2 Homework 2

The homework will NOT be collected and graded, it is just intended to help you understand the material better. Most of them are well known exercises in Stochastic Processes and Probability, so it is very likely that some of you encountered them before.

Exercise 2.1 Consider a Brownian Motion $(B_t)_{t\geq 0}$. Using the distribution of the Brownian Motion and the properties of conditional expectation, show that $e^{\lambda B_t - \frac{\lambda^2}{2}t}$ is a martingale (with respect to the same filtration for which B is a Brownian Motion) for each $\lambda \in \mathbb{R}$.

Exercise 2.2 (Hitting times for Brownian motion and Brownian Motion with drift). Consider a Brownian Motion W and denote by $W_t^\mu = W_t + \mu t$, for $\mu \in \mathbb{R}$. This is called the Brownian Motion with drift μ. Denote by T_b^μ the first hitting time of level $b > 0$ for the BM with drift:

$$T_b^\mu = \inf\{t | W_t^\mu = b\}.$$

According to the previous exercise, for each λ the process $e^{\lambda W_t^\mu - (\frac{\lambda^2}{2} + \lambda\mu)t}$ is a martingale. Fix $\alpha > 0$. If we solve the equation $\frac{\lambda^2}{2} + \lambda\mu = \alpha$, we find the (only) positive solution $\lambda = -\mu + \sqrt{\mu^2 + 2\alpha}$. For this positive α and λ, we can apply the Optional Sampling Theorem to the martingale $e^{\lambda W_t^\mu - \alpha t}$ up to time $t \wedge T_b^\mu$:

$$1 = \mathbb{E}[e^{\lambda W^\mu_{t\wedge T_b^\mu} - \alpha(t \wedge T_b^\mu)}].$$

Let $t \to \infty$ (and use appropriate bounds for the integrand), to obtain that

$$\mathbb{E}[e^{-\alpha T_b^\mu}] = e^{\mu b - b\sqrt{\mu^2 + 2\alpha}}, \alpha > 0. \tag{2.2}$$

This is the Laplace transform of the hitting time. Please note that the above expression includes the possibility that the hitting time is infinite, since $\alpha > 0$. Now let $\alpha \to 0$ to conclude that

$$\mathbb{P}(T_b^\mu < \infty) = e^{\mu b - b|\mu|}$$

Now take the derivative in (2.2) and then let $\alpha \to 0$ to compute $\mathbb{E}[T_b^\mu]$. Note that the whole idea works for $b < 0$ as well, since the Brownian Motion is symmetric.

Exercise 2.3 Using the notation of the previous exercise, if $\mu > 0$ show that:

$$\mathbb{P}^\mu[\sup_{0\leq t<\infty} (W_t^{-\mu}) \in db] = 2\mu e^{-2\mu b} db.$$