Multiscale Finite Element Methods for Heterogeneous Porous Media

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Flow in Porous Media

Darcy's Law and Permeability

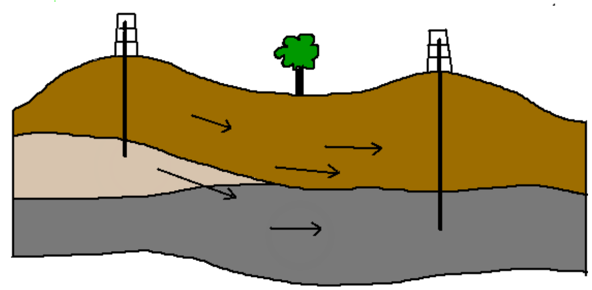
Darcy's empirical law (1856). The fluid velocity is proportional to the pressure gradient

$$\boxed{\mathbf{u} = -K\nabla p}$$

where

$\mathbf{u}(\mathbf{x})$ is the volumetric flux (the Darcy velocity)
$K(\mathbf{x})$ is the measured rock permeability divided by the fluid viscosity
$p(\mathbf{x})$ is the fluid pressure

Governing Equations of Single-phase Flow



Combined with conservation of mass, we obtain the second order elliptic system

$$\begin{cases} \mathbf{u} = -K\nabla p & \text{in } \Omega \quad \text{(Darcy's law)} \\ \nabla \cdot \mathbf{u} = f & \text{in } \Omega \quad \text{(conservation)} \\ \mathbf{u} \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

where

$f(\mathbf{x})$ is the source or sink term (i.e., wells)

Objective: Approximate $\mathbf{u}$ (and p) accurately.

A Variational Formulation

The differential problem:

$$\begin{cases} -\nabla \cdot K\nabla p = f & \text{in } \Omega \\ -K\nabla p \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

Function Space:

$$X = H^1/\mathbb{R} = \left\{ w \in L^2 : \nabla w \in (L^2)^3, \int_\Omega w\, dx = 0 \right\}$$

$$(\psi, \phi) = \int_\Omega \psi(\mathbf{x}) \cdot \phi(\mathbf{x})\, dx \qquad \text{(Inner-product)}$$

A variational problem:

Find $p \in X$ such that

$$a(p, w) \equiv (K\nabla p, \nabla w) = (f, w) \qquad \forall\ w \in X$$

Theorem: The two problems are equivalent, and there exists a unique solution.

Let $X_h \subset X$ be a finite dimensional subspace.

An approximate variational problem:

Find $p_h \in X_h$ such that

$$a(p_h, w_h) = (f, w_h) \qquad \forall\ w_h \in X_h$$

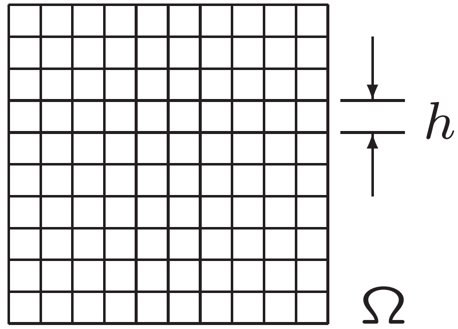
Theorem: There is $C > 0$ such that

$$\|p - p_h\|_1 \le C \min_{w_h \in X_h} \|p - w_h\|_1$$

where

$$\|w\|_1 = \left\{ \int_\Omega \left(|w|^2 + |\nabla w|^2 \right) dx \right\}^{1/2}$$

That is, up to C, the approximation is optimal.



Construction of X_h: Define a grid over Ω. Over each grid *element* E, let $w_h \in X_h$ be a polynomial. Piece them together so they are continuous.

Theorem: For polynomials of degree k,

$$\min_{w_h \in X_h} \|p - w_h\|_1 \le C\|p\|_{k+1} h^k \quad \text{where} \quad \|w\|_k = \left\{ \sum_{|\alpha| \le k} \int_\Omega |D^\alpha w|^2 \, dx \right\}^{1/2}$$

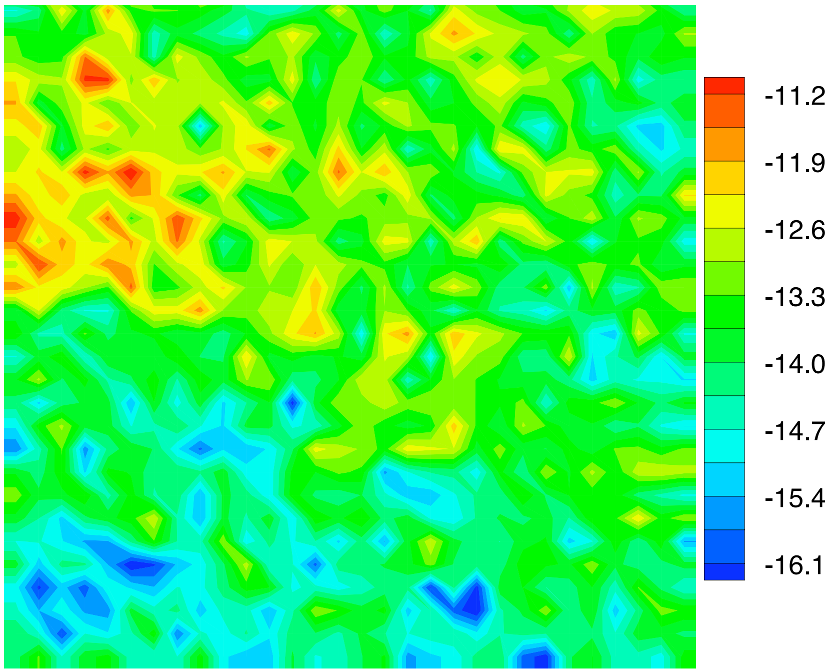
Corollary: $p_h \to p$ as $h \to 0$. In fact,

$$\|p - p_h\|_1 \le C\|p\|_{k+1} h^k = \mathcal{O}(h^k)$$

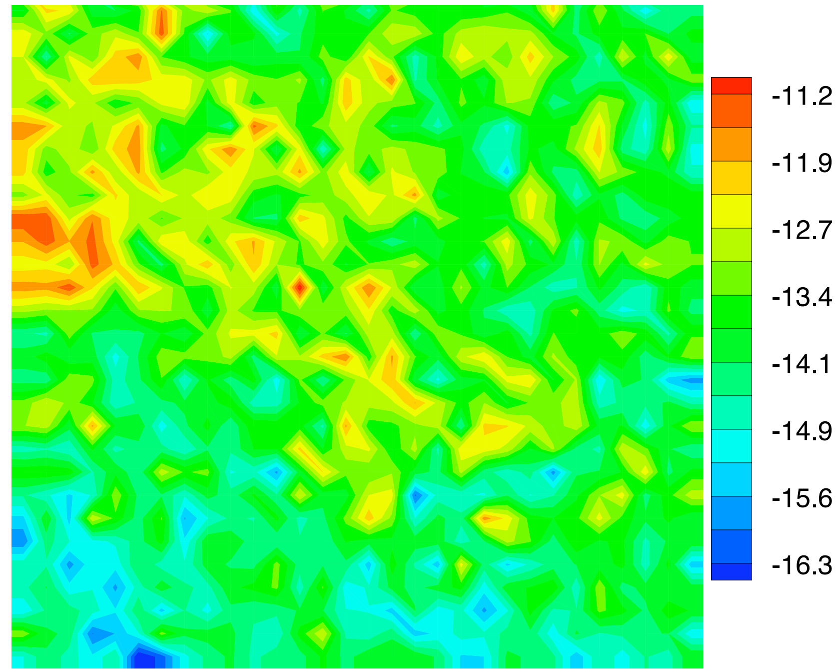
Heterogeneity and Problems of Scale

Meter-Scale Natural Heterogeneity

Log10 X Permeability of Lawyer Canyon



Log10 Z Permeability of Lawyer Canyon



Lawyer Canyon data, meter scale
(ranges by a factor of 10^6)

Difficulty: Fine-scale variation in K (the *permeability*) leads to fine-scale variation in the solution $(\mathbf{u}, p)$.

The Problem of Scale

Suppose K varies on the scale ϵ. Then

$$|\nabla p| = \mathcal{O}(\epsilon^{-1}) \qquad \text{and} \qquad |D^k p| = \mathcal{O}(\epsilon^{-k})$$

Typical error estimates. From polynomial approximation theory, the best approximation on a finite element partition $\mathcal{T}_h$ is

$$\inf_{q \in \mathbb{P}_{k-1}(\mathcal{T}_h)} \|p - q\|_0 \le C\|p\|_k h^k \sim C\left(\frac{h}{\epsilon}\right)^k$$

- If $h > \epsilon$, this is *not* small!
- To resolve p, we need a spatial discretization $h < \epsilon$.
  That is, we must resolve K in some way!

Multiscale Finite Element Methods

Multiscale Approaches

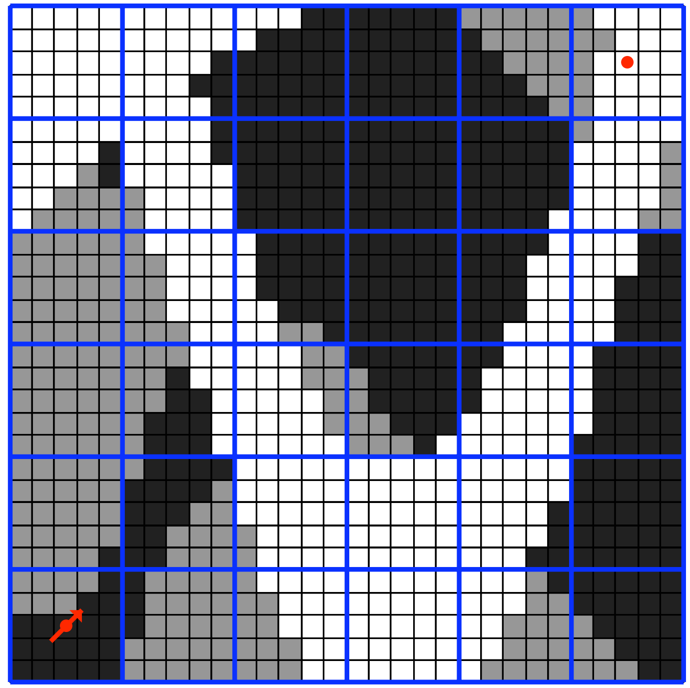
Objective. We want to solve the problem in a way that:

- does not fully incorporate the problem dynamics (i.e., solves some global coarse scale problem to resolution $H > \epsilon$),
- yet captures significant features of the solution, by taking into account the micro-structure (to resolution $h < \epsilon$).

Possible solutions. (Sorry, this is a very incomplete list!)

- **Multiscale finite elements**
  1. Babuška, Caloz & Osborn 1994
  2. Hou & Wu 1997
  3. Hou, Wu & Cai 1999
  4. Efendiev, Hou & Wu 2000
  5. Strouboulis, Copps & Babuška 2001
  6. Chen & Hou 2003
  7. Aarnes 2004
  8. Aarnes, Krogstad & Lie 2006
- **Multiscale finite volumes**
  1. Jenny, Lee & Tchelepi 2003
  2. He & Ren 2004
- **Multiscale basis optimization**
  1. Rath 2007 (Ph.D. dissertation)
- **Variational multiscale analysis**
  1. Hughes 1995
  2. Hughes, Feijóo, Mazzei & Quincy 1998
  3. Arbogast, Minkoff & Keenan 1998
  4. Brezzi 1999
  5. Arbogast 2004
  6. Arbogast & Boyd 2006
- **Multiscale mortar methods**
  1. Arbogast, Pencheva, Wheeler & Yotov 2007
- **Heterogeneous multiscale methods**
  1. E & Engquist 2003

Remark. These are really the same general method!

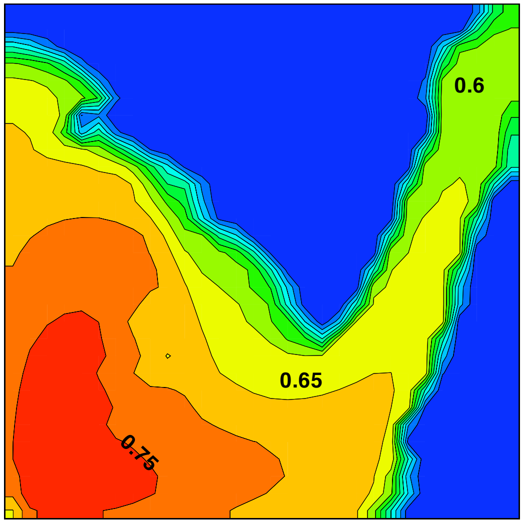


K = 0.1 D
K = 1.0 D
K = 10.0 D

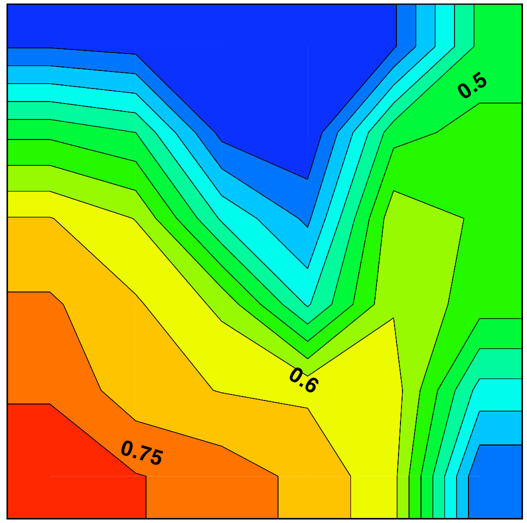
Permeability field

(White and Horne, 1987)

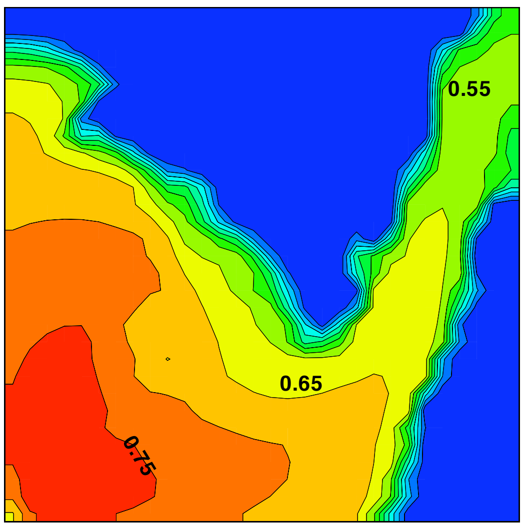
A Fluvial Subsurface Environment—2



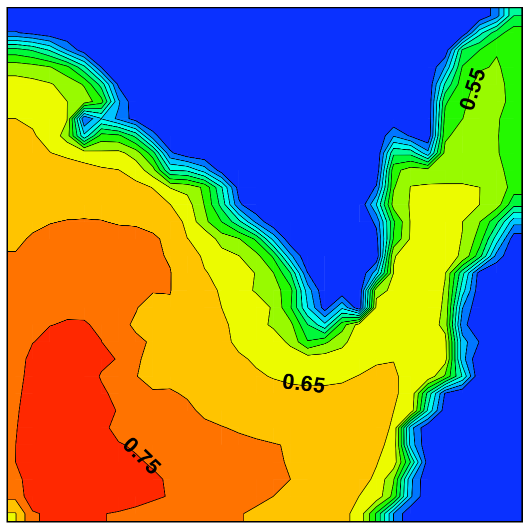
Fine 30×30



Average K 6×6



Upscaled to 6×6



Upscaled to 3×3

Water saturation contours

Using average parameters smears the solution (as expected).

This is problematic for nonlinearities!

$$F(\text{avg}(x)) \neq \text{avg}\big(F(x)\big)$$

Variational Multiscale Methods

The Variational Multiscale Method
(Hughes et al., 1995, 1998; Brezzi, 1999)

Goal: Find the part of the solution that is *unresolved* in standard finite element approximation.

Problem: Find $u \in X$ such that

$$a(u, v) = f(v) \qquad \forall v \in X$$

Direct sum decomposition: Define *coarse* and *fine* (i.e., *subgrid*) scales

$$X = \bar{X} \oplus X'$$

Then $u = \bar{u} + u'$ is uniquely decomposed.

Separating scales: Find $\bar{u} \in \bar{X}$ and $u' \in X'$ such that

$$a(\bar{u} + u', \bar{v}) = f(\bar{v}) \qquad \forall \bar{v} \in \bar{X} \qquad \text{(coarse scale)}$$
$$a(\bar{u} + u', v') = f(v') \qquad \forall v' \in X' \qquad \text{(subgrid scale)}$$

Closure Operator

We can define $u' : \bar{X} \rightarrow X'$ by

$$a(\bar{v} + u'(\bar{v}), v') = f(v') \qquad \forall v' \in X'$$

Affine representation: Define the linear operator $\widehat{u}' : \bar{X} \rightarrow X'$ by

$$a(\bar{v} + \widehat{u}'(\bar{v}), v') = 0 \qquad \forall v' \in X'$$

and constant term $\tilde{u}' \in X'$ by

$$a(\tilde{u}', v') = f(v') \qquad \forall v' \in X'$$

Then

$$u' = u'(\bar{u}) = \widehat{u}'(\bar{u}) + \tilde{u}'$$

Remark: Given the coarse scale, we recover the fine-scale. In upscaling theory, closure operators are often *assumed* rather than being *derived*. Hence the term *subgrid upscaling*.

Upscaling the Problem

Upscaled problem: Find $\bar{u} \in \bar{X}$ such that

$$a(\bar{u} + \hat{u}'(\bar{u}), \bar{v}) = f(\bar{v}) - a(\tilde{u}', \bar{v}) \qquad \forall \bar{v} \in \bar{X}$$

or, in symmetric form,

$$a(\bar{u} + \hat{u}'(\bar{u}), \bar{v} + \hat{u}'(\bar{v})) = f(\bar{v}) - a(\tilde{u}', \bar{v}) \qquad \forall \bar{v} \in \bar{X}$$

Change of scale results in modifying both a and f:

$$\mathcal{A}(\bar{u}, \bar{v}) = F(\bar{v}) \qquad \forall \bar{v} \in \bar{X}$$

where

$$\begin{array}{lll} \mathcal{A} : \bar{X} \times \bar{X} \rightarrow \mathbb{R} & \text{is} & \mathcal{A}(\bar{u}, \bar{v}) = a(\bar{u} + \hat{u}'(\bar{u}), \bar{v} + \hat{u}'(\bar{v})) \\ F : \bar{X} \rightarrow \mathbb{R} & \text{is} & F(\bar{v}) = f(\bar{v}) - a(\tilde{u}', \bar{v}) \end{array}$$

Full two-scale solution:

$$u = \bar{u} + u'(\bar{u}) = \bar{u} + \hat{u}'(\bar{u}) + \tilde{u}'$$

Approximation

Upscaled problem: Find $\bar{u} \in \bar{X}$ such that

$$a(\bar{u} + \hat{u}'(\bar{u}), \bar{v} + \hat{u}'(\bar{v})) = f(\bar{v}) - a(\tilde{u}', \bar{v}) \qquad \forall \bar{v} \in \bar{X}$$

Finite Element Approximation: Find $\bar{u}_h \in \bar{X}_h \subset \bar{X}$ such that

$$a(\bar{u}_h + \hat{u}'(\bar{u}_h), \bar{v}_h + \hat{u}'(\bar{v}_h)) = f(\bar{v}_h) - a(\tilde{u}', \bar{v}_h) \qquad \forall \bar{v}_h \in \bar{X}_h$$

Multiscale Finite Element Space: Let

$$\hat{X}_h = \{\bar{v}_h + \hat{u}'(\bar{v}_h) : \bar{v}_h \in \bar{X}_h\}$$

Note that $\dim \hat{X}_h = \dim \bar{X}_h$.

Equivalent form: Find $u_h \in \hat{X}_h + \tilde{u}'$ such that

$$a(u_h, \hat{v}_h) = f(\hat{v}_h) \qquad \forall\ \hat{v}_h \in \hat{X}_h$$

Remark: The key is to find a decomposition $X = \bar{X} \oplus X'$ so that we can efficiently compute the upscaling operator $\hat{u}'$ on $\bar{X}_h$.

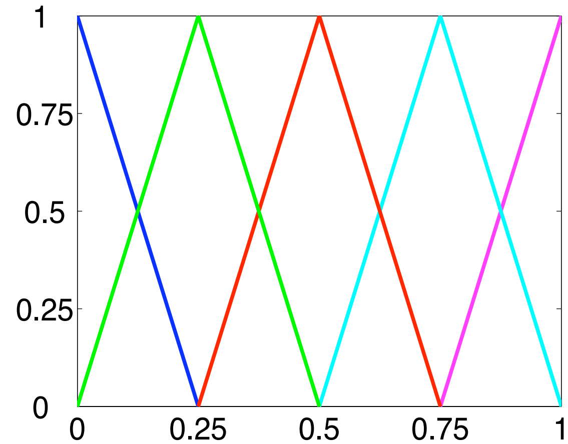
A Simple Example—1

(Babuška and Osborn, 1983; Hou and Wu, 1997)

Differential problem.

$$\begin{cases} -\dfrac{d}{dx}\left(K\dfrac{dp}{dx}\right) = 0, \quad 0 < x < 1 \\ p(0) = 0 \quad \text{and} \quad p(1) = 1 \end{cases}$$

Standard finite elements.



The solution $p(x) \in X + x$,

$$X = \left\{ w \in H^1 : w(0) = w(1) = 0 \right\}$$

satisfies

$$a(p, w) \equiv (Kp_x, w_x) = 0 \qquad \forall w \in X.$$

Constructing $\bar{X}_h$:

- Choose a uniform grid of five points: $x_i = i/4$, $i = 0, 1, 2, 3, 4$.
- Let $\bar{X}_h$ be linear on each element.

Two-scale decomposition: $X = \bar{X} \oplus X'$.

Let X' be the "bubble functions" over the grid

$$X' = \{w \in H^1 : w(x_i) = 0,\ \ i = 0, 1, 2, 3, 4\}$$

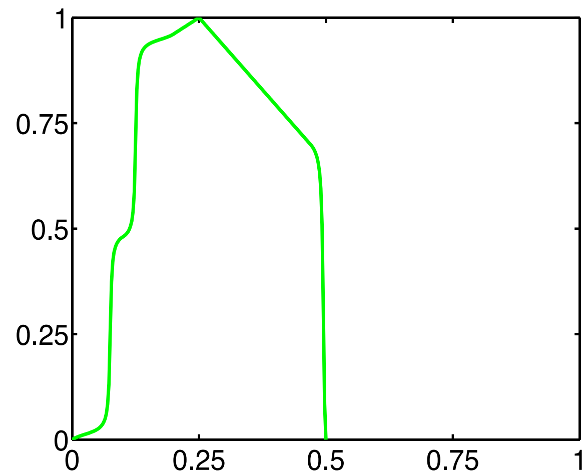
Localization: $\hat{u}'(\bar{v})$ breaks into 4 small or localized problems!

$$a(\bar{v} + \hat{u}'(\bar{v}), v') = 0 \quad \forall v' \in X' \quad \text{on } (x_{i-1}, x_i),\ \ i = 1, 2, 3, 4$$

Constructing X_h: $\psi = \bar{w} + \hat{u}'(\bar{w})$

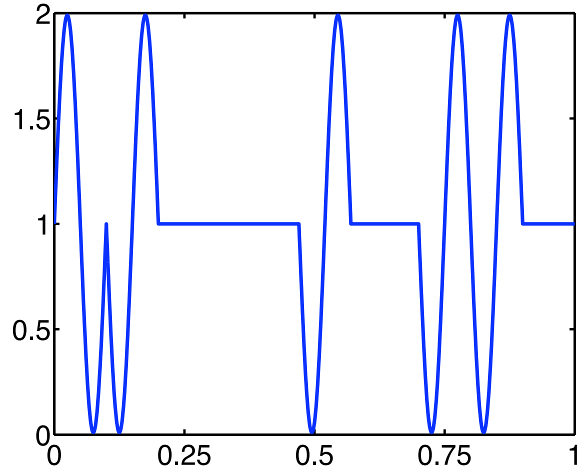
$$\begin{cases} -\dfrac{d}{dx}\left(K\dfrac{d\psi}{dx}\right) = 0 \quad 0 < x < 0.25 \\ \psi(0) = 0 \quad \text{and} \quad \psi(0.25) = 1 \end{cases}$$

$$\begin{cases} -\dfrac{d}{dx}\left(K\dfrac{d\psi}{dx}\right) = 0 \quad 0.25 < x < 0.5 \\ \psi(0.25) = 1 \quad \text{and} \quad \psi(0.5) = 0 \end{cases}$$

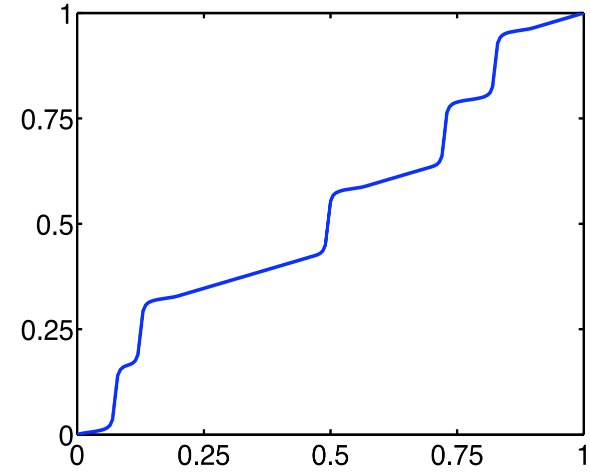


Multiscale basis function

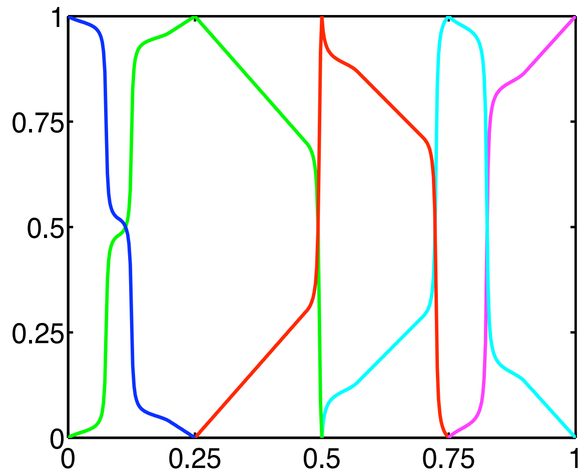
A Simple Example—3



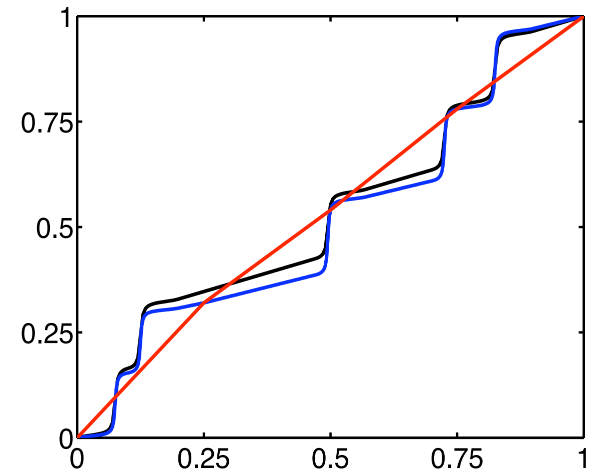
Coefficient K



True solution p



Multiscale basis functions



Multiscale vs. Standard solution

Mixed Variational Multiscale Methods

Upscaling Second Order Elliptic PDE'S in Mixed Form
(Arbogast et al., 1998; Arbogast, 2000; 2004)

$$\begin{cases} K^{-1}\mathbf{u} = -\nabla p & \text{in } \Omega \quad \text{(Darcy's law)} \\ \nabla\cdot\mathbf{u} = f & \text{in } \Omega \quad \text{(conservation)} \\ \mathbf{u}\cdot\nu = 0 & \text{on } \partial\Omega \end{cases}$$

Spaces:

$$W = L^2/\mathbb{R}$$

$$\mathbf{V} = H(\mathrm{div}) = \{\mathbf{v} \in (L^2)^3 : \nabla\cdot\mathbf{v} \in L^2,\ \mathbf{v}\cdot\nu = 0 \text{ on } \partial\Omega\}$$

$$(\psi,\phi) = \int_\Omega \psi(\mathbf{x})\cdot\phi(\mathbf{x})\,dx \qquad \text{(Inner-product)}$$

A mixed variational formulation:

Find $p \in W$ and $\mathbf{u} \in \mathbf{V}$ such that

$$(K^{-1}\mathbf{u},\mathbf{v}) = (p,\nabla\cdot\mathbf{v}) \qquad \forall\ \mathbf{v}\in\mathbf{V} \qquad \text{(Darcy's law)}$$

$$(\nabla\cdot\mathbf{u},w) = (f,w) \qquad \forall\ w\in W \qquad \text{(conservation)}$$

A Two-Scale Expansion

Define a coarse computational grid on Ω.

Pressure space: $W = \bar{W} \oplus W'$

$$\bar{W} \supset \{\bar{w} \in W : \bar{w} \text{ is constant } \forall \text{ coarse elements } E_c\}$$
$$W' = \bar{W}^{\perp}$$

Velocity space: $\mathbf{V} = \bar{\mathbf{V}} \oplus \mathbf{V}'$

$$\bar{\mathbf{V}} \subset \{\mathbf{v} \in \mathbf{V} : \nabla \cdot \mathbf{v} \in \bar{W}\} \quad \text{(conservation)}$$
$$\mathbf{V}' = \{\mathbf{v}' \in \mathbf{V} : \nabla \cdot \mathbf{v}' \in W',\ \mathbf{v}' \cdot \nu = 0 \text{ on } \partial E_c \ \forall\ E_c\} \quad \text{(locality)}$$

such that

$$(a) \quad \nabla \cdot \bar{\mathbf{V}} = \bar{W} \quad \text{(coarse conservation)}$$
$$(b) \quad \nabla \cdot \mathbf{V}' = W' \quad \text{(subgrid conservation)}$$

Separation of Scales

Separate scales uniquely via the direct sum as

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}' \in \bar{\mathbf{V}} \oplus \mathbf{V}'$$
$$p = \bar{p} + p' \in \bar{W} \oplus W'$$

Coarse:

$$(K^{-1}(\bar{\mathbf{u}} + \mathbf{u}'), \bar{\mathbf{v}}) = (\bar{p}, \nabla \cdot \bar{\mathbf{v}}) \qquad \forall\ \bar{\mathbf{v}} \in \bar{\mathbf{V}}$$
$$(\nabla \cdot \bar{\mathbf{u}}, \bar{w}) = (f, \bar{w}) \qquad \forall\ \bar{w} \in \bar{W}$$

Subgrid:

$$(K^{-1}(\bar{\mathbf{u}} + \mathbf{u}'), \mathbf{v}') = (p', \nabla \cdot \mathbf{v}') \qquad \forall\ \mathbf{v}' \in \mathbf{V}'$$
$$(\nabla \cdot \mathbf{u}', w') = (f, w') \qquad \forall\ w' \in W'$$

The Closure Operator

Constant part: Define $(\tilde{p}', \tilde{\mathbf{u}}') \in W' \times \mathbf{V}'$ by

$$(K^{-1}\tilde{\mathbf{u}}', \mathbf{v}') = (\tilde{p}', \nabla \cdot \mathbf{v}') \qquad \forall\ \mathbf{v}' \in \mathbf{V}'$$

$$(\nabla \cdot \tilde{\mathbf{u}}', w') = (f, w') \qquad \forall\ w' \in W'$$

Linear part: For $\bar{\mathbf{v}} \in \bar{\mathbf{V}}$, define $(\hat{p}', \hat{\mathbf{u}}') \in W' \times \mathbf{V}'$

$$(K^{-1}(\bar{\mathbf{v}} + \hat{\mathbf{u}}'), \mathbf{v}') = (\hat{p}', \nabla \cdot \mathbf{v}') \qquad \forall\ \mathbf{v}' \in \mathbf{V}'$$

$$(\nabla \cdot \hat{\mathbf{u}}', w') = 0 \qquad \forall\ w' \in W'$$

Then

$$p' = \hat{p}'(\bar{\mathbf{u}}) + \tilde{p}'$$
$$\mathbf{u}' = \hat{\mathbf{u}}'(\bar{\mathbf{u}}) + \tilde{\mathbf{u}}'$$

The Upscaled Equation

The coarse scale equation, in symmetric form, is:

Find $(\bar{p}, \bar{\mathbf{u}}) \in \bar{W} \times \bar{\mathbf{V}}$ such that

$$\begin{aligned} &(K^{-1}(\bar{\mathbf{u}} + \hat{\mathbf{u}}'(\bar{\mathbf{u}})), (\bar{\mathbf{v}} + \hat{\mathbf{u}}'(\bar{\mathbf{v}}))) \\ &\qquad = (\bar{p}, \nabla \cdot \bar{\mathbf{v}}) - (K^{-1}\tilde{\mathbf{u}}', \bar{\mathbf{v}}) \qquad \forall\ \bar{\mathbf{v}} \in \bar{\mathbf{V}} \\ &(\nabla \cdot \bar{\mathbf{u}}, \bar{w}) = (f, \bar{w}) \qquad \forall\ \bar{w} \in \bar{W} \end{aligned}$$

Full solution:

$$p = \bar{p} + \hat{p}'(\bar{\mathbf{u}}) + \tilde{p}'$$
$$\mathbf{u} = \bar{\mathbf{u}} + \hat{\mathbf{u}}'(\bar{\mathbf{u}}) + \tilde{\mathbf{u}}'$$

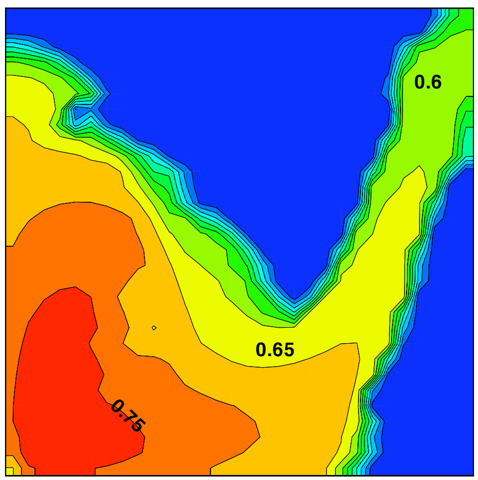
We can also rewrite the problem as

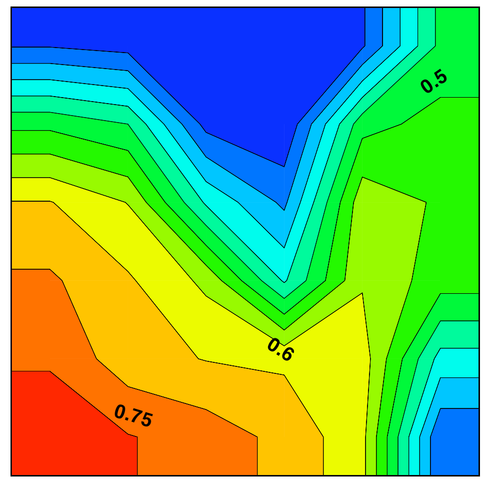
Find $(\bar{p}, \bar{\mathbf{u}}) \in \bar{W} \times \bar{\mathbf{V}}$ such that

$$\begin{aligned}(K^{-1}\bar{\mathbf{u}}, \bar{\mathbf{v}}) &- (K^{-1}\hat{\mathbf{u}}'(\bar{\mathbf{u}}), \hat{\mathbf{u}}'(\bar{\mathbf{v}})) \\ &= (\bar{p}, \nabla \cdot \bar{\mathbf{v}}) - (K^{-1}\tilde{\mathbf{u}}', \bar{\mathbf{v}}) \qquad \forall\ \bar{\mathbf{v}} \in \bar{\mathbf{V}}\end{aligned}$$

$$(\nabla \cdot \bar{\mathbf{u}}, \bar{w}) = (f, \bar{w}) \qquad \forall\ \bar{w} \in \bar{W}$$

Thus the subscale correction is antidiffusive on the coarse scale.





Fine 30×30 Average K coarse 6×6

Numerical Approximation

Choose any mixed space $\bar{\mathbf{V}}_H \times \bar{W}_H$ on the coarse mesh.

Formulation 1: Find $(\bar{\mathbf{u}}_H, \bar{p}_H) \in \bar{\mathbf{V}}_H \times \bar{W}_H$ such that

$$\begin{aligned} &(K^{-1}(\bar{\mathbf{u}}_H + \hat{\mathbf{u}}'(\bar{\mathbf{u}}_H)), (\bar{\mathbf{v}}_H + \hat{\mathbf{u}}'(\bar{\mathbf{v}}_H))) \\ &\qquad = (\bar{p}_H, \nabla \cdot \bar{\mathbf{v}}_H) - (K^{-1}\tilde{\mathbf{u}}', \bar{\mathbf{v}}_H) && \forall\ \bar{\mathbf{v}}_H \in \bar{\mathbf{V}}_H \\ &(\nabla \cdot \bar{\mathbf{u}}_H, \bar{w}_H) = (f, \bar{w}_H) && \forall\ \bar{w}_H \in \bar{W}_H \end{aligned}$$

Then

$$\begin{aligned} \mathbf{u} \approx \mathbf{u}_H &= \bar{\mathbf{u}}_H + \hat{\mathbf{u}}'(\bar{\mathbf{u}}_H) + \tilde{\mathbf{u}}' \\ p \approx p_H &= \bar{p}_H + \hat{p}'(\bar{\mathbf{u}}_H) + \tilde{p}' \end{aligned}$$

Formulation 2: Define

$$\hat{\mathbf{V}}_H = \{\bar{\mathbf{v}}_H + \hat{\mathbf{u}}'(\bar{\mathbf{v}}_H) \ :\ \bar{\mathbf{v}}_H \in \bar{\mathbf{V}}_H\}$$

Find $\mathbf{u}_H \in \hat{\mathbf{V}}_H + \tilde{\mathbf{u}}'$ and $\bar{p}_H \in \bar{W}_H$ such that

$$\begin{aligned} (K^{-1}\mathbf{u}_H, \hat{\mathbf{v}}_H) &= (\bar{p}_H, \nabla \cdot \hat{\mathbf{v}}_H) && \forall\ \hat{\mathbf{v}}_H \in \hat{\mathbf{V}}_H \\ (\nabla \cdot \mathbf{u}_H, \bar{w}_H) &= (f, \bar{w}_H) && \forall\ \bar{w}_H \in \bar{W}_H \end{aligned}$$

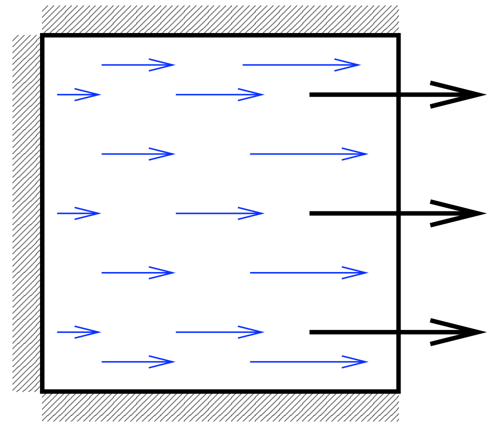
Remark: We have some multiscale finite elements!

The Lowest Order Mixed Finite Elements

On a coarse element E with edge e.

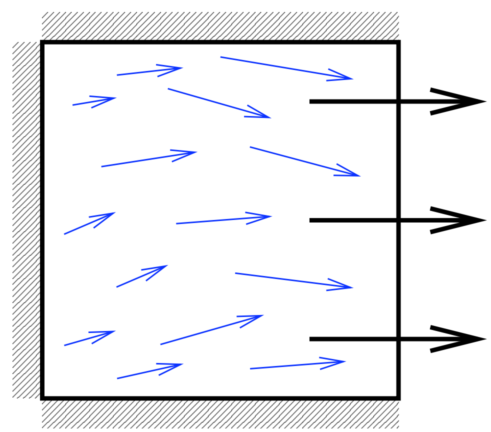
Standard Raviart-Thomas (RT0) finite element.

$$\begin{cases} R_e = -\nabla\omega \\ \nabla\cdot R_e = 1/|E| \\ R_e\cdot\nu = \begin{cases} 1/|e| & \text{on } e \\ 0 & \text{otherwise} \end{cases} \end{cases}$$



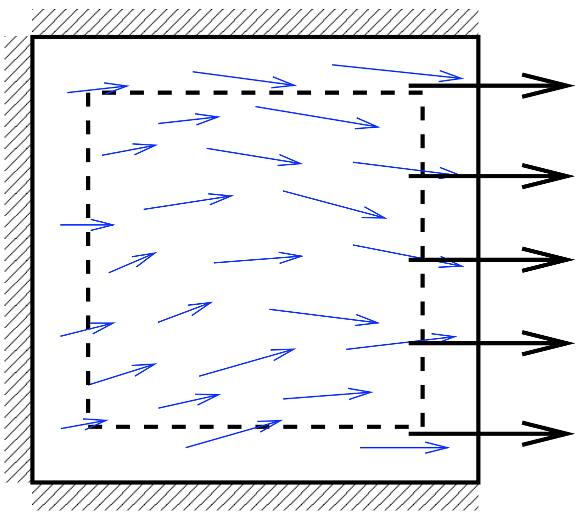
Variational multiscale finite element: $R_e^{\mathsf{MS}} = R_e + \hat{\mathbf{u}}_e'$

$$\begin{cases} R_e^{\mathsf{MS}} = -K\nabla\omega \\ \nabla\cdot R_e^{\mathsf{MS}} = 1/|E| \\ R_e^{\mathsf{MS}}\cdot\nu = \begin{cases} 1/|e| & \text{on } e \\ 0 & \text{otherwise} \end{cases} \\ \nabla\cdot\hat{\mathbf{u}}_e' = 0 \\ \hat{\mathbf{u}}_e'\cdot\nu = 0 \end{cases}$$

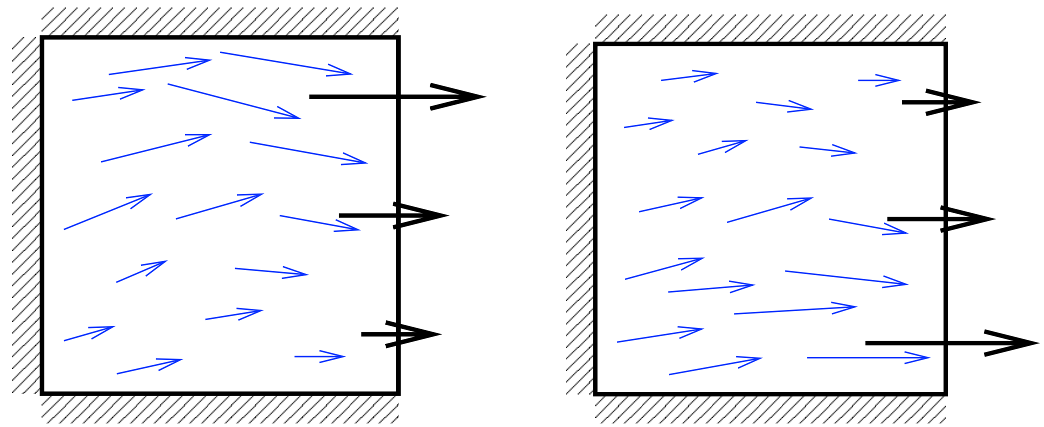


Boundary Conditions for Local Subproblems

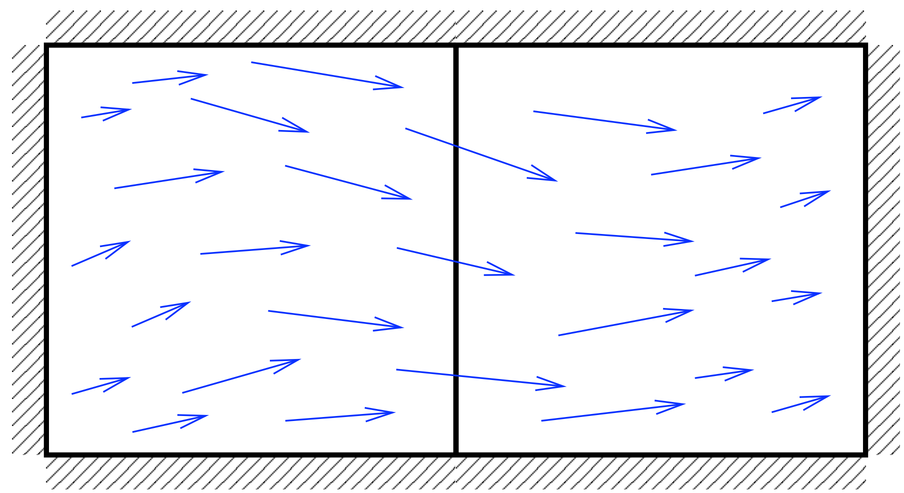
Neumann BCs for constant outflow, but oversample. (Hou et al., 1997, 2003) Results in a nonconforming method.



Neumann BCs for linear outflow. (Arbogast, 2000)



Dual element problem with source and sink terms. (Aarnes et al., 2004)



Estimates of the Pressure and Velocity Errors

Optimal Error Estimates

$$\mathbf{u} \approx \mathbf{u}_H = \bar{\mathbf{u}}_H + \hat{\mathbf{u}}'(\bar{\mathbf{u}}_H) + \tilde{\mathbf{u}}'$$
$$p \approx p_H = \bar{p}_H + \hat{p}'(\bar{\mathbf{u}}_H) + \tilde{p}'$$

Theorem (A., 2004).

$$\|K^{-1/2}(\mathbf{u} - \mathbf{u}_H)\|_0 \leq \inf_{\substack{\mathbf{v}_H \in \bar{\mathbf{V}}_H + \mathbf{V}' \\ \nabla\cdot\mathbf{v}_H = f}} \|K^{-1/2}(\mathbf{u} - \mathbf{v}_H)\|_0$$

$$\nabla \cdot \mathbf{u}_H = f$$

Remark: We have assumed that the upscaling operator is solved exactly, since it can be well resolved on a fine grid.

Let L be the order of approximation of the coarse mixed finite element velocity space used. Typically:

$L = 1$ for lowest order Raviart-Thomas (RT0) spaces
$L = 2$ for lowest order Brezzi-Douglas-Marini (BDM1) spaces

Theorem (A., 2004).

$$\|\mathbf{u} - \mathbf{u}_H\|_0 \leq C\|\mathbf{u}\|_L H^L = O(H^L)$$
$$\nabla \cdot \mathbf{u}_H = f$$
$$\|p - p_H\|_0 \leq C\|\mathbf{u}\|_L H^{L+1} = O(H^{L+1})$$

Suppose that K is locally periodic of period ϵ. Then

$$K(x) = \kappa(x, x/\epsilon)$$

where $\kappa(x, y)$ is periodic in y of period 1 on the unit cube Y.

Let K_0 be the homogenized permeability matrix, defined by

$$K_{0,ij}(x) = \int_Y \kappa(x, y) \left(\delta_{ij} - \frac{\partial \chi^j(x, y)}{\partial y_i} \right) dy$$

where, for fixed x, $\chi^j(x, y)$ is the Y-periodic solution of

$$\nabla_y \cdot (\kappa \nabla_y \chi^j) = \frac{\partial \kappa}{\partial y_j}$$

Homogenized solution: Let $(\mathbf{u}_0, p_0)$ solve

$$\begin{cases} \mathbf{u}_0 = -K_0 \nabla p_0 & \text{in } \Omega \\ \nabla \cdot \mathbf{u}_0 = f & \text{in } \Omega \\ \mathbf{u}_0 \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

Then $(\mathbf{u}_0, p_0)$ is a smooth “approximation” of $(\mathbf{u}, p)$.

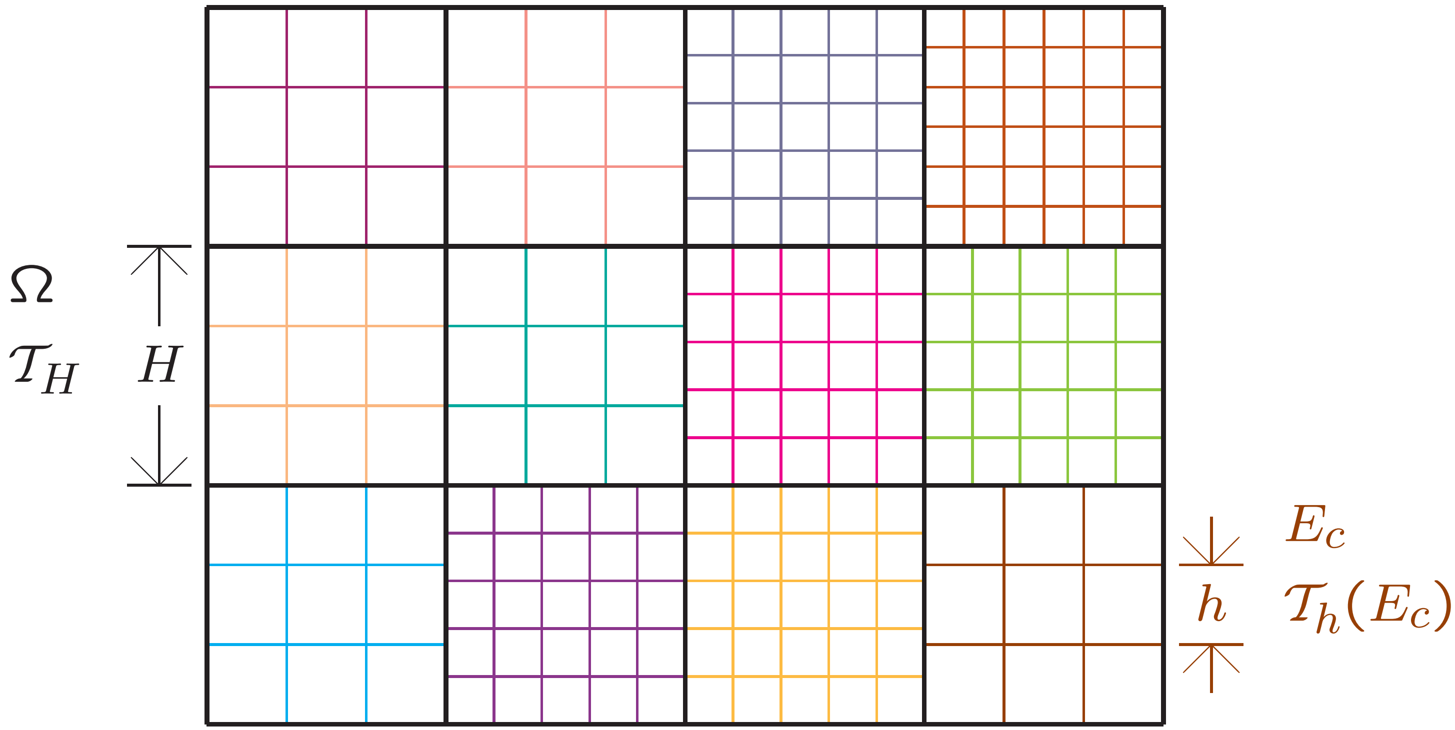
Theorem (Chen and Hou, 2003; A. and Boyd, 2005). Assuming periodicity and the mixed variational multiscale method with $L = 1$ (RT0) or 2 (BDM1):

$$\|\mathbf{u} - \mathbf{u}_H\|_0 \le C\left\{\epsilon\|p_0\|_2 + \sqrt{\frac{\epsilon}{H}}\|p_0\|_{1,\infty} + H^L\left(\|\mathbf{u}_0\|_L + \|f\|_{L-1}\right)\right\}$$
$$= O(H^L + \sqrt{\epsilon/H})$$
$$\|p - p_H\|_0 \le C\left(\epsilon + (\epsilon/H)^{1/d-\eta} + H\right)\|\mathbf{u} - \mathbf{u}_H\|_0$$

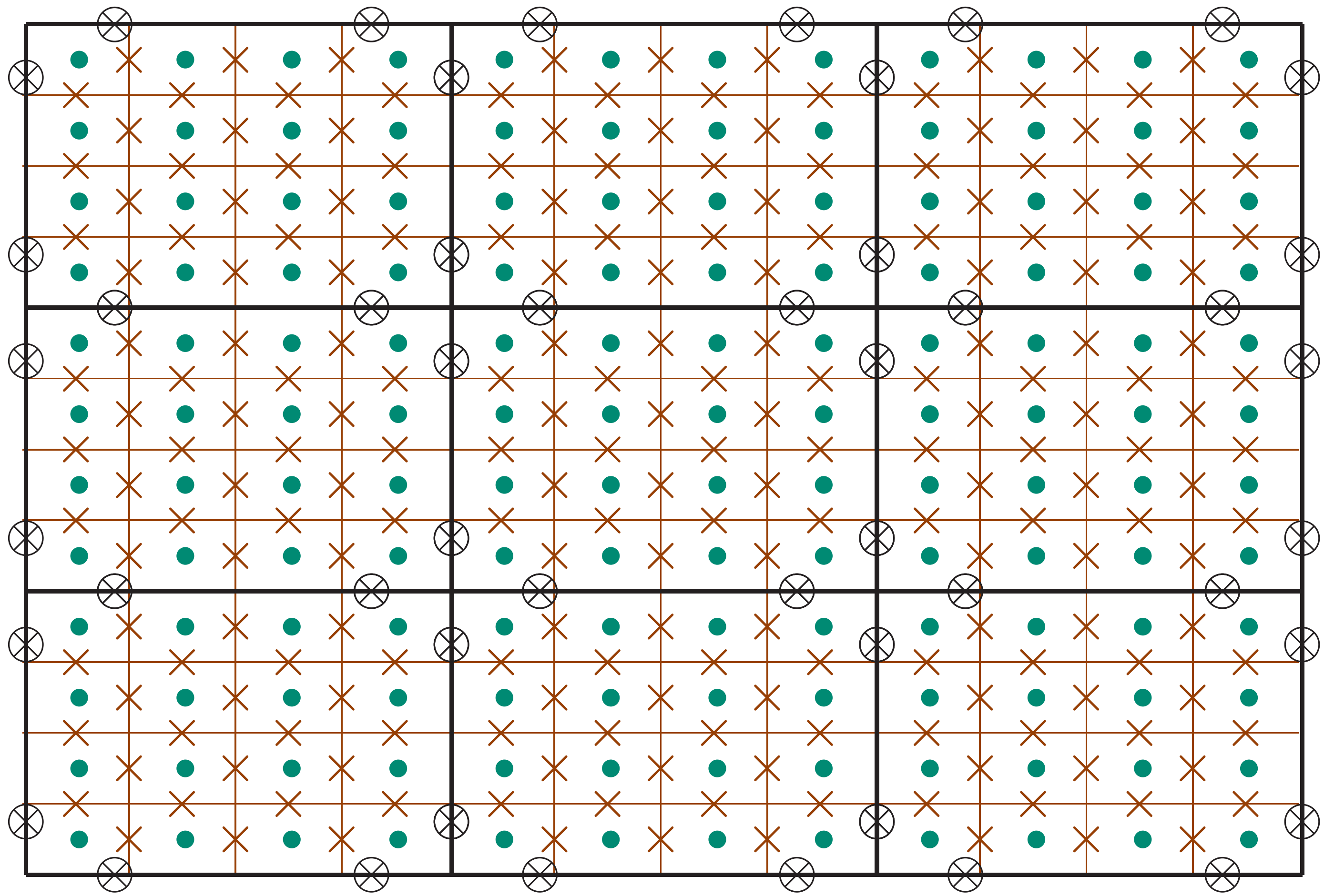
where d is the space dimension and $\eta > 0$ if $d = 2$ and $\eta = 0$ if $d = 3$.

Approximation of the Subgrid

Approximate the subgrid part of the basis functions by a mixed method on a fine grid of spacing $h \sim \epsilon$.



Composite Numerical Grid for BDM1–RT0

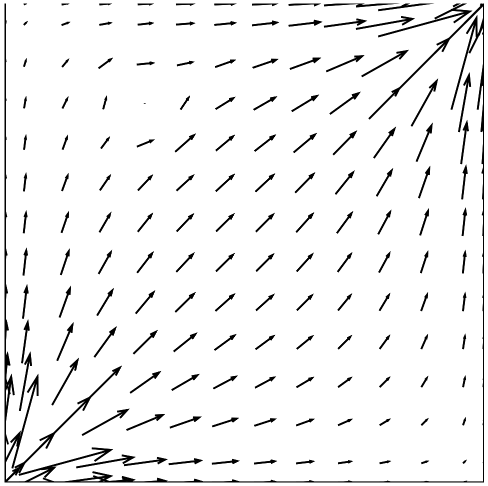


⊗ Coarse velocity (linear)　　× Subgrid velocity

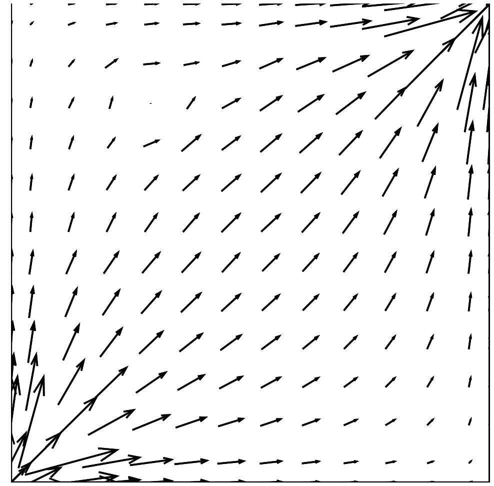
• Pressure

Numerical Examples and Application to Subsurface Flow Simulation

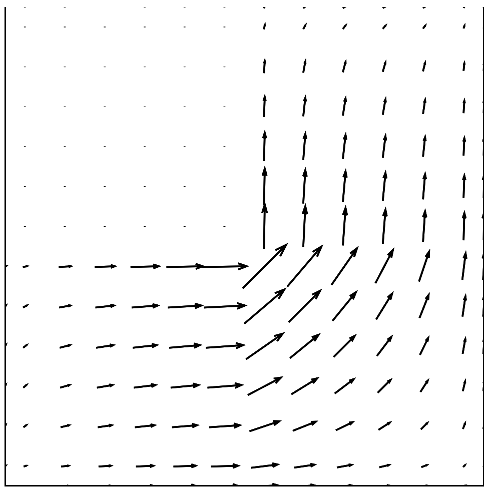
Low Premeability Spot (10^{-16})



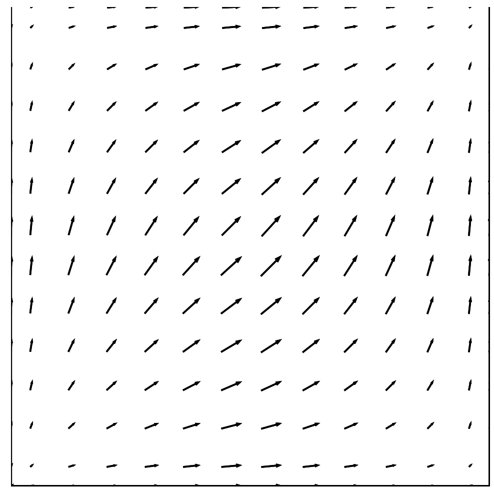
12×12 fine scale



2×2 upscaled

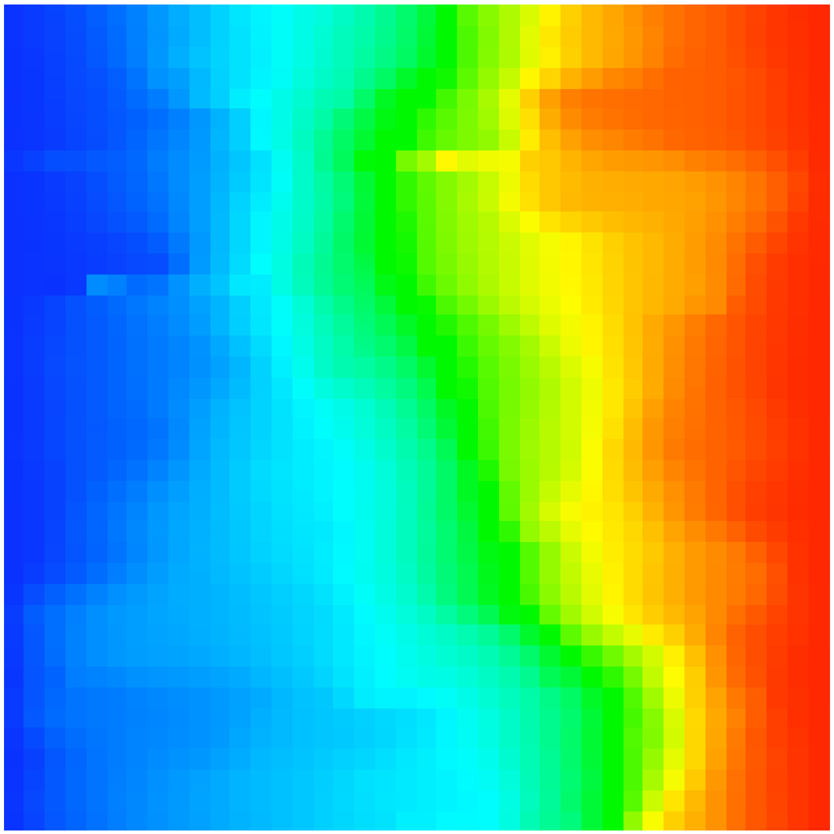


2×2 coarse scale

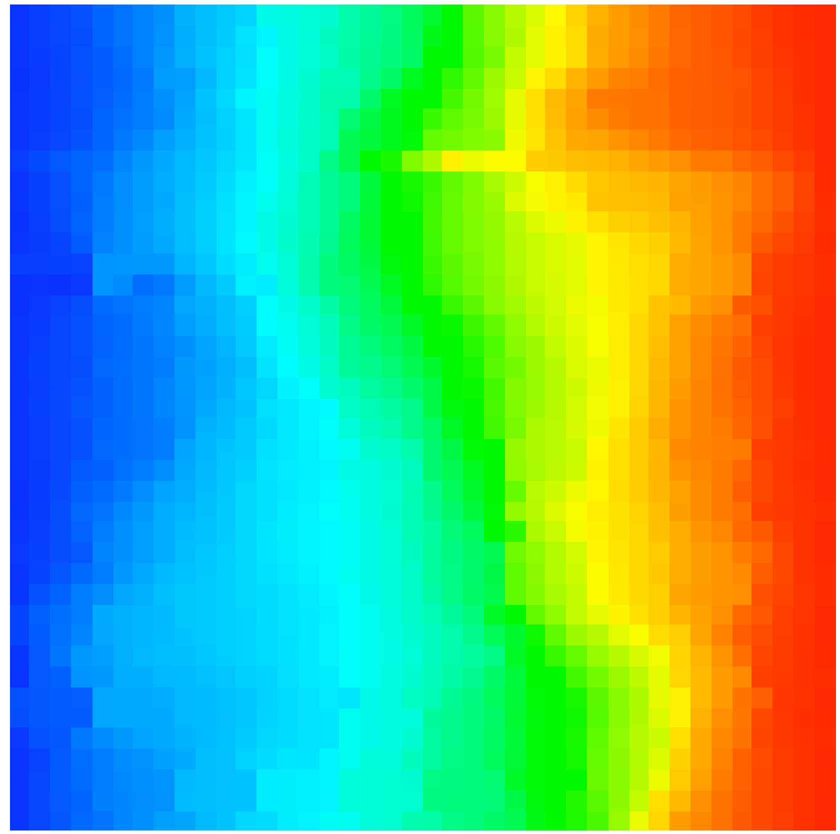


2×2 coarse scale using coarse average K

Pressure contours for a horizontal flood



Fine 40×40 solution



Upscaled to 10×10

Application to Waterflood Simulation

Use standard equations and sequential solution.

Pressure equation: Global pressure formulation.

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{u} = q(P)$$
$$\mathbf{u} = -K\lambda(S)\Big(\nabla P - \rho(S) g \mathbf{e}_3\Big)$$

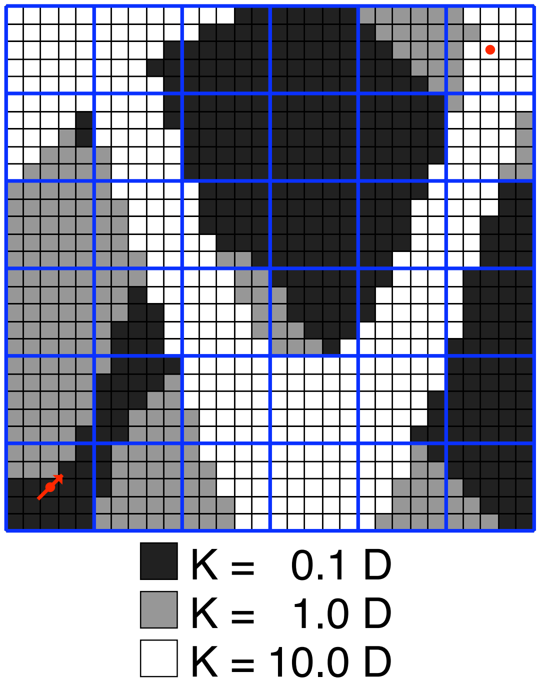
Upscale this equation. Use BDM1/RT0 unless otherwise noted.

Saturation equation: Kirchhoff formulation.

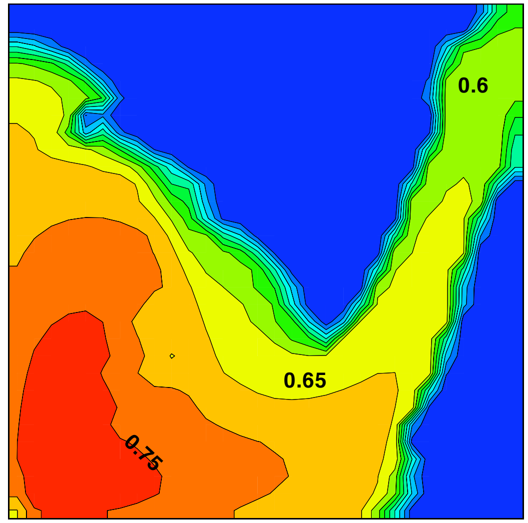
$$\frac{\partial \phi S}{\partial t} + \nabla \cdot \mathbf{u}_w = q_w(S)$$
$$\mathbf{u}_w = -K\nabla Q(S) + c(\mathbf{u}, S)$$

Solve on the fine scale.

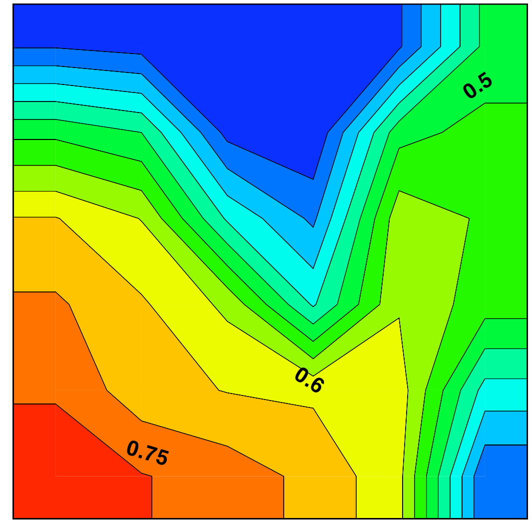
A Fluvial Subsurface Environment–1



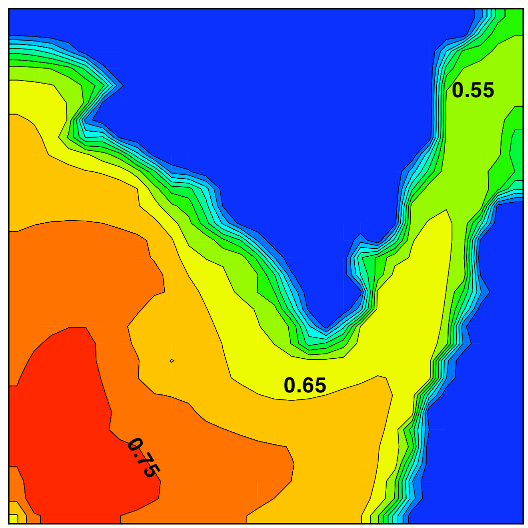
Permeability field
(White & Horne, 1987)



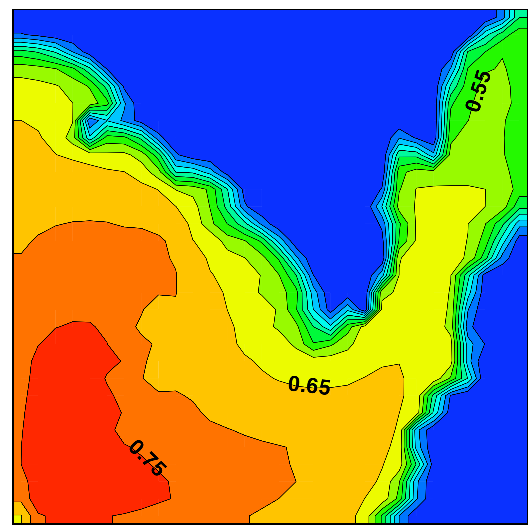
Fine 30 × 30



Average K 6 × 6

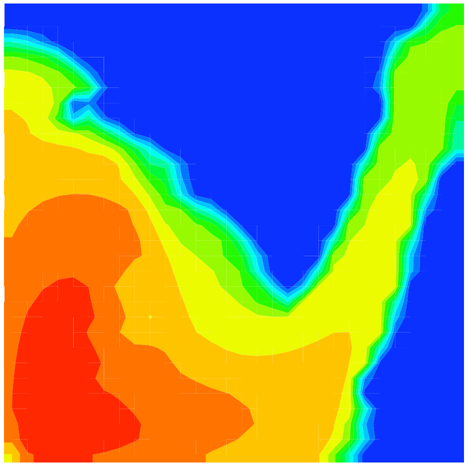


Upscaled to 6 × 6

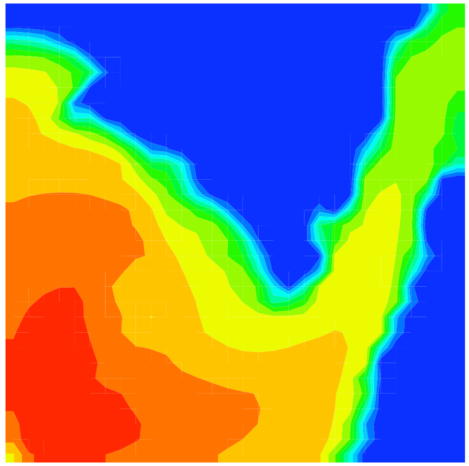


Upscaled to 3 × 3

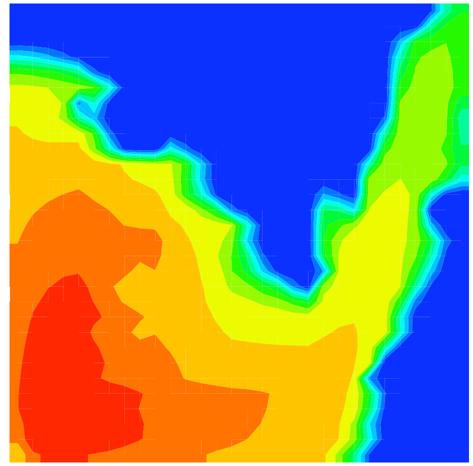
Fluvial Water Saturation Contours at 200 days–2



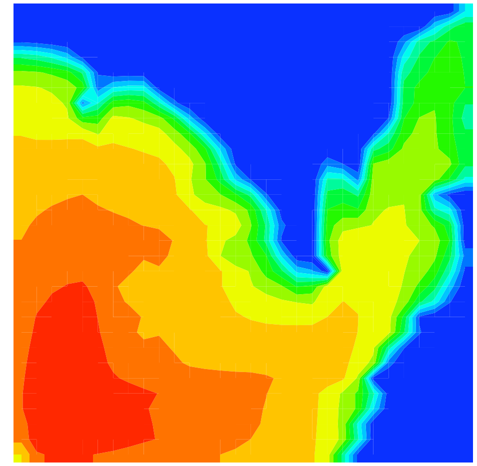
30×30 Fine



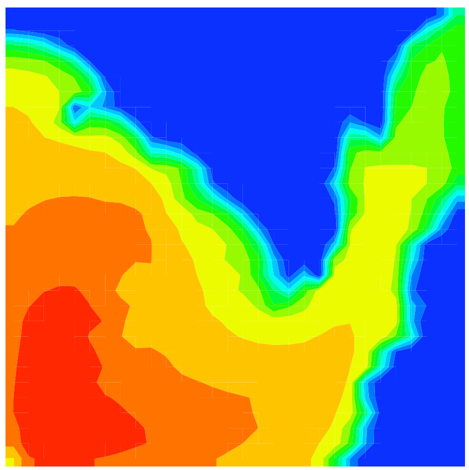
6×6 BDM1/RT0



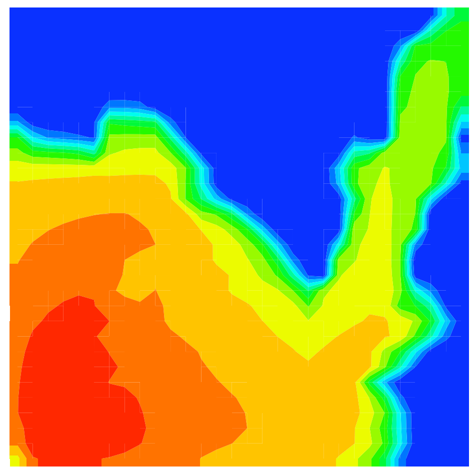
6×6 Dual



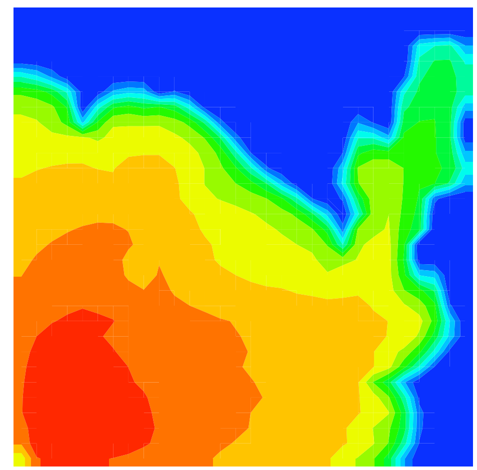
6×6 RT0/RT0



3×3 BDM1/RT0



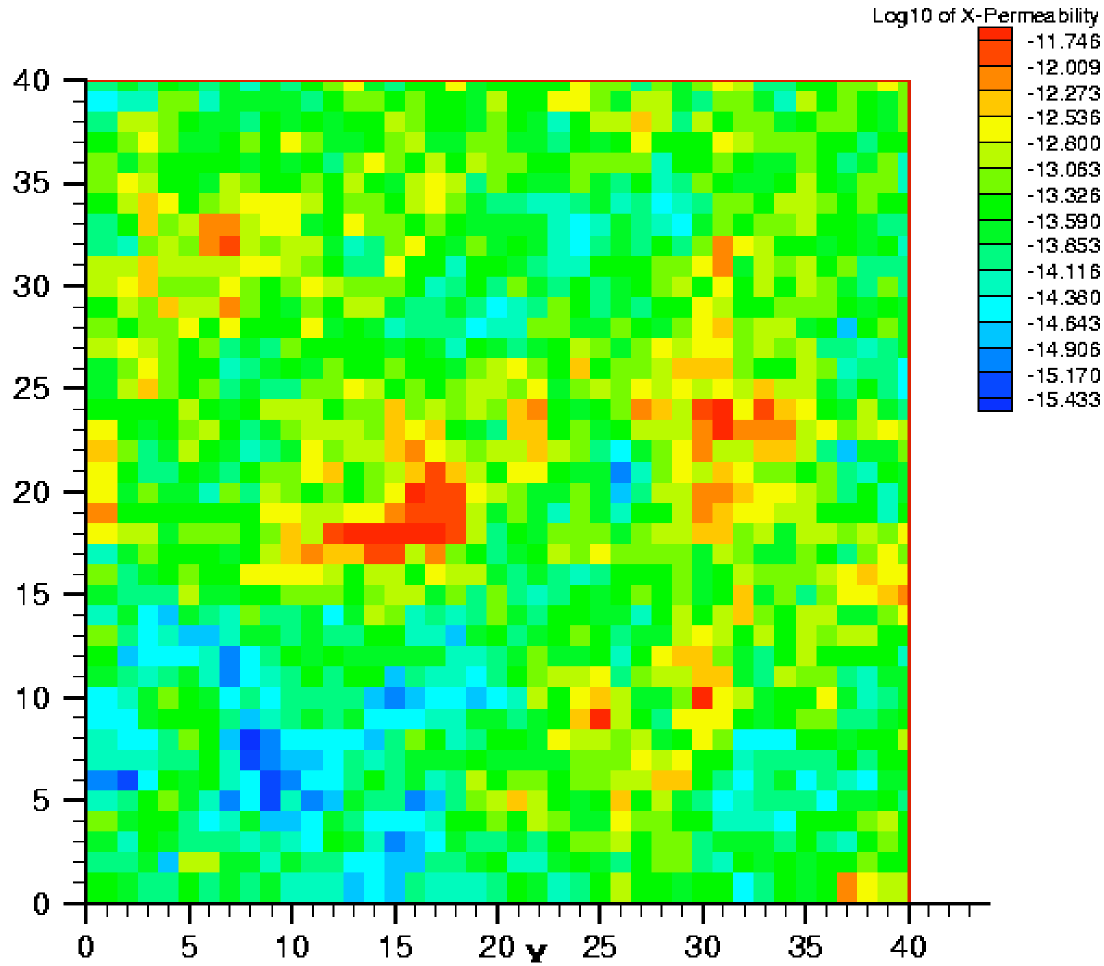
3×3 Dual



3×3 RT0/RT0

A Quarter Five-spot Oil Reservoir Waterflood—1

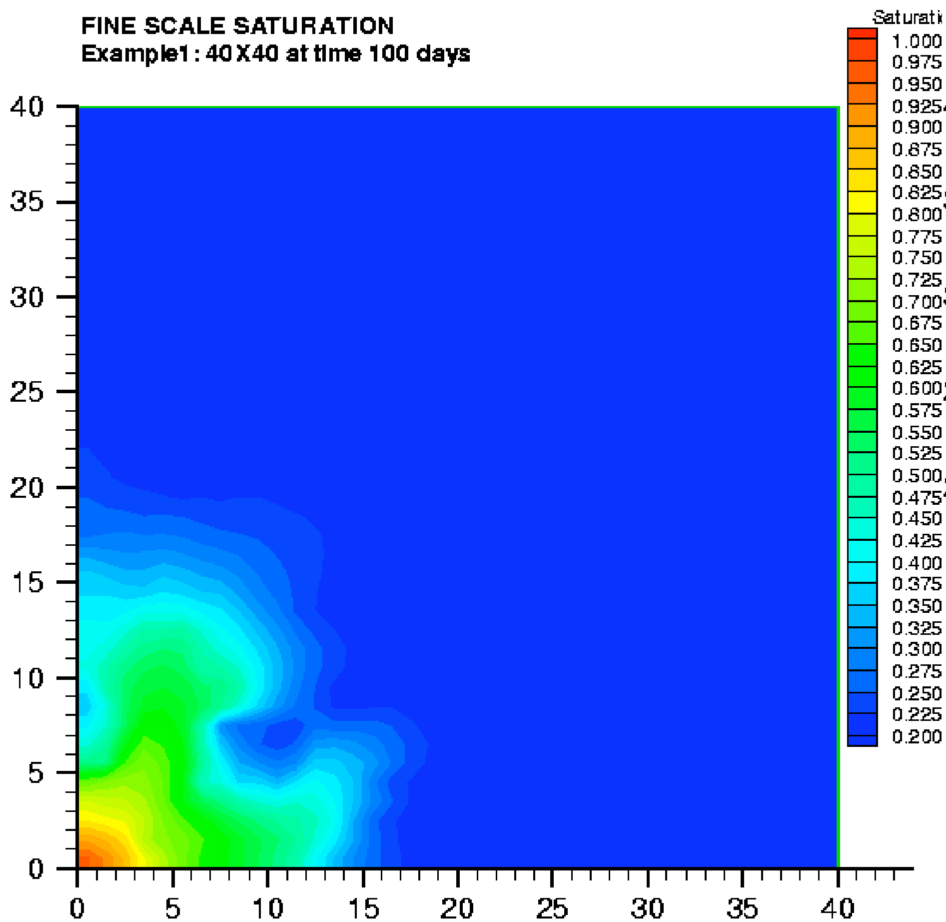
Logarithm of the permeability



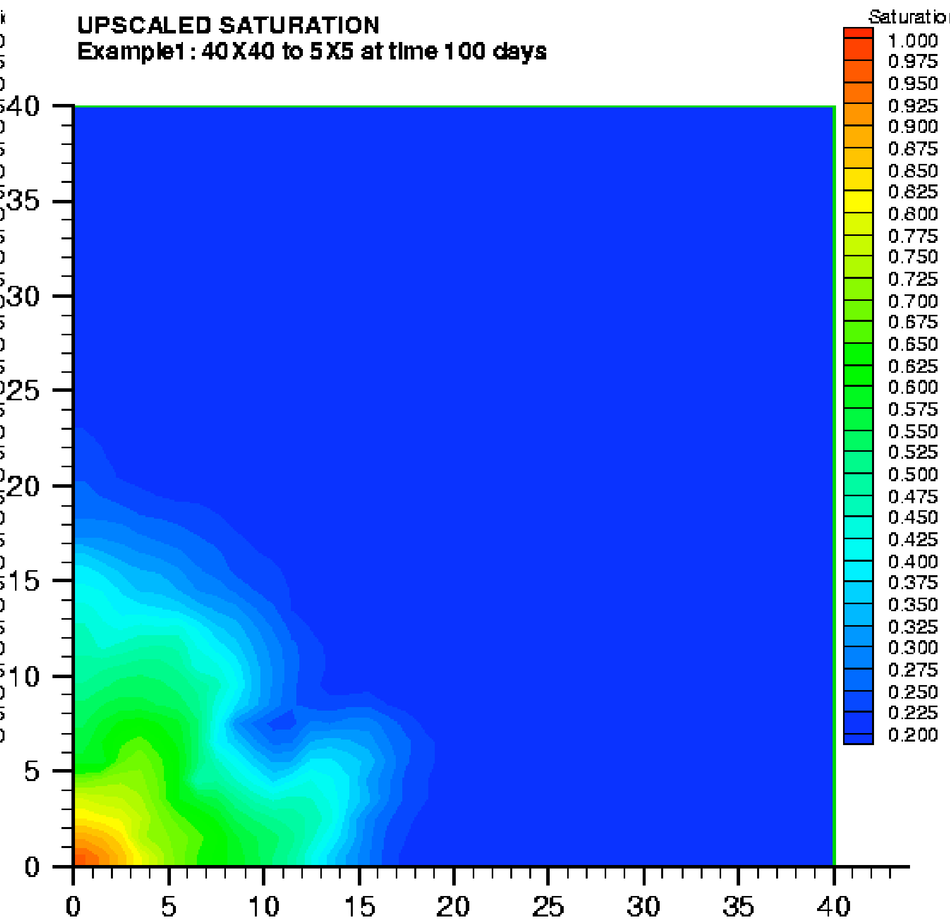
Fine 40×40

A Quarter Five-spot Oil Reservoir Waterflood—2

Water saturation contours at 100 days



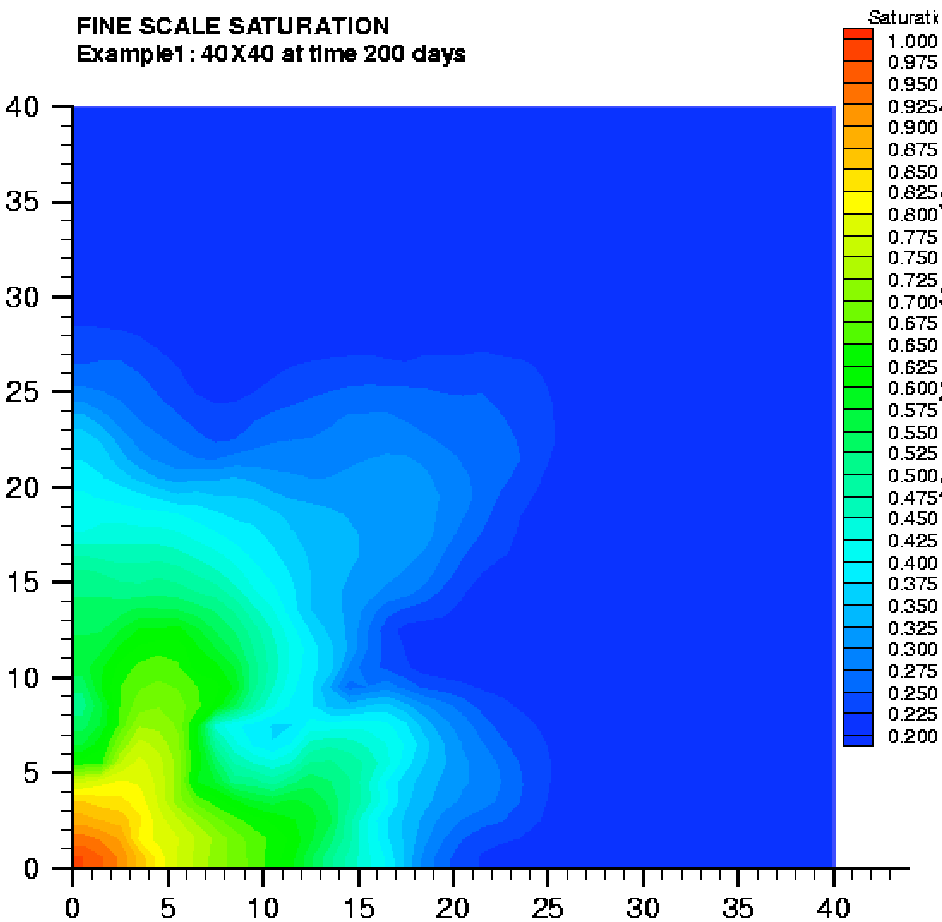
Fine 40 $\times$ 40



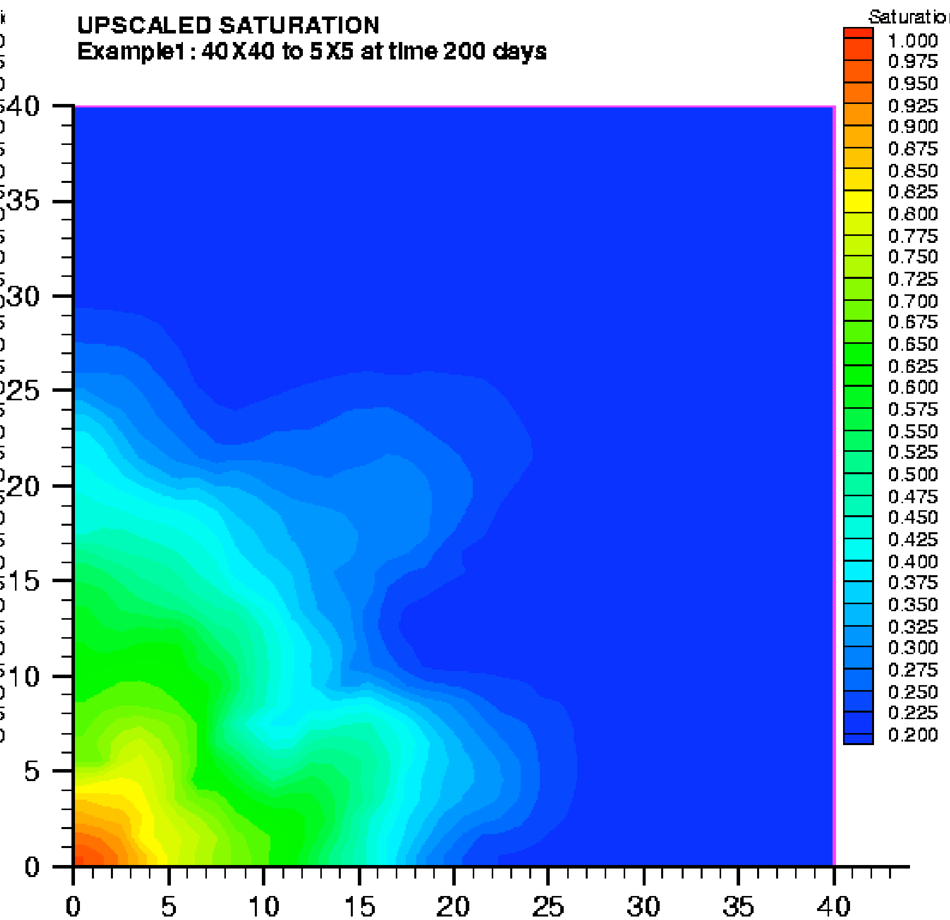
Fully upscaled to 5 $\times$ 5

A Quarter Five-spot Oil Reservoir Waterflood—3

Water saturation contours at 200 days



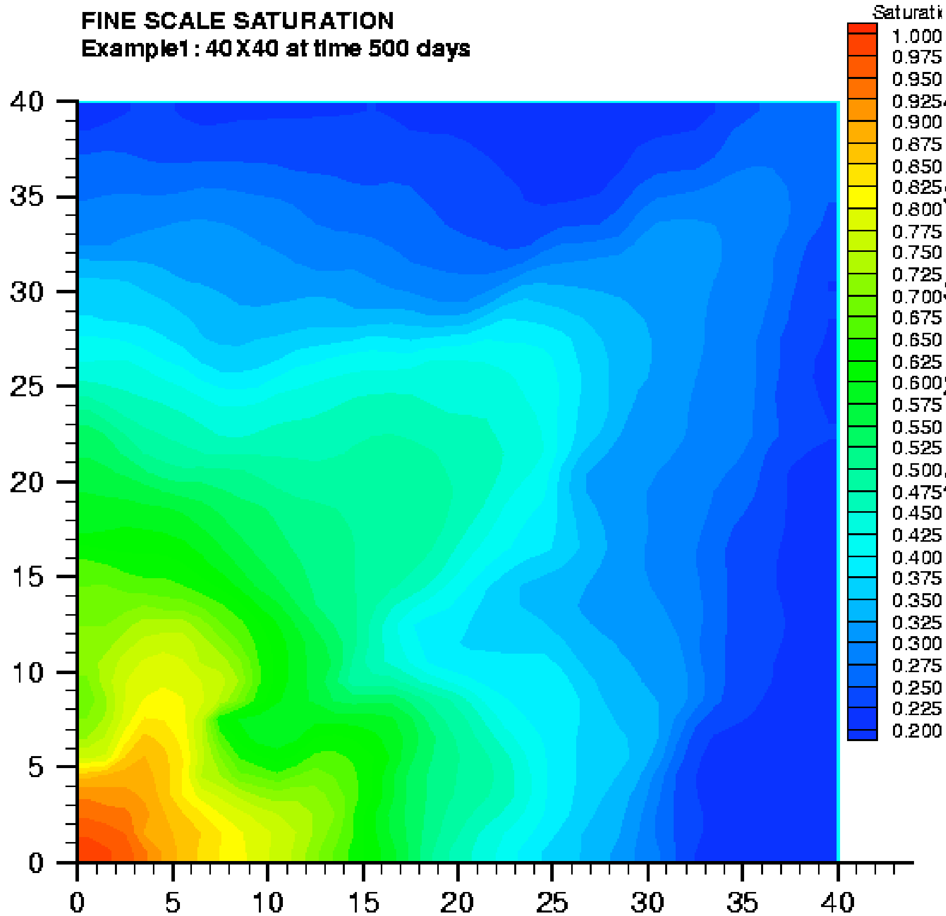
Fine 40×40



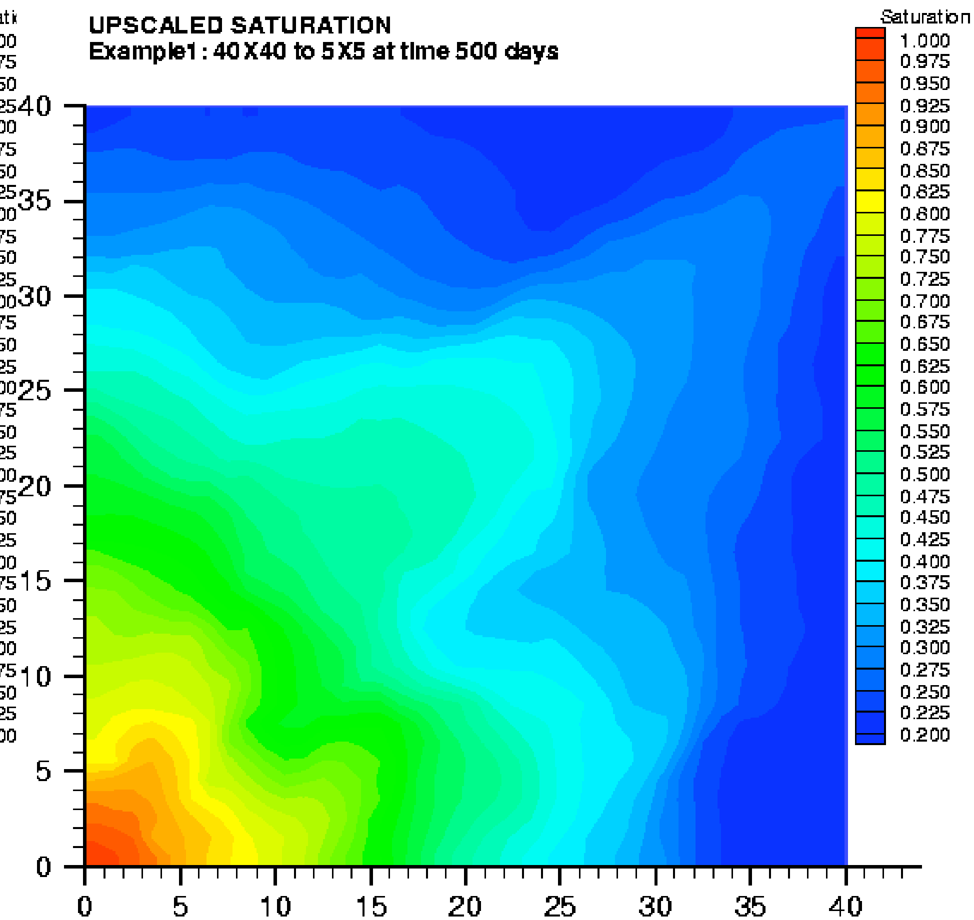
Fully upscaled to 5×5

A Quarter Five-spot Oil Reservoir Waterflood—4

Water saturation contours at 500 days



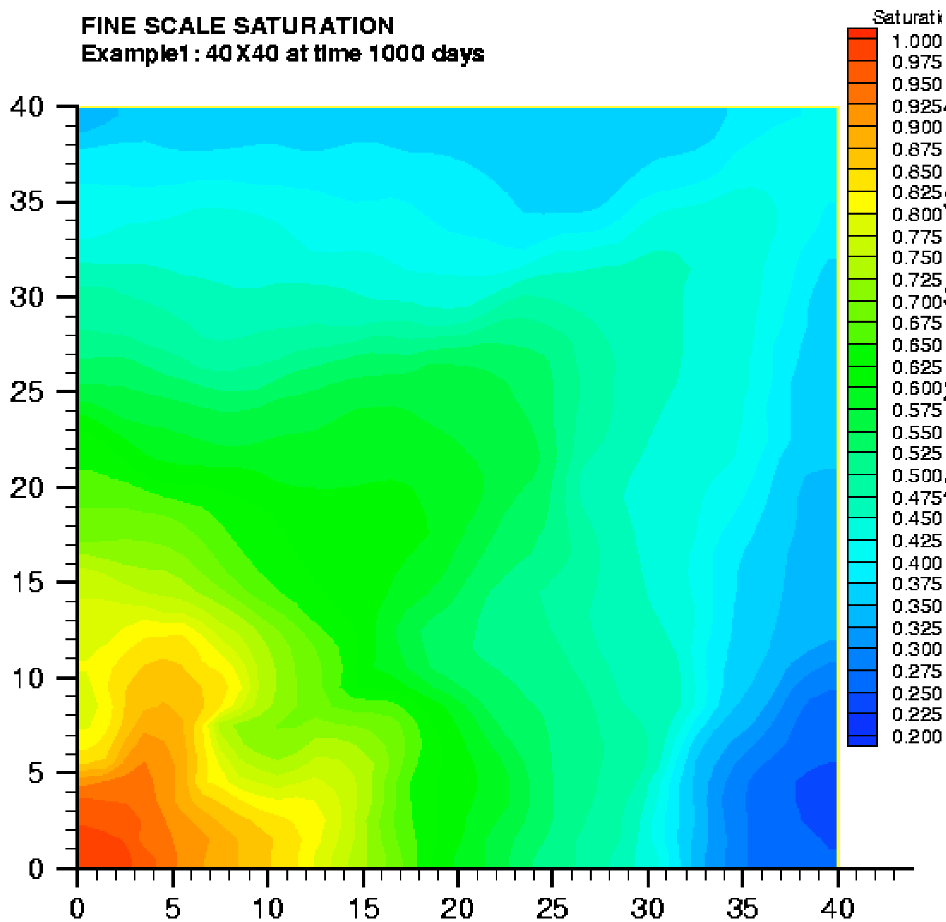
Fine 40×40



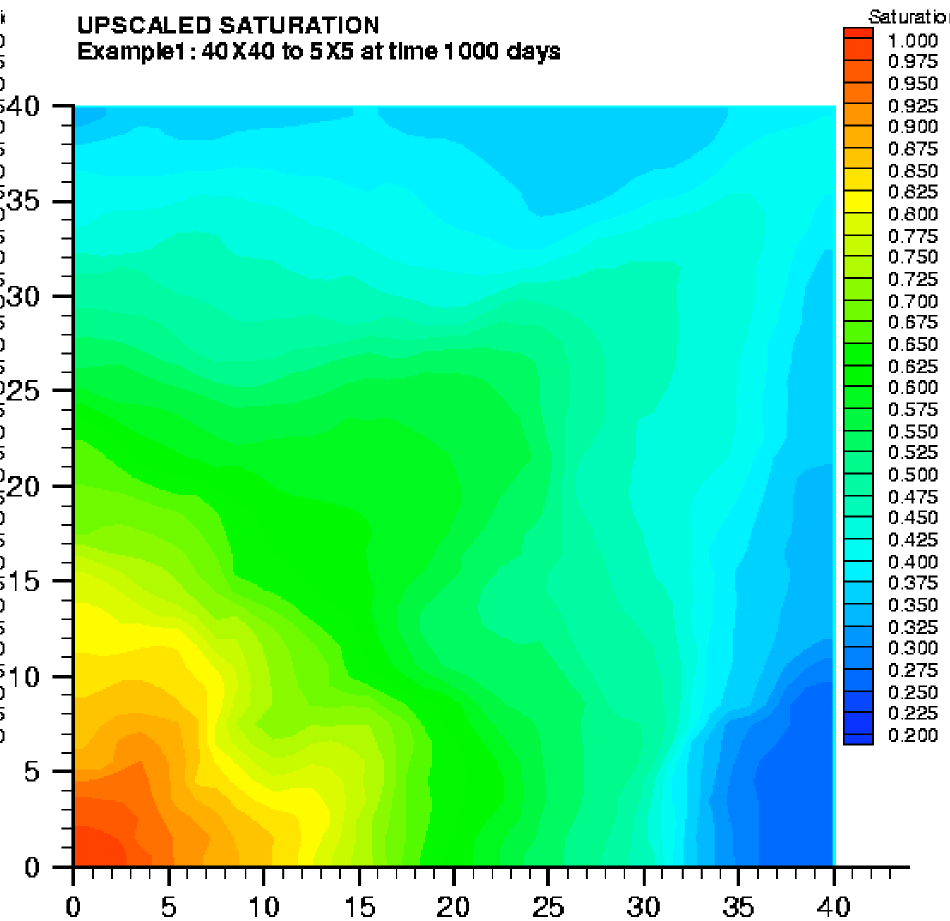
Fully upscaled to 5×5

A Quarter Five-spot Oil Reservoir Waterflood—5

Water saturation contours at 1000 days



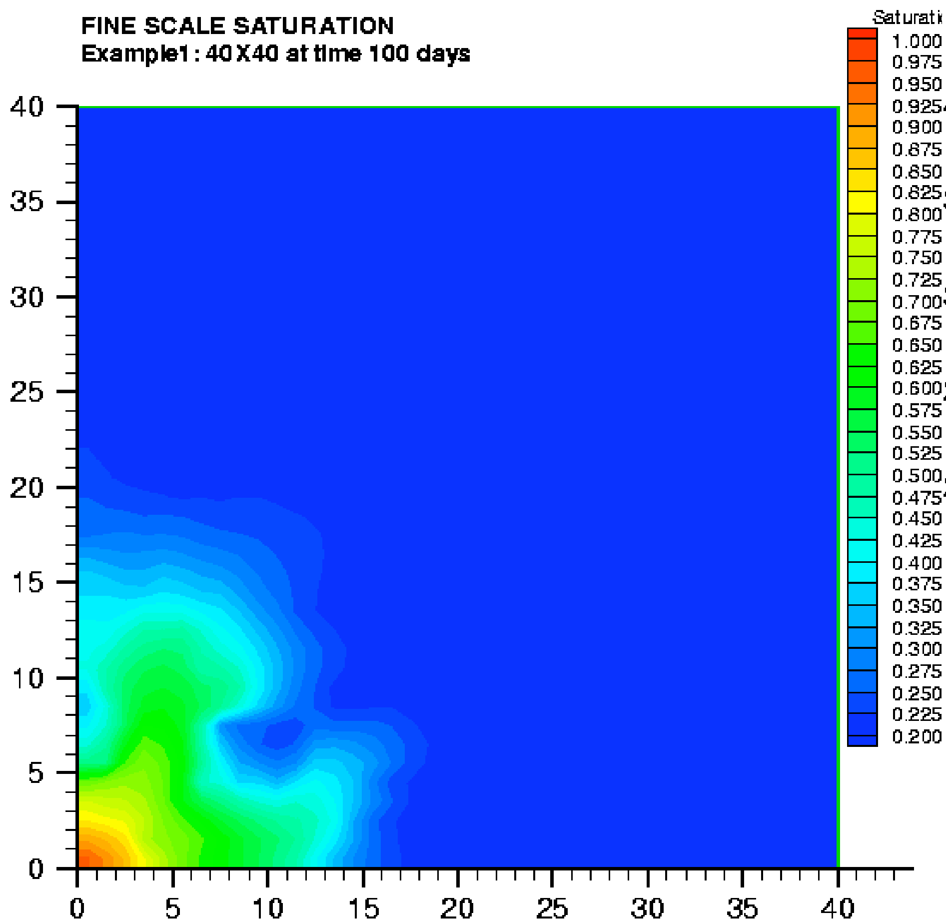
Fine 40 × 40



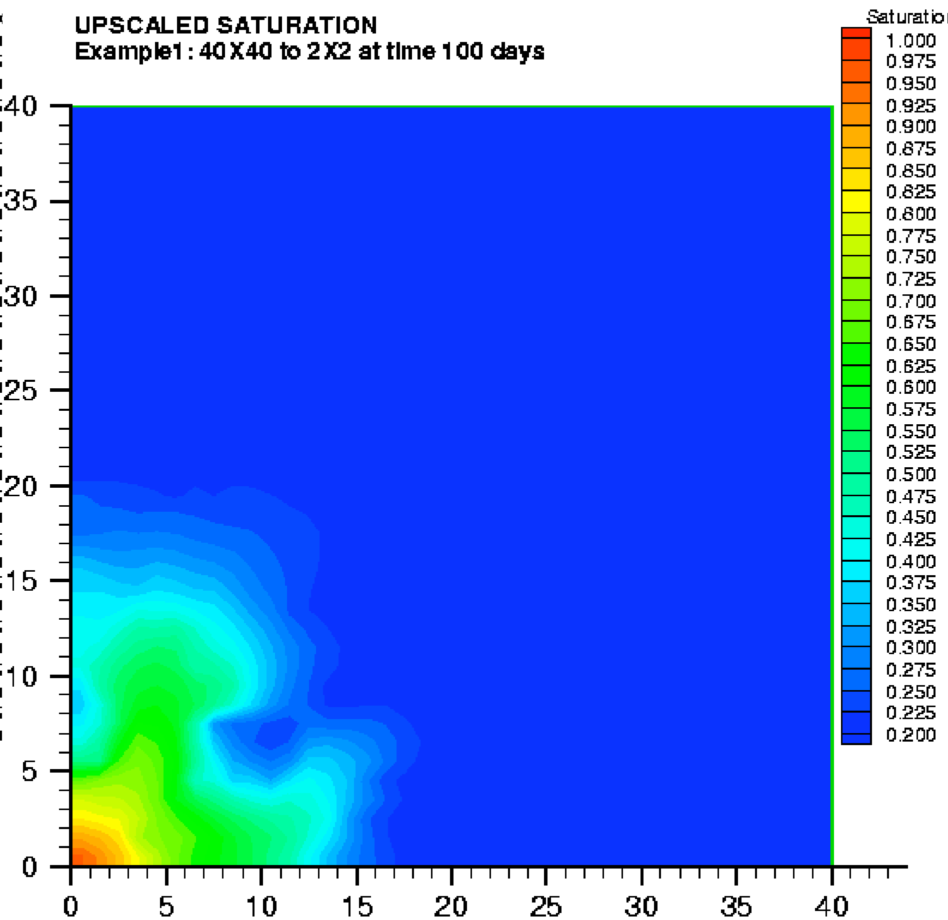
Fully upscaled to 5 × 5

A Quarter Five-spot Oil Reservoir Waterflood—6

Water saturation contours at 100 days



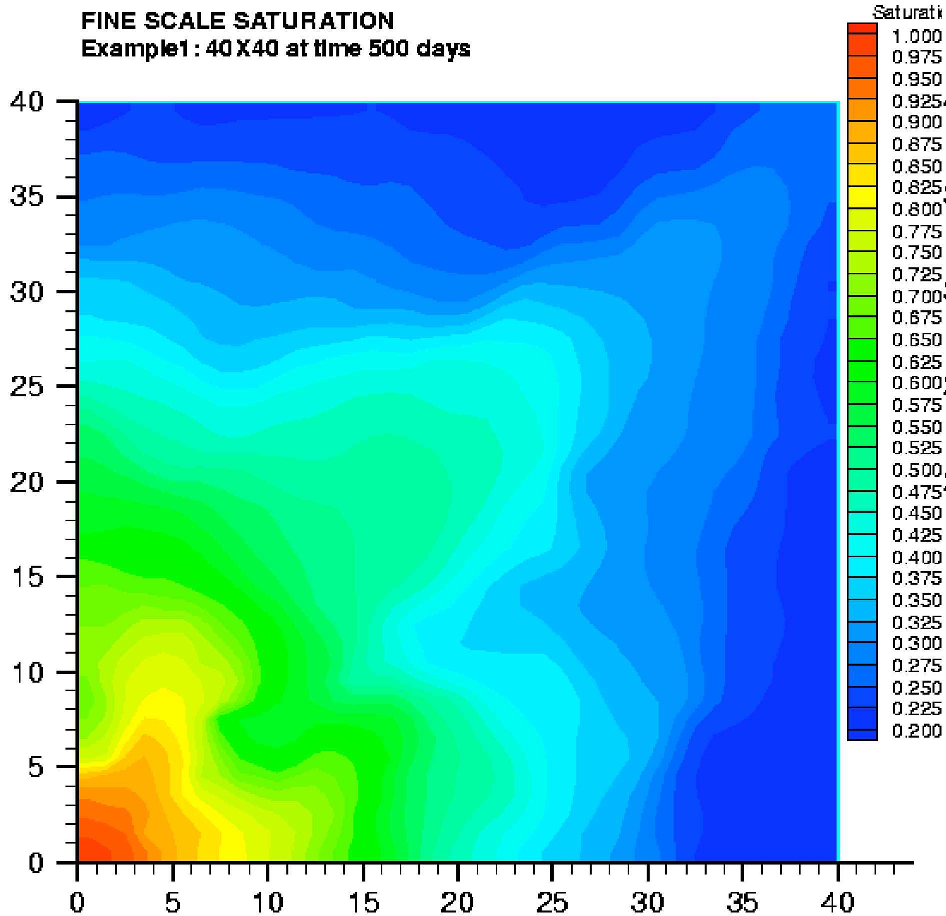
Fine 40×40



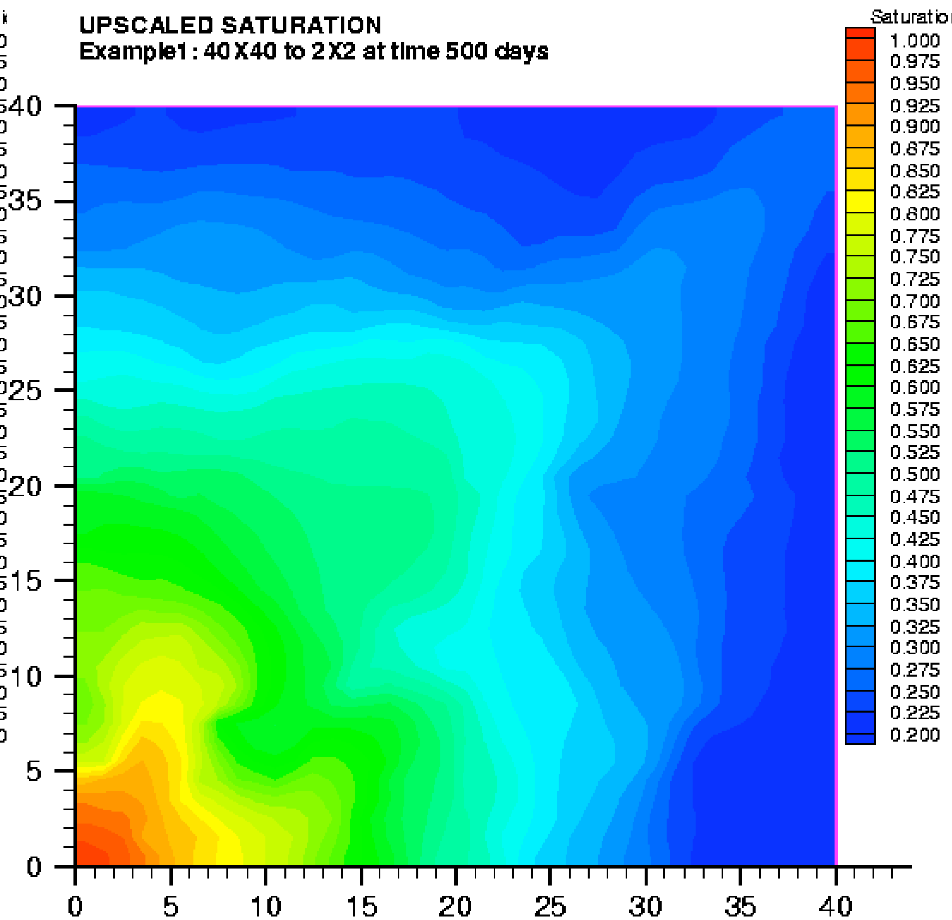
Fully upscaled to 2×2

A Quarter Five-spot Oil Reservoir Waterflood—7

Water saturation contours at 500 days



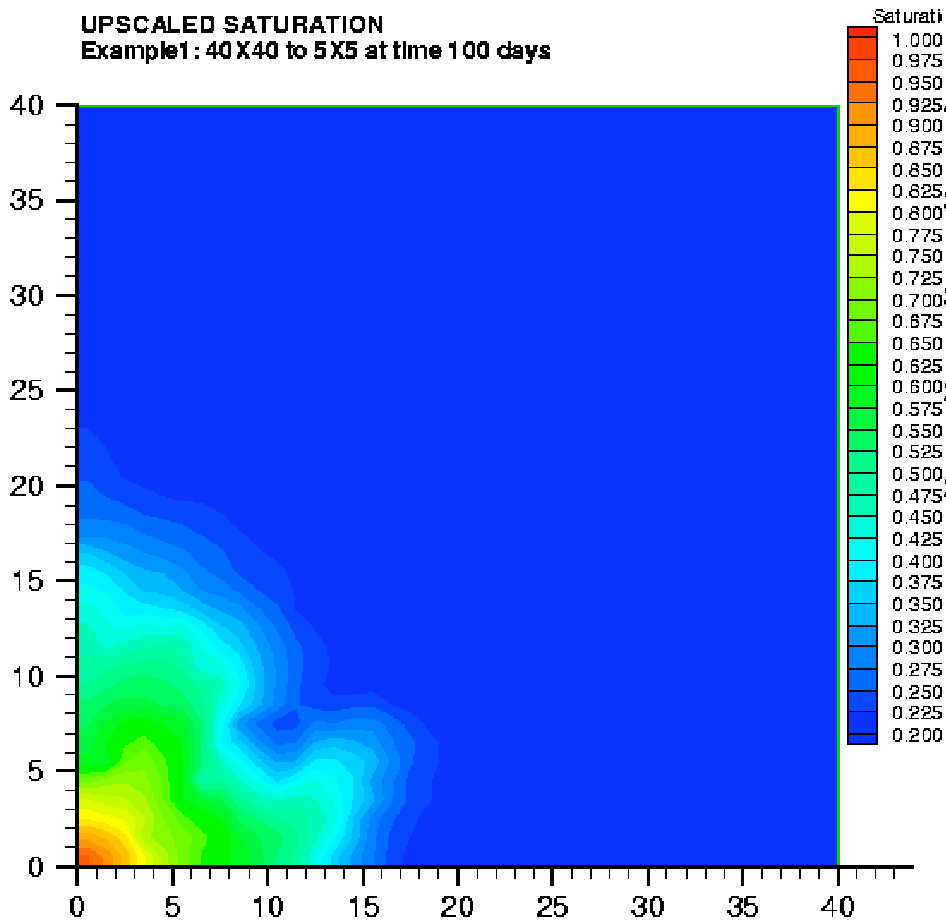
Fine 40 × 40



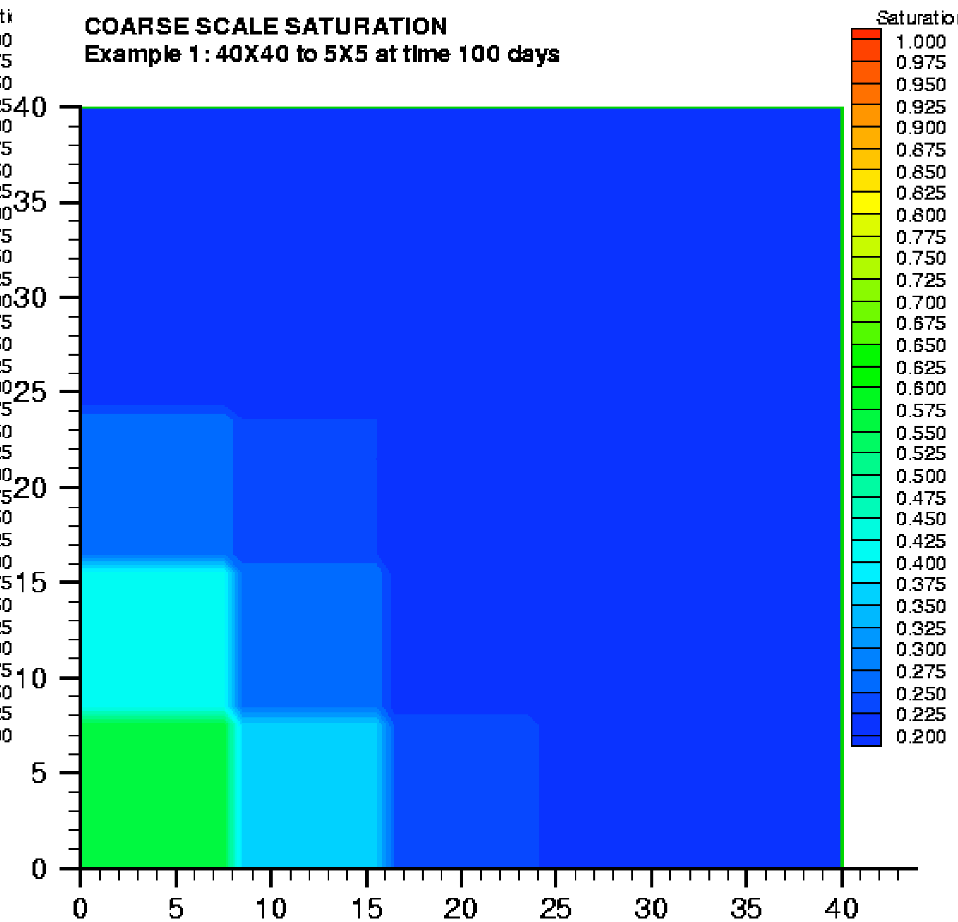
Fully upscaled to 2 × 2

A Quarter Five-spot Oil Reservoir Waterflood—8

Water saturation contours at 100 days



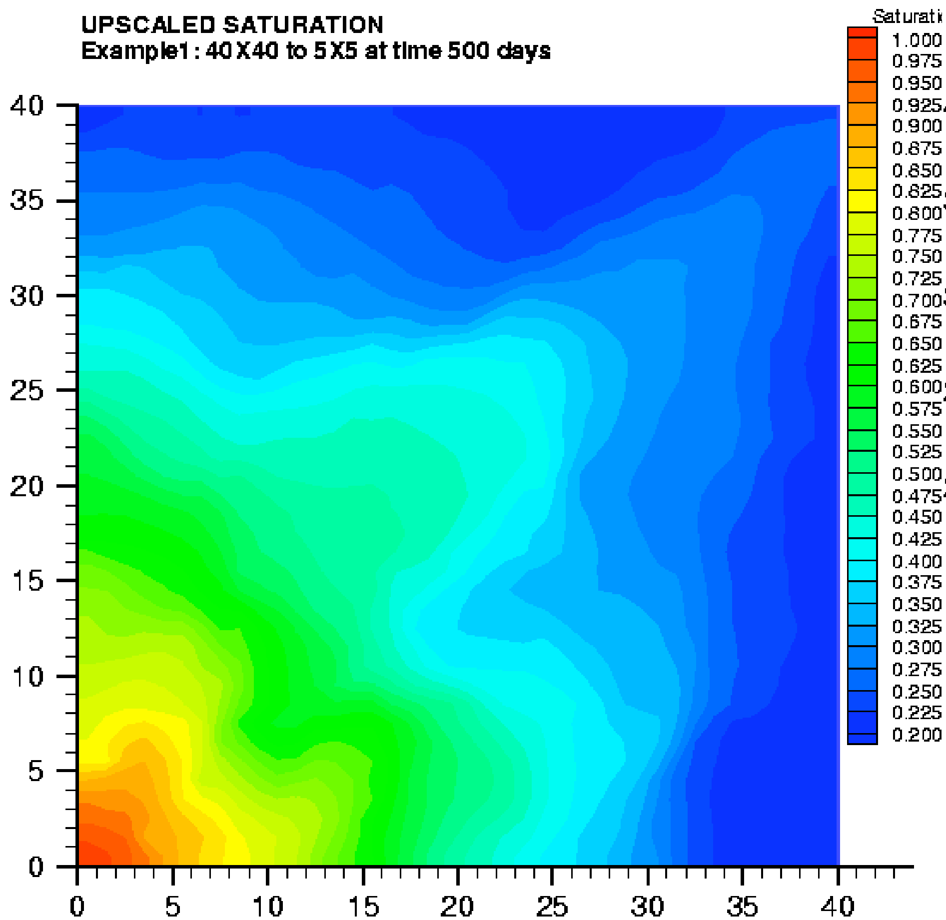
Fully upscaled to 5×5



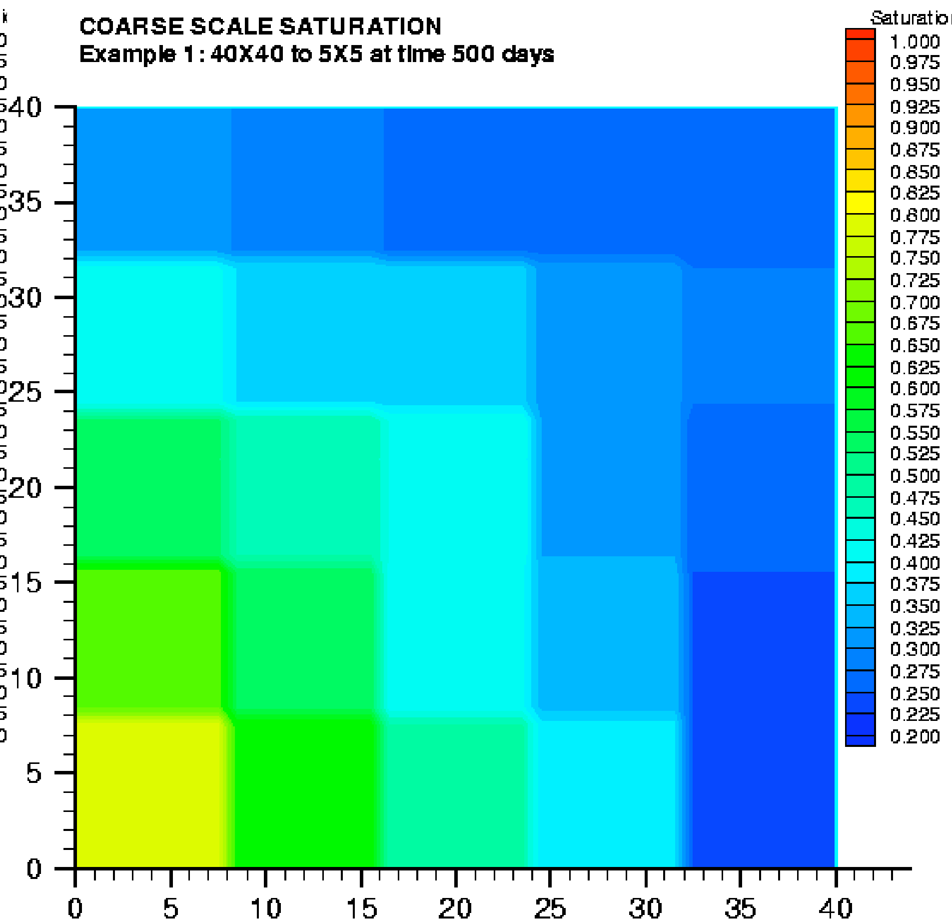
Coarse 5×5

A Quarter Five-spot Oil Reservoir Waterflood—9

Water saturation contours at 500 days



Fully upscaled to 5×5



Coarse 5×5

Conclusions

Conclusions—1

1. Natural porous media is highly heterogeneous, so standard finite element (or other) approximation is inaccurate, since it fails to resolve all the relevant scales adequately on the coarse grids we are forced to use.
2. Multiscale finite element basis functions can partially resolve the fine scales on coarse grids.
3. The Variational Multiscale Method is a framework that formally separates coarse and subgrid parts of the velocity and pressure spaces to obtain
   - conservation of mass on coarse and subgrid scales (physics),
   - locality of the subgrid operators (numerics).
4. The fine scales introduce antidiffusion into the system, and so cannot be modeled in any simple way.

Conclusions—2

5. The method achieves optimal order accuracy and accuracy with respect to the scale of heterogeneity ϵ.

	$\mathbf{u}$	p
Polynomial BDM1	H^2/ϵ^2	H^3/ϵ^2
Multiscale BDM1	$H^2 + \sqrt{\epsilon/H}$	$H^3 + (\epsilon/H)^{1/2+1/d}$

6. The method parallelizes naturally, and so is very efficient.
7. The numerical examples show that the methods can capture significant detail on coarse grids.
8. The variational multiscale method allows us to solve the main components of the flow for very large problems on very coarse grids, even though we under-resolve the fine scales themselves.