1 Homework 1

The homework will NOT be collected and graded, it is just intended to help you understand the material better. Most of them are well known exercises in Stochastic Processes and Probability, so it is very likely that some of you encountered them before.

Exercise 1.1 Let $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G} \subseteq \mathcal{F}$. Show that the conditional expectation $\mathbb{E}[X|\mathcal{G}]$ is the unique random variable $\hat{Y} \in L^2(\Omega, \mathcal{G}, \mathbb{P})$ which attains the minimum

$$\min_{Y \in L^2(\Omega, \mathcal{G}, \mathbb{P})} \mathbb{E}[(X - Y)^2].$$

In other words, for $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, the conditional expectation is the orthogonal projection on $L^2(\Omega, \mathcal{G}, \mathbb{P})$ with respect to the L^2-inner product.

Exercise 1.2

1. (Conditional Jensen inequality) Using the property that a convex function $\psi : \mathbb{R} \to \mathbb{R}$ admits the representation

   $$\psi(x) = \sup_{l \in \mathcal{L}_\psi} l(x),$$

   where $\mathcal{L}_\psi$ is the set of all linear functions $l \leq \psi$, show that

   $$\psi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\psi(X)|\mathcal{G}],$$

   (Please note that some integrability conditions are required: it is left to you to figure them out)

2. using the previous item, show that if (M_n) is a martingale and ψ is convex (such that $\psi(M_n)$ is integrable for each n), then $(\psi(M_n))_n$ is a submartingale. If ψ is concave we get a supermartingale.

   Note: this property explains the names sub/super mart's, since a convex function is *subharmonic.*

Exercise 1.3

1. (Doob's decomposition). Let $\{X_n, \mathcal{F}_n\}$ be a submartingale. Show that it can be decomposed uniquely as

   $$X_n = M_n + A_n, \quad n = 0, 1, \ldots$$

   where $\{M_n, \mathcal{F}_n\}$ is a martingale and the process $\{A_n\}_n$ is increasing, predictable with respect to the filtration $\{\mathcal{F}_n\}_n$ and $A_0 = 0$. Find the processes M and A in terms of the process X.

2. Let $X_n = \sum_{k=1}^{n} I_{B_n}$ for $B_n \in \mathcal{F}_n$. What is the Doob decomposition for X_n?

Exercise 1.4 (L^2-martingales, different types of quadratic variation) Let M_n a martingale such that $M_n \in L^2$ for each n. Therefore, by Jensen's inequality, $(M_n^2)_n$ is a submartingale.

1. (quadratic variation) Define the process V_n by $V_0 = 0$ and $V_n = (M_1 - M_0)^2 + \ldots (M_n - M_{n-1})^2$. It is obvious that V is INCREASING AND ADAPTED. Show that $(M_n^2 - V_n)_n$ is a martingale. The process V is called QUADRATIC VARIATION of M.

2. (predictable quadratic variation) Show that the predictable process A in the DOOB DECOMPOSITION OF $(M_n^2)_n$ has the form

$$A_0 = 0, A_n = \mathbb{E}[(M_1 - M_0)^2|\mathcal{F}_0] + \dots \mathbb{E}[(M_n - M_{n-1})^2|\mathcal{F}_{n-1}].$$

   The process A is called the PREDICTABLE QUADRATIC VARIATION

   Remark: note that $V - A$ is a martingale, so A is ALSO THE PREDICTABLE COMPENSATOR (the predictable part in the Doob decomposition) of V.

Exercise 1.5 Assume that you play repeatedly against an opponent (the house) a game where the probability that you win is p and the probability that the house wins is q. You start with a dollars and the house with b dollars and every time the game is played you win/lose one dollar. The game is played until either you or the house is broke. What is the probability that you go broke before the house does?

Hint: you can model your wealth as an asymmetric random walk M_n starting at $M_0 = a$ going up or down one unit with probabilities p and q. If we denote by

$$\tau_x = \inf\{n|M_n = x\},$$

what you are looking for is $\mathbb{P}(\tau_0 < \tau_{a+b})$. This can be computed the following way

1. find a function $\phi : \mathbb{Z} \to \mathbb{R}$ such that $(\phi(M_n))_n$ is a martingale

2. use the Optional Sampling Theorem to compute $\mathbb{P}(\tau_0 < \tau_{a+b})$