 = \|A_2^{-1}\|\|A_1\|$. In this case, the assumptions in Theorem 3.1 imply that $\gamma_1 < 1$. Hence, we can choose the $\delta_1 = 1$ so that $\|D\mathcal{T}[\Phi]\|_{C^0} \leq \delta_1$. Once we have that, we can choose $\delta_0 > \delta_1$. Using the mean value theorem and the normalization $\mathcal{T}[\Phi](0) = 0$, we obtain that

$$\mathcal{T}(\chi_{\delta_0,\ldots,\delta_k;\delta^\alpha}) \subset \chi_{\delta_0,\ldots,\delta_k;\delta^\alpha}.$$

In the case $r = 1+\alpha$, $0 < \alpha \leq \mathrm{Lip}$, we choose $\delta_0 = 1$, $\delta_1 = 1$, $\delta^\alpha = 1$. By the chain rule, we have $D\mathcal{T}[\Phi] = (D\Phi) \circ \Gamma D\Gamma$. Hence, using (7),

$$H_\alpha[D\mathcal{T}[\Phi]] \leq \|A_2^{-1}\| \big(H_\alpha[D\Phi] \,\mathrm{Lip}(\Gamma)^\alpha \|D\Gamma\|_{C^0} + \|D\Phi\|_{C^0} H_\alpha D\Gamma \big).$$

Note that $H_\alpha[D\Gamma] \leq H_\alpha[DN]$ and, since $\alpha > 0$, this can be made arbitrarily small by rescaling.

Hence, we have for functions Φ in $\chi_{1,1;1}$

$$H_\alpha[D\mathcal{T}[\Phi]] \leq \gamma + \varepsilon$$

where $\gamma \equiv \|A_2^{-1}\| \|D\Gamma\|_{C^0}^{1+\alpha} < 1$ by assumption. Adjusting that ε is small enough, we obtain the desired result.

This finishes the proof of Lemma 4.6. □

We emphasize that the conditions of smallness that we have imposed on $\|B\|$, $\|N\|_{C^2}$, are independent of r. The way to ensure that the high order derivatives get trapped is by choosing the δ_i $i \geq 2$ and δ^α entering in the definition of the $\chi_{\delta_0,\ldots,\delta_k;\delta^\alpha}$ to be large enough.

Note also that the argument relied heavily on the fact that we could have all the γ_i smaller than one with conditions that are independent of i. This is certainly true in the stable case, but will be false in the pseudo-stable case.

Remark 4.8. In case that we can apply Ascoli-Arzelá theorem applied to the spaces χ – e.g. when X_1 is a separable space, we obtain a proof of the existence of solutions of (18) just by applying the Schauder-Tychonov theorem because the space χ is compact and, clearly convex. The operator $\mathcal{T}$ is continuous because it has a closed graph.

This gives a very short proof of the existence of the invariant manifolds since we avoid the estimates to obtain that $\mathcal{T}$ is a contraction. On the other hand, we do not obtain the uniqueness properties. Nevertheless, we observe that, in the case that the manifold we are establishing is the stable manifold, there are geometric arguments [Shu87], [LI83] which establish that the stable manifold is unique.

Hence, a possible argument to prove the stable manifold for finite dimensional spaces is to apply the Schauder theorem once one establishes the propagated bounds. Then, one can establish the uniqueness by geometric arguments as in the references above. □

4.4.4. *Contraction properties of the operator $\mathcal{T}$ on χ.*

Lemma 4.9. *In the condition of Lemma 4.6 $\mathcal{T}$ is a contraction on $\chi_{\delta_0 \cdots \delta_r}$ with the norm (22).*

Recall that

$$\begin{aligned} D\mathcal{T}[\Phi](y) = A_2^{-1} D\Phi\big(A_1 y + B\Phi(y) + N_1(y, \Phi(y))\big) \\ \cdot \big(A_1 + B\Phi(y) + D_1 N_1(y, \Phi(y))\big) \\ + A_2^{-1}\big[D_1 N_2(y, \Phi(y)) + D_2 N_2(y, \Phi(y)) D\Phi(y)\big] \end{aligned} \tag{31}$$

From (31) it will be rather straightforward to estimate $|||\mathcal{T}[\Phi] - \mathcal{T}[\tilde{\Phi}]|||$ by adding and subtracting terms appropriately.

We will also use that $|y| \leq 1$, $||\Phi|| \leq 1$, $||D\Phi|| \leq 1$, $||D\Phi|| \leq 1$.

Remark 4.10. As a heuristic guide for the subsequent estimates is that, ignoring all the objects that can be made arbitrarily small, the formula (31) for the derivative amounts to just

$$A_2^{-1} D\Phi(A_1(y)) A_1. \tag{32}$$

As we will see, since all the terms are small, the proof of contraction will be obtained by adding and substracting terms from the proof of contraction in (32).

The contraction of (32) is proved using the improved contraction estimates. We have:

$$\begin{aligned} ||A_2^{-1} D\Phi(A_1 y) A_1 - A_2^{-1} D\tilde{\Phi}(A_1 y) A_1|| \\ \leq ||A_2^{-1}|| ||A_1|| |D\Phi(A_1 y) - D\tilde{\Phi}(A_1 y)| \\ \leq ||A_2^{-1}|| ||A_1|| |||\Phi - \tilde{\Phi}||| |A_1 y| \\ \leq ||A_2^{-1}|| ||A_1||^2 |||\Phi - \tilde{\Phi}||| |y|. \end{aligned}$$

From the above estimates, it follows that the main part of $\mathcal{T}$ (as in (32) is a contraction in $|||\cdot|||$. □

Now we turn to proving estimates for the full $\mathcal{T}$ and not just (32). Roughly speaking we will try to proceed along the lines indicated in the Remark 4.10, but we will have to pay attention to estimating systematically all the other terms which will turn out to be arbitrarily small with the adjustments in Section 4.1.

$$\begin{aligned} |D_1 N(y, \Phi(y)) - D_1 N(x, \tilde{\Phi}(y))| \leq ||D_1 D_2 N||_{C^0(B_1)} |\Phi(y) - \tilde{\Phi}(y)| \\ \leq \varepsilon |||\Phi - \tilde{\Phi}||| |y|^2. \end{aligned} \tag{33}$$

A fortiori, similar bounds are true for N_1, N_2 in place of N.

Moreover,

$$\begin{aligned}
|D_2N(y,&\Phi(y))D\Phi(y) - D_2N(y,\tilde\Phi(y))D\tilde\Phi(y)| \\
&\le |D_2N(y,\Phi(y))||D\Phi(y) - D\tilde\Phi(y)| \\
&\qquad + |D_2N(y,\tilde\Phi(y)) - D_2N(y,\tilde\Phi(y))||D\tilde\Phi(y)| \\
&\le \varepsilon \|\Phi - \tilde\Phi\| |y|
\end{aligned}$$

where we have used again (33) to estimate the second factor.

Again, we note that, the same estimates remain valid for N_1, N_2 in place N.

We denote by

$$\begin{aligned}
\Gamma[\Phi](y) &= A_1 y + B\Phi(y) + N_1(y,\Phi(y)) \\
D\Gamma[\Phi](y) &= A_1 + BD\Phi(y) + D_1N_1(y,\Phi(y)) + D_2N(y,\Phi(y))
\end{aligned}$$

We estimate

$$||D\Gamma[\Phi]||_{C^0(B_1)} \le ||A_1|| + \varepsilon\ .$$

Hence

$$|\Gamma[\Phi](y)| \le \big(|A_1| + \varepsilon\big)||y||\ .$$

Moreover, we have

$$|D\Gamma[\Phi](y) - D\Gamma[\tilde\Phi](y)| \le \varepsilon \|\Phi - \tilde\Phi\| |y|$$

$$|\Gamma[\Phi](y) - \Gamma[\tilde\Phi](y)| \le \varepsilon \|\Phi - \tilde\Phi\| |y|$$

The only terms left to estimate in $D\mathcal{T}[\Phi] - D\mathcal{T}[\tilde\Phi]$ can be expressed as

$$A_2^{-1}(D\Phi\circ\Gamma[\Phi](y)D\Gamma[\Phi](y) - D\tilde\Phi\circ\Gamma[\tilde\Phi](y)D\Gamma[\tilde\Phi](y)). \tag{34}$$

The norm of (34) is bounded by (we recall that $s = \min(2,r)$ entered in the definition of $\|\cdot\|$)

$$\begin{aligned}
||A_2^{-1}||\big[&|D\Phi\circ\Gamma[\Phi](y) - D\tilde\Phi\circ\Gamma[\tilde\Phi](y)||D\Gamma[\Phi](y)| \\
&+ |D\tilde\Phi\circ\Gamma[\Phi](y) - D\tilde\Phi\circ\Gamma[\tilde\Phi](y)||D\Gamma[\Phi](y)| \\
&+ |D\tilde\Phi\circ\Gamma[\tilde\Phi](y)||D\Gamma[\Phi](y) - D\Gamma[\tilde\Phi](y)|\big] \\
\le ||A_2^{-1}||\big[&\|\Phi - \tilde\Phi\|\big(||A_1|| + \varepsilon\big)|y|^{s-1}\big(||A_1|| + \varepsilon\big) \\
&+ |\Gamma[\Phi](y) - \Gamma[\tilde\Phi](y)|\big(||A_1|| + \varepsilon\big) + \varepsilon\|\Phi - \tilde\Phi\||y|^{s-1}\big] \\
\le \big(||A_2^{-1}||&(||A_1|| + \varepsilon)^s + \varepsilon\big)\|\Phi - \tilde\Phi\||y|^{s-1}.
\end{aligned}$$

Collecting the previous estimates, we obtain that as indicated in Remark 4.10, the operator $\mathcal{T}$ has a Lipschitz constant in $\|\cdot\|$ which is $||A_2^{-1}||||A_1||^s + \varepsilon$. By the assumptions in Theorem 3.1, this is a contraction on $\chi_{\delta_0,\dots,\delta_k;\delta^\alpha}$. Hence, we obtain that there a fixed point of $\mathcal{T}$ – therefore a solution of (18) – in the closure of χ for the $\|\cdot\|$. Applying Lemma 2.5 in [LI73], we conclude that this fixed point is $C^{\tilde{r}}$.

Hence, we have established the conclusions of Theorem 3.1 for the case $\|A_1\| < 1$, $r \in \mathbb{N} + [0, \mathrm{Lip}]$.

Remark 4.11. We note that, since we have used a contraction argument, we have shown that there is exactly one fixed point in $\overline{\chi_{\delta_0,\dots,\delta_k;\delta^\alpha}}$, where $^-$ denotes the closure in the topology we considered.

Note that if $\ell > k$

$$\chi_{\delta_0,\dots,\delta_k,\delta_{k+1},\dots\delta_\ell;\delta^\beta} \subset \chi_{\delta_0,\dots,\delta_k;\delta^\alpha}. \tag{35}$$

It is important to realize that our choice of the δ's for different regularties is done by induction in the regularity. In this way, the δ of high index are added without changing the δ of low order. In this way, when we consider different regularities, the spaces that get mapped into themselves are nested as in (35).

Notice also that the conditions to obtain contraction do not change with r – we only establish contraction in a low regularity norm – and the conditions to get the space mapped into itself do not change either with higher r for $r \geq 2$. This is because the smallness assumptions that we take do not depend on derivatives higher than the second.

Notice that when we take the assumption (11), we obtain contraction only on C^2 spaces or spaces with higher regularity. As we will see in Example 5.3, this conclusion is sharp . Even in cases that we obtain a unique $C^{2-\delta}$ manifold, it is possible to obtain infinitely many manifolds which are $C^{2-2\delta}$.

We furthermore observe that if we obtain uniqueness in a ball, the uniqueness propagates to the basin of attraction of the origin. Given two invariant manifolds contained in the basin of attraction, their intersection with any bounded set will be eventually mapped into the ball taking enough iterations. Their images will also be invariant manifolds, which, by the previous uniqueness statement will have to agree. □

4.4.5. *Proof of Theorem 3.1 in the stable case when $r = \infty, \omega$.* The case C^∞ is a very simple consequence of the observations made in Remark 4.11.

We note that, we can find a sequence $\{\delta_i\}_{i\in\mathbb{N}}$ such that, for every $k \in \mathbb{N}$, $\chi_{\delta_0,\delta_1,\dots,\delta_k}$ is mapped into itself to $\mathcal{T}$ and $\mathcal{T}$ is a contraction in the distance $\|\!|\cdot|\!\|$.

Because of the nesting (35), the fixed points in those spaces χ have to coincide. The fixed point, therefore is in $\bigcap_k C^{\tilde{k}} = C^\infty$.

The analytic case is very simple. We work in a complex Banach space and we recall that differentiable functions in a complex Banach space are analytic (see [Kat95]).

If we carry out the C^2 proof as above, but in a complex ball, we obtain that the function is differentiable in the unit ball and, therefore analytic.

$\square$

4.5. **Proof of Theorem 3.1 in the Pseudo-stable case.** The proof follows very similar lines as the proof in the stable case. Nevertheless we have to make different choices of spaces and of norms. Indeed, the proof of the pseudo-stable cases is in many ways somewhat simpler since the devices of weighted norms that we used to in the contractive case are not useful for the case of pseudo-stable manifolds, so that the proofs will be more straighforward and we will not have to distinguish the cases of different regularities.

4.5.1. *The operator $\mathcal{T}$ is well defined.* The main difference with the stable case is that we no not have that $|D\Gamma| < 1$. Hence, we cannot ensure that $\Phi \circ \Gamma$ is defined in a ball if Φ is.

Hence, we have to consider spaces of functions Φ defined in the whole X_1 this in turns, forces us to consider non-linearities N that are uniformly small on $X_1 \times B_1(X_2)$. This is precisely what was accomplished in Section 4.1.1.

We will, therefore assume in the following that N is defined and small in $X_1 \times B_1(X_2)$. In this circumstances, the operator $\mathcal{T}$ is well defined.

4.5.2. *Some spaces of functions.* The spaces that we will consider are very similar to the spaces χ that we considered in Section 4.4.1.

We consider $r = k + \alpha$ as usual and consider the spaces

(36)
$$\begin{aligned} \tilde{\chi}_{\delta_0 \cdots \delta_k; \delta^\alpha} = \big\{ \Phi : X_1 \to X_2, \Phi \in C^r, &\|D^i \Phi\|_{C^0(X_1)} \leq \delta_i, i = 0, \ldots, r, \\ &H_\alpha(D^k \Phi) \leq \delta^\alpha \\ &\Phi(0) = 0, D\Phi(0) = 0 \} \end{aligned}$$

The spaces $\tilde{\chi}$ differ from the χ spaces because the functions in $\tilde{\chi}$ are defined in the whole space X_1 rather than just on the unit ball.

The main technical difference with the spaces $\tilde{\chi}$ is that we consider the $\tilde{\chi}$ spaces endowed with the $C^0(X_1)$ topology.

In the pseudo-stable case, we will not use weighted norms, since in this case, the extra factors $\|D\Gamma\|_{C^0}$ do not help since they are bigger than 1.

The analogues of Propositions 4.3 and 4.4 are the following results which are rather obvious.

Proposition 4.12. *Let* $\Gamma : X_1 \to X_1$ *be a* C^1 *function such that* $\Gamma(0) = 0$, $\|D\Gamma\|_{C^0(X_1)} \le 1$.

Then,

$$\|\Phi \circ \Gamma\|_{C^0(X_1)} \le \|\Phi\|_{C^0(X_1)}. \tag{37}$$

Proposition 4.13. *Let* $N_2 : X_1 \times B_1(X_2) \to X_2$ *satisfy* $DN_2(0,0) = 0$. *Assume that* $N_2 \in C^{1+\alpha}$, $0 \le \alpha \le \mathrm{Lip}$. *Let* Φ *be a* $C^{1+\alpha}$ *function,* $\Phi(0) = 0$, $D\Phi(0) = 0$.

Then, the function $\eta(y) = N_2(y, \Phi(y))$ *satisfies*

$$\|\eta\|_{C^0(X_1)} \le \|N_2\|_{C^0(X_1 \times B_1(X_2))}.$$

4.5.3. *The range of the operator* $\mathcal{T}$ *on the spaces* $\tilde{\chi}$. This section is an analogue in the pseudo-stable case of the results in Section 4.4.3. As we will see the methods are very similar even if the conclusions are different due to the fact that, possibly $\|A_1\| \ge 1$, hence high powers of of A_1 are not contractions and $\|A_2^{-1}\|\|A_1\|^k$ will not be a contraction for high k.

Lemma 4.14. *In the conditions of Theorem 3.1, assume that*

$$\|A_2^{-1}\|\|A_1\|^r < 1.$$

Denote $r = k + \alpha$ *as usual.*

Then, after making the adjustments in Section 4.1 so that $\|B\|$, $\|N\|_{C^1(X_1 \times B_1(X_2))}$ *is small enough, it is possible to find* $\delta_0, \dots, \delta_k; \delta^\alpha$, $\delta_0 = \delta_1 = 1$, $\delta_i \ge 0$, *is such a way that*

$$\mathcal{T}(\tilde{\chi}_{\delta_0,\delta_1 \cdots \delta_k;\delta^\alpha}) \subset \tilde{\chi}_{\delta_0,\dots,\delta_k;\delta^\alpha}.$$

Proof. The proof is very similar to – but simpler than – the proof of Lemma 4.6.

We just note that, because of (20) we have a direct analogue of (28) (we just need to change the domains where we carry out the estimates).

$$\|D^i\mathcal{T}[\Phi]\|_{C^0(X_1)} \le \|D\Phi^i\|_{C^0(X_1)}\big(\|A_1\| + 3\varepsilon\big)^i\|A_2^{-1}\| + P_i(\delta_0, \dots, \delta_{i-1}) \tag{38}$$

where P_i is a real polynomial with positive coefficients.

We also have an analogue of (29) just changing the domains.

$$H_{\alpha,N_1}(D^i\mathcal{T}[\Phi]) \le H_{\alpha,N_1}(D\Phi^i)\big(\|A_1\| + 3\varepsilon\big)^{i+\alpha}\|A_2^{-1}\| + P_i(\delta_1, \dots, \delta_{i-1}) \tag{39}$$

Hence, we proceed by induction assuming that we have set

$$\delta_0 = 1, \quad \delta_1 = 1.$$

As in the contractive case, we just note that this can be arranged if we arrange that $\|N\|_{C^1}, \|B\|$ are sufficiently small.

Assume inductively that we have determined $\delta_0 = 1, \delta_1 = 1, \dots, \delta_i$, Then, choose δ_{i+1} in such a way that

$$\delta_{i+1} \leq \gamma_{i+1}\delta_i + P_{i+1}(\delta_1, \dots, \delta_1) \tag{40}$$

where $\gamma_{i+1} = \|A_2^{-1}\|\|\|\Gamma\|_{C^1}^{i+1}$.

This choice of δ_{i+1} satisfying (40) is possible provide that $\gamma_{i+1} < 1$. By the assumptions in Theorem 3.1, we see that we can adjust that $\gamma_{i+1} < 1$ for $i \leq k$.

It is clear that if δ_{i+1} satisfies (40) and $\mathcal{T}(\chi_{\delta_0,\dots,\delta_i}) \subset \chi_{\delta_0,\dots,\delta_i}$, then $\mathcal{T}(\chi_{\delta_0,\dots,\delta_i,\delta_{i+1}}) \subset \chi_{\delta_0,\dots,\delta_i,\delta_{i+1}}$. Hence we can recursively, find the δ_i.

Similarly, using (39), we can ensure that we can find δ^α provided that $\|A_2^{-1}\|\|\|\Gamma\|_{C^1}^{k+\alpha} < 1$.

□

Remark 4.15. The main contrast with the proof of Lemma 4.6 is that in Lemma 4.6, the γ_i were decreasing as i increased. In the pseudo-stable case considered here, the γ_i are increasing with i and for large enough i the condition $\gamma_i < 1$ is violated. This is what makes the induction finding the δ stop and, hence, makes the proof stop for high regularity. As we will see in Example 5.4, this is not an artifact and there are examples where one cannot get more regularity than the regularity predicted by this argument. □

4.5.4. *Contraction properties of the operator $\mathcal{T}$ on $\tilde{\chi}$.* This section is quite analogous to Section 4.4.4. The main result is Lemma 4.16 whose proof goes along very similar lines as the proof of Lemma 4.9.

Lemma 4.16. *In the condition of Lemma 4.6 $\mathcal{T}$ is a contraction on $\chi_{\delta_0 \cdots \delta_r}$ with the norm* (22).

Again we remark that a useful heuristic idea is to note that, after we have performed all the preliminary adjustments in Section 4.1, the main part of the operator $\mathcal{T}$ is just

$$\mathcal{R}[\Phi] \equiv A_2^{-1}\Phi \circ A_1. \tag{41}$$

Hence, we expect that the estimates for $\mathcal{T}$ are similar to the estimates of the very simple operator

$$\|\mathcal{R}[\Phi] - \mathcal{R}[\tilde{\Phi}]\|_{C^0(X_1\times B_1(X_2))} \leq \|A_2\|^{-1}\|\Phi - \tilde{\Phi}\|_{C^0(X_1\times B_1(X_2))}$$

Now, we proceed to estimate the terms in the full operator $\mathcal{T}$. The most difficult terms will be those which include Φ evaluated at two different points – which depend on Φ. For this terms we will need to assume that Φ is at least Lipschitz. This is arranged by choosing the $\tilde{\chi}$ in such a way that the Lipschitz constant is bounded.

We note that if $\Phi, \tilde{\Phi} \in \tilde{\chi}$, we have

$$|N(y, \Phi(y)) - N(y, \tilde{\Phi}(y))| \le \varepsilon \|\Phi - \tilde{\Phi}\|_{C^0}$$

where, as before, we denote by ε terms that can be made as small as desired by choosing the adjustments in Section 4.1. A fortiori, we have similar bounds for the N_1, N_2.

Similarly

$$\begin{aligned} &|[A_1 y + B\Phi(y) + N_1(y, \Phi(y)) - N_2(y, \Phi(y))] \\ &\quad - [A_1 y + B\tilde{\Phi}(y) + N_1(y, \tilde{\Phi}(y)) - N_2(y, \tilde{\Phi}(y))]| \\ &\quad \le \varepsilon \|\Phi - \tilde{\Phi}\|_{C^0} \end{aligned}$$

Finally,

$$\begin{aligned} |\Phi(A_1 y + B\Phi(y) + N_1(y, \Phi(y)))) &- \tilde{\Phi}(A_1 y + B\tilde{\Phi}(y) + N_1(y, \tilde{\Phi}(y))))| \\ \le \Big| &\Phi(A_1 y + B\Phi(y) + N_1(y, \Phi(y)))) \\ &- \tilde{\Phi}(A_1 y + B\Phi(y) + N_1(y, \Phi(y))))\Big| \\ + \Big| &\tilde{\Phi}(A_1 y + B\Phi(y) + N_1(y, \Phi(y)))) \\ &- \tilde{\Phi}(A_1 y + B\tilde{\Phi}(y) + N_1(y, \tilde{\Phi}(y))))\Big| \\ \le \varepsilon &\|\Phi - \tilde{\Phi}\|_{C^0} + \mathrm{Lip}(\tilde{\Phi})\varepsilon \|\Phi - \tilde{\Phi}\|_{C^0}. \end{aligned}$$

Remark 4.17. In the proof we have presented, we only use the properties (12) to obtain that the spaces $\tilde{\chi}$ get mapped into each other. Nevertheless, the contraction part of the argument only uses that $\|A_2^{-1}\| < 1$.

It is possible – but we will not carry out the details here – to show that the operator $\mathcal{T}$ is a contraction on C^s. This property is sometime useful if one wants to validate the results of some numerical computations or to prove smooth dependence on parameters. □

Remark 4.18. Notice that we have established uniqueness of the fixed point in the spaces $\tilde{\chi}$ under the assumption that $\|N\|_{C^1}, \|B\|$ are small.

We note that this can be arranged by scaling and cut-off as indicated in Section 4.1. Nevertheless, it is important to note that the cut-off may affect the invariant manifold arbitrarily close to the origin.

Note that if $\|A_1\| \ge 1$, the invariance equation (18) can propagate an small disturbance.

We will illustrate this phenomenon in Example 5.3.

The uniqueness claimee here is obtained only in the spaces $\tilde{\chi}$ which incorporate some conditions of growth at infinity of the functions Φ. This conditions are very different from the conditions we obtained for the stable manifolds which were just regularity of the manifolds at the origin. □

Remark 4.19. A very well known consequence of the fact that the uniqueness is obtained only after we impose a cut-off is that center manifolds may fail to be C^∞. This comes about because the fixed points produced in a C^r space by carrying out some cut-offs are different from those of C^{r+k}, which requires a different cut-off. □

5. Some examples

In this section, we collect some examples that show that the results claimed in Theorem 3.1 cannot be improved in certain directions. Some of these examples are related to examples in [dlLW95], [dlL97], [CFdlL03a], [CFdlL03c].

In the first example, we show that the spectral gap conditions (11) cannot be improved.

Example 5.1. *Consider the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by:*

$$F(x_1, x_2) = \left(\frac{1}{2}x_1, \frac{1}{4}x_2 + x_1^2\right). \tag{42}$$

Then, the map does not have any C^2 invariant manifold tangent to the space $X_1 = \{(x, 0) | x \in \mathbb{R}\}$.

Note that the example satisfies (11) with inequality replaced by equality. All the other hypothesis of the theorem are satisfied.

Proof. Any C^2 invariant manifold can be tangent to X_1 at the origin can be written locally as the graph of an function $\Phi : X_1 \to X_2$.

The function Φ should satisfy the equation (17), which in our case reads

$$\Phi\left(\frac{1}{2}x_1\right) = \frac{1}{4}\Phi(x_1) + x_1^2 \tag{43}$$

Taking derivatives of (43) twice and evaluating at the origin, we obtain:

$$\left(\frac{1}{2}\right)^2 D^2\Phi(0) = \frac{1}{4}D^2\Phi(0) + 2$$

Clearly, this shows that there is no function Φ satisfying (43) and which has two derivatives at the origin. A fortiori, there is no C^2 function. □

Remark 5.2. We note that the main reason why Example 5.1 works is because $(1/2)^2 = 1/4$. Hence, there is a resonance. Indeed, the papers [dlL97], [ElB01], [CFdlL03a] study systems that satisfy non-resonance conditions rather than spectral gap conditions. □

Example 5.3. *Consider the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ in example 5.1.*

Given $\sigma > 0$, it admits infinitely many $C^{2-\sigma}$. invariant manifolds which are tangent to the space $X_1 = \{(x, 0)| x \in \mathbb{R}\}$.

Proof. Again, it suffices to produce solutions for (43).

We will recursively determine the functions on the intervals $I_i = [2^{-i-1}, 2^{-i}]$ starting with an arbitrary choice Φ_0 on the interval I_0 which has support in the interior of the interval.

We will write $\Phi(x_1) = \sum_i \Phi_i(x_1) + x_1^2$ where the Φ_i have support in the interior of I_i.

Notice that the equation of invariance is such that if we have the function Φ_i determined, we can find Φ_{i+1}. Indeed, we obtain that (43) is equivalent to

$$\Phi_{i+1}(x_1) = \frac{1}{4}\Phi_i(2x_1) + x_1^2 \tag{44}$$

Hence, we can define by recursion the Φ_i. By standard estimates, we have from (44)

$$\|\Phi_{i+1}\|_{C^{2-\sigma}(I_{i+1})} \le \frac{1}{4} 2^{2-\sigma} \|\Phi_i\|_{C^{2-\sigma}(I_i)} + 2^{2-\sigma}.$$

This shows that the series giving Φ converges uniformly in $C^{2-\sigma}$. □

Example 5.4. *We consider the function*

$$F(x_1, x_2) = \left(\frac{1}{2}x_1, \frac{1}{3}x_2 + \Psi(x_1)\right) \tag{45}$$

where Ψ is a C^∞ function with support contained in $(1, 2)$ which is not identically zero.

Then, there is one and only one invariant manifold which is a graph of a bounded function. For revery $\sigma > 0$, this function is $C^{\log 3/\log 2-\sigma}$. but not $C^{\log 3/\log 2+\sigma}$.

Proof. The equation for invariance is

$$\Phi(1/2\ x_1) = \frac{1}{3}\Phi(x_1) + \Psi(x_1) \tag{46}$$

Again, we write $\Phi = \sum_{i\in\mathbb{Z}} \Phi_i$ where Φ_i has support on $(2^i, 2^{i+1})$.

The equations for Φ_i are

$$\Phi_{i+1}(x_1) = 3\Phi_i(2x_1) \tag{47}$$

when $i \neq 1$ and

$$\Phi_1(x_1) = 3\Phi_0(2x_1) + \Psi(x_1) \tag{48}$$

Applying (47) repeatedly, we have

$$\Phi_n(x_1) = 3^n \Phi_0(2^n x_1).$$

Hence, the only possibility to make Φ bounded at ∞ is to have $\Phi_1 \equiv 0$. This can be arranged if we choose $\Phi_0 = -\Psi$.

Then, we are forced, applying (47), to have

$$\begin{aligned} \Phi_{-n}(x_1) &= 3^{-n-1}\Phi_1(2^{n+1}x_1) \\ &= 3^{-n-1}\Phi_1(2^{n+1}x_1). \end{aligned} \tag{49}$$

Unless the Ψ is identically zero, we see that (49) is not $C^{\log 3/\log 2+\sigma}$.

□

Note that we can run the argument in the proof of the statements in Example 5.4 run in the opposite direction. That is, we can argue that the only way to obtain a function satisfying (46) which is $C^{\log 3/\log 2+\sigma}$. in a neighborhood of the origin is to have $\Phi_1 \equiv 0$ which in turns forces exponential growth at infinity.

Hence, Example 5.4 is an example in which the two uniqueness conditions in Remark 4.11 and in Remark 4.18 are, so to speak orthogonal. Unless Ψ is zero, if one of them is satisfied the other is going to fail.

We also note that if we cut off the map (45) as indicated in Section 4.1, we may obtain just the linear map. Of course, bounded invariant manifolds for the linear map are just the linear spaces. This shows that, in the pseudo-stable case, the preparations in Section 4.1 do affect the manifolds produced.

Note also that in Example 5.4, we have the phenomenon that once the manifold is more regular than the critical regularity $\log 3/\log 2$ then it is C^∞.

We also note that in the case of the linear map, we have infinitely many invariant manifolds $x_2 = A|x_1|^{\log 3/\log 2}$ which are $C^{\log 3/\log 2}$. Hence uniqueness does not hold even in the critical regularity.

6. Results for flows

We do not formulate precisely the results for flows since the formulation for differentiable vector fields is quite standard. See e.g [CFdlL03a].

In Section 6.1, we will present a rigorous argument that shows that, when we have some uniqueness result for the maps, the results Theorem 3.1 for maps imply results for vector fields.

The main result is that, provided that the time-1 map of the flow verifies the assumptions of Theorem 3.1, the, the manifold invariant under the time-1 map produced by Theorem 3.1 is invariant under the flow.

Even if the argument in Section 6.1 gives a rigorous proof of results for flows by reducing them to results for maps, we think it is interesting to point that it is possible to carry out a proof for flows. In Section 6.2 we present an sketch – not a rigorous proof – of a direct proof of the result for flows. We hope that the reader will be able to fill details easily.

6.1. **Deducing results for flows from Theorem 3.1 for maps.**

Given a flow $\{S_t\}_{t\in\mathbb{R}}$ generate by a smooth vector field Y, we note that the assumptions of Theorem 3.1 can be satisfied for all the maps S_t. We furthermore have that $DS_t(0) = \exp(tDY(0)$. Hence, if $DY(0)$ has an invariant decomposition so does $DS_t(0)$. We will denote $DY(0) = A$ and denote the decomposition as in (14).

We also note that the results in Section 4.1 can be easily adapted for flows. Namely, by scaling we can ensure that $||DS_t(\cdot) - \exp(tA)||_{C^r(X_1)}$ is as small as desired for $t \in [0,1]$.

In the pseudo-stable case, we can proceed as in Section 4.1.1 to cut-off the vector field to ensure that the vector field is bounded in $X_1 \times B_2(X_2)$ and that therefore we can define S_t in $X_1 \times B_1(X_2)$ in such a way that it is close to linear.

If we apply Theorem 3.1 to each of the maps S_t we obtain an invariant manifold. The only thing that we have to worry about is whether these manifolds are the same for all the different values of t.

We note that since $S_t \circ S_s = S_s \circ S_t$, if a manifold $\mathcal{M}$ is invariant under S_t, then so is $S_s\mathcal{M}$.

If we have some uniqueness statement for invariant manifolds, we can conclude that $S_s\mathcal{M} = \mathcal{M}$. That is $\mathcal{M}$ is invariant for the whole flow.

In the stable case, the uniqueness statement that we can use is that if $\mathcal{M}$ is tangent at the origin to X_1, then, clearly so is $S_s\mathcal{M}$. Since, $S_s\mathcal{M}$ is also as regular as $\mathcal{M}$, then, the conclusions of Remark 4.11 allow us to conclude the desired result.

In the pseudo-stable case, the observation is that since $\mathcal{M}$ is the graph of a function $\Phi : X_1 \to X - 2$, which is uniformly bounded and S_s differs from $\exp sA$ by a map of small Lipschitz constant, we obtain, therefore that $S_s\mathcal{M}$ is also a graph.

Using the uniqueness statements in Remark 4.18, we obtain that $S_s\mathcal{M} = \mathcal{M}$.

Of course, for each of the stable or the pseudo-stable cases we have used different uniqueness statements. This is perfectly logical, of course. Nevertheless we point out that, as will be shown in Example 5.4, the manifolds selected by the each of the different uniqueness arguments may be different.

Remark 6.1. An important observation is that the above argument does not use at all the fact that the vector field X is differentiable. We only use the flow generated by the vector field is differentiable.

Important examples of this happen in Partial Differential equations where very often one has unbounded – hence discontinuous – operators generate very smooth flows. This is, of course very well known for a long time – see e.g [Sho97]. □

6.2. **Sketch of a direct proof.** Even if the argument presented in Section 6.1 gives a rigorous proof for flows from the results for maps, it is worth mentioning that one can also give a direct proof of the results for flows.

The method is a variant of the usual Perron's integral equation method. For the sake of completeness, we present an sketch of a direct proof. Even if this will not be a complete proof, we hope that this presentation may be useful for the readers who are more familiar with the proofs for flows than with the proofs for diffeomorphisms.

We write the differential equations generating the flow separating the components.

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + B x_2 + N_1(x_1, x_2); \\ \dot{x}_2 &= A_2 x_2 + N_2(x_1, x_2); \end{aligned} \tag{50}$$

As it is standard, we will first derive an equation for a function whose graph is invariant, we will show that the equation has a solution.

For the sake of simplicity, we will assume that the vector field is differentiable, even if it seems clear that some of the results would go through with the only assumptions that the vector field generates a smooth semigroup (satisfying appropriate growth conditions, which we detail later).

We note that if we have $x_2 = \Phi(x_1)$ we have

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + B\Phi(x_1) + N_1(x_1, \Phi(x_1)); \\ \dot{\Phi}(x_1) &= A_2 \Phi(x_1) + N_2(x_1, \Phi(x_1)); \end{aligned} \tag{51}$$

We note that, if we fix Φ, the first equation becomes an ODE for x_1. If Φ is Lipschitz, we see that the first equation of (51) will have a unique solution. We denote the solution of this equation with initial conditions x_1 by $\Gamma_t^\Phi(x_1)$.

We have that

$$|\Gamma_t^{\Phi}(x_1)| \le \|e^{A_1 t}\| e^{\varepsilon t} \qquad t \ge 0$$

where ε is a number which is arbitrarily small if we choose the arrangements in Section 4.1 to be small enough.

The idea of the proof is the same as the standard proof of the stable manifold by Perron's method (see e.g. [Hal80] Ch. VII pp. 225 ff.).

We use the variation of parameters formula in the second equation of (51) and obtain

$$\Phi(\Gamma_t^{\Phi}(x_1)) = e^{A_2 t}\Phi(x_1) + \int_0^t ds\ e^{A_2(t-s)} N_2(\Gamma_s^{\Phi}(x_1), \Phi(\Gamma_s^{\Phi}(x_1))) \tag{52}$$

Equivalently,

$$e^{-A_2 t}\Phi(\Gamma_t^{\Phi}(x_1)) = \Phi(x_1) + \int_0^t ds\ e^{-sA_2} N_2(\Gamma_s^{\Phi}(x_1), \Phi(\Gamma_s^{\Phi}(x_1))) \tag{53}$$

In the stable case, we note that, if we consider functions Φ which are C^2 and satisfy $\Phi(0) = 0$, $D\Phi(0) = 0$, we have

$$|e^{-A_2 t}\Phi(\Gamma_t^{\Phi}(x_1))| \le C\|e^{-A_2 t}\|(\|e^{A_1 t}\| e^{\varepsilon t})^2 \tag{54}$$

The condition

$$\|e^{-A_2 t}\|(\|e^{A_1 t}\| e^{\varepsilon t})^2 \le Ce^{\eta t} \tag{55}$$

for some $\eta > 0$, that the RHS of (55) goes to zero exponentially, is, clearly an analogue of (11).

Under this assumption (55), we obtain that the first term in (53) goes to 0 as $t \to \infty$. Hence, we obtain that a condition for invariance of the graph of Φ is

$$\Phi(x_1) = -\int_0^{\infty} ds\ e^{-sA_2} N_2(\Gamma_s^{\Phi}(x_1), \Phi(\Gamma_s^{\Phi}(x_1))) \tag{56}$$

Note also that (55) and the quadratic vanishing of N_2, Φ at the origin also imply that the integral in the RHS of (56) converges uniformly.

Hence, we will consider (56) as a fixed point equation for the operator defined by the RHS.

A useful heuristic guide is that this operator is very similar to

$$\int_0^{\infty} ds\ e^{-sA_2} N_2(e^{A_1 s}(x_1), \Phi(e^{A_1 s} x_1)))$$

Again, if we take spaces with weighted norms, we obtain that the operator obtained by composing Φ on the right with a contractive function has a norm which is bounded by the square of the contraction.

To show that the operator $\mathcal{T}$ is a contraction when topologized by the conical norm follows more or less the same line of argument than in the case of diffeomorphisms, namely adding and subtracting terms till we get to the heuristic principle.

The proof of flows is slightly more complicated than in the case of diffeomorphisms since we have to obtain estimates for $\Gamma_s^\Phi(x_1) - e^{A_1 s}x_1$. Of course, these can be obtained from the dependence on the solutions of ODE's on parameters, but they take longer to write than the analogues for diffeomorphisms.

We, of course, also have to show that the operator maps the χ spaces into themselves but the proof is extremely similar once we obtain formulas for the high derivatives of the operator $\mathcal{T}$ which separate the higher derivatives.

The pseudo-stable case is in many ways easier. In this case, we cannot assume that Γ_s^Φ is a contraction, so we need to make sure that the functions Φ have domain in all of X_1 and, therefore that the N are smooth in all of $X_1 \times B_1(X_2)$. This is done in Section 4.1.

Since we are assuming that e^{sA_2} is a contraction for large s, there is no problem showing that the RHS of (56) is a contraction in C^0 norm. Also, using the same formulas for the high derivatives, it is possible to show that the χ spaces get mapped into themselves.

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