Homework 1 - Solutions

Problem 1 (Interest Rate Parity)**.** Let $r_t^d(T)$ denote the domestic (U.S.) interest rate at time t over the time interval $[t,T]$ is the sense that 1 dollar may be borrowed or loaned at time t in exchange for $\$(1 + r_t^d(T))$ dollars at time $T > t$. Similiarly, let $r_t^f(T)$ denote the foreign (Euro zone) interest rate. Let S_t denote the spot exchange rate, so 1 euro is worth S_t dollars at time t. Let $F_t(T)$ denote the forward exhange rate. This means that at time t it is possible to enter into an agreement (with no initial no cost) to buy or sell one euro at time $T > t$ for $F_t(T)$ dollars. Find the relationship between the quantities S_t, F_t, $r_t^f(T)$ and $r_t^d(T)$ if the law of one price holds.
Solution: We must have

$$F_t(T)(1 + r_t^f(T)) = S_t(1 + r_t^d(T)).$$

Problem 2 (Put-Call Parity)**.** Let $B_t(T) \leq 1$ denote the price at time of t for a bond which pays one dollar at time T, and let S_t denote the price at time t of a stock which does not pay dividends. A European[1] call option on a stock with strike K and maturity T is a contract which gives the owner the right to purchase the stock at T for K dollars (regardless of the current price of the stock in the market), and a European put option on a stock with strike K and maturity T is a contract which gives the owner the right to sell the stock at T for K dollars (regardless of the current price of the stock in the market). Let $C_t(T,K)$ denote the price at time t for a European call with strike K and maturity T, and let $P_t(T,K)$ denote the price at time t for a European put with strike K and maturity K. Assume that the law of one price holds, and find the relationship between $C_t(T,K)$, S_t, $P_t(T,K)$, and $B_t(T)$.
Solution: At time T, holding a call with maturity T and strike K, being short a put with maturity T and strike K, and holding K bonds with maturity T is equivalent to holding the stock. The law of one price then implies that $C_t(T,K) - P_t(T,K) + K\,B_t(T) = S_t$.

For the next two problems we will need the following definition.

Definition. If X is a bounded random variable and $\mathcal{F}$ is a σ-field, then we say the random variable Y is a version of $\mathbb{E}^{\mathbb{P}}[X \mid \mathcal{F}]$ if

$$\mathbb{E}^{\mathbb{P}}[\mathbb{I}_A X] = \mathbb{E}^{\mathbb{P}}[\mathbb{I}_A Y] \quad \forall A \in \mathcal{F}$$

Problem 3. Let $Z \geq 0$ be a random variable with $\mathbb{E}^{\mathbb{P}}[Z] = 1$ and $\mathbb{P}(Z = 0) = 0$, and let $(\mathcal{F}_t)_t$ be a filtration. Define $\mathbb{Q}(A) \triangleq \mathbb{E}^{\mathbb{P}}[Z\,\mathbb{I}_A]$ and $Z_t \triangleq \mathbb{E}^{\mathbb{P}}[Z \mid \mathcal{F}_t]$.

1. Show $\mathbb{P}$ and $\mathbb{Q}$ are equivalent.

2. Show that $(Z_t)_t$ is a $\mathbb{P}$-martingale.

[1]Note: the use of the name"European" in the description of this option has nothing to do with the exchange rate.

3. Let X be a bounded random variable. Show that $\mathbb{E}^{\mathbb{P}}[Z\,X \mid \mathcal{F}_t]/Z_t$ is a version of $\mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{F}_t]$.

4. Let M_t be a bounded process. Show that $(M_t)_t$ is a $\mathbb{Q}$-martinale if and only if $(Z_t M_t)_t$ is a $\mathbb{P}$-martingale.

Solutions:

1. Its clear that $\mathbb{P} \gg \mathbb{Q}$, and if $\mathbb{P}(A) > 0$, then $\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}[Z\,\mathbb{I}_A] > 0$, so $\mathbb{Q} \gg \mathbb{P}$ by contrapositive.

2. Immediate from the tower property.

3. If we take $A \in \mathcal{F}_t$, then we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\big[\mathbb{I}_A \mathbb{E}^{\mathbb{P}}[ZX \mid \mathcal{F}_t]/Z_t\big] &= \mathbb{E}^{\mathbb{P}}\big[Z\,\mathbb{I}_A\,\mathbb{E}^{\mathbb{P}}[ZX \mid \mathcal{F}_t]/Z_t\big] \\ &= \mathbb{E}^{\mathbb{P}}\big[\mathbb{E}^{\mathbb{P}}[Z \mid \mathcal{F}_t]\,\mathbb{I}_A\,\mathbb{E}^{\mathbb{P}}[ZX \mid \mathcal{F}_t]/Z_t\big] \\ &= \mathbb{E}^{\mathbb{P}}[\mathbb{I}_A Z X] = \mathbb{E}^{\mathbb{P}}[X].\end{aligned}$$

4. If we take $s < t$, then we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[M_t \mid \mathcal{F}_s] &= \mathbb{E}^{\mathbb{P}}[Z_t M_t \mid \mathcal{F}_s]/Z_s, \text{ and} \\ M_s &= Z_s M_s / Z_s.\end{aligned}$$

Since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, we have equality of the left-hand sides $\mathbb{Q}$-a.s. iff we have equality of the right-hand sides $\mathbb{P}$-a.s.

Problem 4. Let $X_1, X_2, \ldots$ be independent random variables[2] with

$$\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = 1/2.$$

Set $\mathcal{F}_n \triangleq \sigma(X_i : i \le n)$, $S_n \triangleq \sum_{i \le n} X_n$, $C = \log_2(5/4)$, and

$$Z_n \triangleq \frac{2^{S_n}}{(5/4)^n} = 2^{S_n - nC} = \prod_{i \le n} 2^{X_i - C}.$$

Finally, fix some $N > 0$ and define a new measure $\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}(Z_N \mathbb{I}_A)$.

1. Is $(S_n)_n$ a $\mathbb{P}$-martingale?

2. If $m < n$, and $f : \mathbb{R}^{n-m} \to \mathbb{R}$ is bounded, show that $\mathbb{E}^{\mathbb{P}}\big[f(X_{m+1}, \ldots, X_n)\big]$ is a version of $\mathbb{E}^{\mathbb{P}}\big[f(X_{m+1}, \ldots, X_n) \mid \mathcal{F}_m\big]$.

3. Is $(Z_n)_n$ a $\mathbb{P}$-martingale?

[2] Independent here means that if fix any subsets $I \subset \mathbb{N}$ and $J \subset \mathbb{N}$ with $I \cap J = \emptyset$, and we choose any bounded random variables Y^I and Y^J where Y^I is measurable with respect to $\sigma(X_i : i \in I)$ and Y^J is measurable with respect to $\sigma(X_j : j \in J)$, then we have $\mathbb{E}^{\mathbb{P}}[Y^I Y^J] = \mathbb{E}^{\mathbb{P}}[Y^I]\,\mathbb{E}^{\mathbb{P}}[Y^J]$.

4. What is $\mathbb{Q}(X_n = 1)$?

5. Are $X_1, X_2, \ldots$ independent under $\mathbb{Q}$?

6. Write S_n as the sum of $\mathbb{Q}$-martingale and a predictable process.

Solution: For any set $I \subset \mathbb{N}$, define $\mathcal{X}(I) = \Pi_{i \in I} 2^{X_i - C}$, and observe that $\mathbb{E}^{\mathbb{P}}[\mathcal{X}(I)] = \Pi_{i \in I} \mathbb{E}^{\mathbb{P}}[2^{X_i - C}] = 1$.

1. Yes. This follows easily from the independence.

2. If $Y \in \mathcal{F}_m = \sigma(X_1, \ldots, X_m)$, then

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\big[f(X_{m+1}, \ldots, X_n)\, Y\big] &= \mathbb{E}^{\mathbb{P}}[f(X_{m+1}, \ldots, X_n)]\, \mathbb{E}^{\mathbb{P}}[Y] \\ &= \mathbb{E}^{\mathbb{P}}\big[\mathbb{E}^{\mathbb{P}}[f(X_{m+1}, \ldots, X_n)]\, Y\big] \end{aligned}$$

3. Yes. If we fix $m < n$ and set $I = \{m+1, m+2, \ldots, n\}$, then we have

$$\mathbb{E}^{\mathbb{P}}[Z_n \mid \mathcal{F}_m] = Z_m\, \mathbb{E}^{\mathbb{P}}[\mathcal{X}(I) \mid \mathcal{F}_m] = Z_m\, \mathbb{E}^{\mathbb{P}}[\mathcal{X}(I)] = Z_m$$

4. $\mathbb{Q}(X_n = 1) = 4/5$. To see this, set $I = \{1, 2, \ldots, i-1, i+1, \ldots, n\}$, so $i \notin I$. Then

$$\begin{aligned} \mathbb{Q}(X_n = 1) &= \mathbb{E}^{\mathbb{P}}[\mathbb{I}_{\{X_n = 1\}} Z_N] \\ &= \mathbb{E}^{\mathbb{P}}[\mathbb{I}_{\{X_n = 1\}} 2^{X_i - C}\, \mathcal{X}(I)] \\ &= \mathbb{E}^{\mathbb{P}}[\mathbb{I}_{\{X_n = 1\}} 2^{X_i - C}]\, \mathbb{E}^{\mathbb{P}}[\mathcal{X}(I)] \\ &= \mathbb{E}^{\mathbb{P}}[\mathbb{I}_{\{X_n = 1\}} 2^{X_i - C}] = 4/5 \end{aligned}$$

5. Yes. Fix and $I \subset \mathbb{N}$ and $J \subset \mathbb{N}$ with $I \cap J = \emptyset$, and set $K = \{1, 2, \ldots, N\}$, so $Z_N = \mathcal{X}(K)$. If $Y^I \in \sigma(X_i : i \in I)$, and $Y^J \in \sigma(X_j : j \in J)$, then

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[Y^I Y^J] &= \mathbb{E}^{\mathbb{P}}[Y^I Y^J Z_N] \\ &= \mathbb{E}^{\mathbb{P}}[Y^I \mathcal{X}(I \cap K) Y^J \mathcal{X}(J \cap K) \mathcal{X}(K \setminus (I \cup J))] \\ &= \mathbb{E}^{\mathbb{P}}[Y^I \mathcal{X}(I \cap K)]\, \mathbb{E}^{\mathbb{P}}[Y^J \mathcal{X}(J \cap K)] \\ &= \mathbb{E}^{\mathbb{P}}[Y^I \mathcal{X}(I \cap K)]\, \mathbb{E}^{\mathbb{P}}[\mathcal{X}(K \setminus I)]\, \mathbb{E}^{\mathbb{P}}[Y^J \mathcal{X}(J \cap K)]\, \mathbb{E}^{\mathbb{P}}[\mathcal{X}(K \setminus J)] \\ &= \mathbb{E}^{\mathbb{P}}[Y^I Z_N]\, \mathbb{E}^{\mathbb{P}}[Y^J Z_n] = \mathbb{E}^{\mathbb{Q}}[Y^I]\, \mathbb{E}^{\mathbb{Q}}[Y^J] \end{aligned}$$

6. It enough to notice that $\mathbb{E}^{\mathbb{Q}}[X_i - 3/5] = 0$, so we can write

$$S_n = \underbrace{\big(S_n - n(3/5)\big)}_{\mathbb{Q}\text{-martingale}} + \underbrace{n(3/5)}_{\text{predicatable}}.$$