3 Homework 3

The homework will NOT be collected and graded, it is just intended to help you understand the material better. Most of them are well known exercises in Stochastic Processes and Probability, so it is very likely that some of you encountered them before.

Exercise 3.1 Recall that the quadratic variation of a continuous martingale M is a continuous increasing and adapted process (defined as the limit of sums of square increments) and characterized by $M_t^2 - \langle M \rangle_t$ being a martingale. Use this to show that

$$\mathbb{E}[(M_t - M_s)^2 | \mathcal{F}_s] = \langle M \rangle_t - \langle M \rangle_s, \quad 0 \le s \le t.$$

Exercise 3.2 This exercise is actually the rigorous proof for the fact that $\langle B \rangle_t = t$. Consider a Brownian motion B, fix a time t and choose a partition Π given by $0 = t_0 < t_1 < \cdots < t_n = t$ of the interval $[0, t]$. Consider the sum of the square of increments

$$Q_\Pi = \sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2.$$

First show that $\mathbb{E}[Q_\Pi] = t$, and then show that

$$\lim_{\|\Pi\| \to 0} \mathbb{E}[(Q_\Pi - t)^2] = 0,$$

to conclude that $Q_\Pi \to t$ in L^2 and in probability. You will need to use the distribution of the increments of Brownian Motion, together with the fourth moment of normal random variable, namely $\mathbb{E}[N^4] = 3\sigma^4$ if $\mathbb{E}[N] = 0$, $\mathbb{E}[N^2] = \sigma^2$ and N is normal.

Exercise 3.3 We know that the so called stochastic exponential $\mathcal{E}(M)_t = \exp(M_t - \frac{1}{2}\langle M \rangle_t)$ is a solution to the equation

$$dX_t = X_t dM_t, \quad X_0 = 1,$$

if $M_0 = 0$ and M is a continuous martingale. This can be proven using Itô formula (in case it was not done in class already, please prove that).

Now, either using a stochastic integrating factor (which as also a stochastic exponential) or applying Itô to $\log(\mathcal{E}(M)_t)$, show that this is the UNIQUE solution.

Exercise 3.4 Using stochastic integrating factors, find the solution of the linear equation

$$\begin{cases} dX_t = (a(t)X_t + b(t))dt + (c(t)X_t + d(t))dW_t, \\ X_0 = \xi \end{cases}$$

Consider the particular case $a(t) = -a$ for $a > 0$ and $b(t) = b$, $c(t) = 0$ and $d(t) = \sigma$. This is called the Ornstein-Uhlenbeck process (and is called mean reverting).

Exercise 3.5 Consider the OU process described above by

$$dX_t = (b - aX_t)dt + \sigma dW_t.$$

If $f(t, x)$ is a $C^{1,2}$ function, what conditions do we need on f so that the process $f(t, X_t)$ is a (local) martingale?