PDE Prep Talk 7/24/09

Parabolic Equations

- unknown function $u(t,x)$
- $t \in \mathbb{R}^*$, $x \in \Omega$ $\Omega \subseteq \mathbb{R}^n$ $t \in [0,T]$
- initial conditions + boundary conditions just right to guarantee $\exists!$ sol u.

examples

Heat eq. (a.k.a. diffusion eq.)

$$\begin{cases} u_t - \Delta u = 0 & \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \\ u(0,x) = f(x) & \\ u(t,x) = g(x) & x \in \partial\Omega \quad t \in [0,T] \end{cases}$$

a^{ij} - diffusion
b^{j} - transport
c creation or depletion

$$(\star) \begin{cases} u_t + Lu = F(x) & \text{on } [0,T] \times \Omega \\ u(0,x) = f(x) & \text{on } \{0\} \times \Omega \\ u(t,x) = g(x) & \text{on } [0,T] \times \partial\Omega \end{cases}$$

$$u_t - \sum_{i,j=1}^{n} a^{ij}(x,t)\, u_{x_i x_j} + \sum_{i=1}^{n} b^{i}(x,t)\, u_{x_i} + c(x,t)\, u = F(x)$$

Def We say $(\star)$ is parabolic if $\exists$ constant $\theta > 0$ s.t.

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\, \xi_i \xi_j > \theta |\xi|^2 \quad \forall (t,x) \in [0,T] \times \Omega, \quad \xi \in \mathbb{R}^n$$

✓ Heat eq.

$$a^{ij}(x,t) = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & & \\ 0 & & 1 & \end{pmatrix} = Id$$

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i \xi_j = \sum_{j=1}^{n} \xi_j \xi_j = |\xi|^2 \geq 1 \cdot |\xi|^2$$

$\theta = 1$ works here.

- constant coefficients.

<u>example</u>

$$u_t - \sum_{j=1}^{n} \underbrace{(1+e^{x_j})}\, u_{x_j x_j} = 0$$

$$a^{ij}(t,x) = \begin{pmatrix} 1+e^{x_1} & & 0 \\ & 1+e^{x_i} & \\ 0 & & \ddots \end{pmatrix}$$

$$\sum a^{ij}(x,t)\xi_i \xi_j = \sum_{j=1}^{n} (1+e^{x_j})\xi_j \xi_j \geq |\xi|^2$$

$$\xi \begin{pmatrix} & \end{pmatrix} \xi$$

<u>example</u> "backwards" parabolic $\boxed{\Omega = \mathbb{R}}$

$$\begin{cases} u_t - ru + au_x + \frac{1}{2}\sigma^2 u_{xx} = 0 \\ \qquad\qquad u(T, x) = f(x) \end{cases}$$ terminal cond

$s = T - t$
$\partial_s = -\partial_t$

$\therefore$ backwards in time switches signs on u_t term.

✓ parabolic

- 1-D $\Rightarrow$ 1×1 matrix $\left(\frac{1}{2}\sigma^2\right)$

$\frac{1}{2}\sigma^2 \xi^2 \geq \theta \xi^2$ for $\theta = \frac{1}{2}\sigma^2$

How to solve it?

Fourier Transform Methods $\quad f : \mathbb{R} \to \mathbb{R}$

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-ix\cdot\xi} f(x)\, dx \qquad \xi \in \mathbb{R}$$

Claim 1 $\quad \widehat{\partial_x f}(\xi) = i\xi \hat{f}(\xi)$

$$\widehat{\partial_x f}(\xi) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-ix\xi} \partial_x f(x)\, dx$$

$$= -\frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \partial_x \left(e^{-ix\xi}\right) f(x)\, dx$$

$$= \frac{i\xi}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx = i\xi \hat{f}(\xi).$$

$\square$

– In frequency space differentiation is algebraic –

Other Important Properties

① $\exists$ inverse transformation

$$\check{g}(x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{ix\xi} g(\xi)\, d\xi$$

$$\hat{\check{g}}(\xi) = g(\xi) \qquad \check{\hat{f}}(x) = f(x).$$

<u>claim 2</u> $\quad \widehat{f * h} = (2\pi)^{1/2} \hat{f}\, \hat{h}$

$$\widehat{f*h}(\xi) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-ix\xi} \int_{\mathbb{R}} f(y)\, h(x-y)\, dy\, dx$$

$$= \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-iy\xi} f(y)\, e^{-i(x-y)\xi} h(x-y)\, dy\, dx$$

$$= \frac{1}{(2\pi)^{1/2}} \int e^{-iy\xi} f(y) \left(\int e^{-i(x-y)\xi} h(x-y)\, dx \right) dy$$

$$= \frac{1}{(2\pi)^{1/2}} \int e^{-iy\xi} f(y) \int e^{-iz\xi} h(z)\, dz\, dy \qquad z = x-y$$

$$= \int e^{-iy\xi} f(y)\, dy\; \hat{h}(\xi) = (2\pi)^{1/2} \hat{f}(\xi)\, \hat{h}(\xi)$$

$\square$

Computing the Heat Kernel:

$u_t - u_{xx} = 0 \qquad u(0,x) = f(x) \qquad x \in \mathbb{R}$ ∞-rod

$\hat{u}_t + \xi^2 \hat{u} = 0$

$\hat{u}_t = -\xi^2 \hat{u} \implies \hat{u}(t,\xi) = e^{-\xi^2 t} \hat{f}(\xi)$

$\hat{\Psi}(t,\xi) \hat{f}(\xi)$

$\hat{\Psi}(t,\xi) = e^{-\xi^2 t}$

$$\Psi(t,x) = \frac{1}{(2\pi)^{1/2}} \int e^{ix\cdot\xi} e^{-\xi^2 t} \, d\xi = \cancel{*}$$

$$z^2 = \xi^2 t - ix\xi + \gamma \qquad \gamma = \frac{-x^2}{t}$$

$$= \left(\xi t^{1/2} - \frac{ix}{t^{1/2}}\right)^2 \qquad dz = t^{1/2} d\xi$$

$$\Psi(t,x) = \frac{1}{(2\pi)^{1/2}} \int_\Gamma e^{-z^2} e^{\gamma} \frac{dz}{t^{1/2}}$$

$$= \frac{e^{\gamma}}{(2\pi)^{1/2}} \left(\frac{\pi}{t}\right)^{1/2} - \frac{e^{-x^2/t}}{2t^{1/2}}$$

heat kernel

$$u(t,x) = \Psi(t,x) * f(x)$$

$$= \frac{1}{2\sqrt{t}} \int e^{-(x-y)^2/t} f(y) \, dy$$

$$\begin{cases} u_t - ru + au_x + \frac{1}{2}\sigma^2 u_{xx} = 0 \\ u(T, x) = f(x) \end{cases} \qquad \Omega = \mathbb{R}$$

WTF: $u(t, x)$ for $0 \le t \le T$

$$\hat{u}_t - r\hat{u} + ia\xi\hat{u} - \frac{1}{2}\sigma^2\xi^2\hat{u} = 0$$

Have them do each term.

$$\hat{u}_t = \left(r - ia\xi + \frac{1}{2}\sigma^2\xi^2\right)\hat{u} \qquad \text{fix } \xi \in \mathbb{R}$$

ODE constant coefficient

$s = T - t$

$$\hat{u}(t, \xi) = e^{(r - ia\xi + \frac{1}{2}\sigma^2\xi^2)t}\,\hat{u}(0, \xi)$$

- backwards time

$$\hat{u}(t, \xi) = e^{-(r - ia\xi + \frac{1}{2}\sigma^2\xi^2)(T-t)}\hat{f}(\xi)$$

$$= \hat{K}(t, \xi)\hat{f}(\xi)$$

- back to original space

<u>Thm</u> $\hat{g}(\xi)\hat{f}(\xi) = \widehat{(g * f)}$ convolution

$$g * f(x) = \int_{\mathbb{R}} g(x-y) f(y)\, dy$$

$$\hat{K}(t,\xi) := e^{-(r - ia\xi + \frac{1}{2}\sigma^2\xi^2)(T-t)}$$

$$K(t,x) = \frac{1}{(2\pi)^{1/2}} \int e^{ix\cdot\xi}\, e^{-(r - ia\xi + \frac{1}{2}\sigma^2\xi^2)(T-t)}\, d\xi$$

Let: $s = (T-t)$

$\alpha = x + as$

$\beta = \frac{1}{2}\sigma^2 s$

$$K(t,x) = \frac{e^{-rs}}{(2\pi)^{1/2}} \int e^{i\alpha\xi - \beta\xi^2}\, d\xi = \cancel{*}$$

$$z^2 = \beta\xi^2 - i\alpha\xi + \gamma \qquad \gamma = -\frac{\alpha^2}{4\beta} =$$

$$= \left(\beta^{1/2}\xi - \frac{i\alpha}{2\beta^{1/2}}\right)^2 \Rightarrow z = \beta^{1/2}\xi - \frac{i\alpha}{2\beta^{1/2}} \qquad dz = \beta^{1/2}d\xi$$

$$\cancel{*} = \frac{e^{-rs}}{(2\pi)^{1/2}} \int_\Gamma e^{-z^2} e^{\gamma} \frac{dz}{\beta^{1/2}}$$

$\Gamma = \left\{ \mathrm{Im}(z) = -\frac{\alpha}{2\beta^{1/2}} \right\}$

line in the complex plane.

$$= \frac{e^{-rs+\gamma}}{(2\pi\beta)^{1/2}} \int_\Gamma e^{-z^2} dz$$

we can shift

$$= \frac{e^{-rs+\gamma}}{(2\pi\beta)^{1/2}} (\pi)^{1/2} = \frac{e^{-rs+\gamma}}{(2\beta)^{1/2}}$$

$$\gamma = -\frac{\alpha^2}{4\beta} = -\frac{(x + as)^2}{2\sigma^2 s}$$

$$(2\beta)^{1/2} = (\sigma^2 s)^{1/2} = \sigma s^{1/2}$$

$$-rs + \gamma = -rs - \frac{(x + as)^2}{2\sigma^2 s}$$

$$= -rs - \frac{x^2 + 2axs + a^2 s^2}{2\sigma^2 s}$$

$$= -rs - \frac{x^2}{2\sigma^2 s} - \frac{ax}{\sigma^2} - \frac{a^2}{2\sigma^2} s$$

$$= -\frac{x^2}{2\sigma^2 s} - \left(r + \frac{a^2}{2\sigma^2}\right) s - \frac{ax}{\sigma^2}$$

$\sigma \to r - \sigma$

$$K(t, x) = \frac{1}{\sigma (T-t)^{1/2}} \exp\left\{ -\frac{x^2}{2\sigma^2 (T-t)} - \frac{ax}{\sigma^2} - \left(r + \frac{a^2}{2\sigma^2}\right)(T-t) \right\}$$

and we know

$$u(t, x) = K(t, x) * f(x)$$

convolution in x.

$$u(t, x) = \int_{\mathbb{R}} K(t, x-y) f(y)\, dy$$

Important Rmks:

→ these transform methods won't work for bounded Ω.

→ $\exists!$ on bounded Ω ... even if coefficients are <u>not</u> constant