Homework 2 - Solutions

Problem 1. Let W be a Brownian motion under $\mathbb{P}$, set $M_t = 1 + W_t$, and let $\tau = \inf\{t : M_t \leq 0\}$ denote the first time that M hits 0. Now fix some $T > 0$, and set $Z = M_{T\wedge\tau}$.

1. Show that $\mathbb{E}^{\mathbb{P}}[Z] = 1$.
2. Now define $\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}(\mathbb{I}_A Z)$. Compute $\mathbb{Q}(\tau \leq T)$?
3. Are $\mathbb{P}$ and $\mathbb{Q}$ equivalent?

Solution:

1. Set $N_t = M_{t\wedge\tau}$. Then N is a martingale by the optional stopping theorem, so
$$\mathbb{E}^{\mathbb{P}}[Z] = \mathbb{E}^{\mathbb{P}}[N_T] = \mathbb{E}^{\mathbb{P}}\big[\mathbb{E}^{\mathbb{P}}[N_T \mid \mathcal{F}_0]\big] = \mathbb{E}^{\mathbb{P}}[N_0] = 1$$
2. We have
$$\begin{aligned}\mathbb{Q}(\tau \leq T) &= \mathbb{E}^{\mathbb{P}}[\mathbb{I}_{\{\tau\leq T\}} M_{T\wedge\tau}]\\ &= \mathbb{E}^{\mathbb{P}}[\mathbb{I}_{\{\tau\leq T\}} M_{\tau}] = 0\end{aligned}$$
3. No. $\mathbb{Q}(\tau \leq T) = 0$, but $\mathbb{P}(\tau \leq T) > 0$

Problem 2. Let W be a Brownian motion under $\mathbb{P}$, let Z solve $\mathrm{d}Z_t = -\lambda_t Z_t \mathrm{d}W_t$, and set $\widehat{W}_t = W_t + \int_0^t \lambda_s \mathrm{d}s$.

1. Compute $\mathrm{d}(1/Z)_t$ to show that process $1/Z$ can be written as a stochastic integral against the process $\widehat{W}_t$
2. Suppose that $\mathrm{d}M_t = H_t \mathrm{d}W_t$. Compute $\mathrm{d}(M/Z)_t$ to show that M/Z can be written as a stochastic integral against the process $\widehat{W}_t$.
3. Let $(\mathcal{F}_t)$ denote the filtration generated by W, and let $\mathbb{Q}$ denote the measure with $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = Z_T$. If all processes are adapted to $\mathcal{F}$, argue that every $\mathbb{Q}$-martingale can be written as a stochastic integral against $\widehat{W}_t$ over the time interval $[0, T]$.

Solutions:

1. We have $\mathrm{d}(1/Z_t) = (\lambda_t/Z_t)\,\mathrm{d}\widehat{W}_t$.
2. We have $\mathrm{d}(M/Z)_t = \dfrac{H_t + \lambda_t M_t}{Z_t}\,\mathrm{d}\widehat{W}_t$.
3. Let $\widehat{M}$ be a $\mathbb{Q}$ martingale. Then MZ is a $\mathbb{P}$ martingale and we can find H such that $M_t Z_t = C + \int_0^t H_s\,\mathrm{d}s$. The previous part then show that we can write $M = MZ/Z$ as a stochastic integral against $\widehat{W}$.

Problem 3 (Brownian Representation)**.** Let (W_t) be a Brownian motion (with $W_0 = 0$) and let $(\mathcal{F}_t)_t$ denote the filtration generated by W. Now fix some $T > 0$, and set $X_t = \mathbb{E}^{\mathbb{P}}[W_T^3 \mid \mathcal{F}_t]$, so X_t is a martingale. Since W has the predictable representation property, there exists a constant C and an integrand H such that $X_t = C + \int_0^t H_u \mathrm{d}W_u$. We would like to find C and H. One approach is outlined below:

1. Use Itô's lemma to write W_T^3 in the form

$$W_T^3 = W_t^3 + \int_t^T A_u \mathrm{d}u + \int_t^T B_u \mathrm{d}W_u$$

   for appropriate A and B.

2. Use the previous expression to compute $\mathbb{E}^{\mathbb{P}}[W_T^3 \mid \mathcal{F}_t]$ explicitly. You may pull integrals over time out of conditional expectations (which are essentially integrals over Ω). This should be enough to determine C

3. Set $f(t, W_t) = \mathbb{E}^{\mathbb{P}}[W_T^3 | \mathcal{F}_t]$ and use the explicit formula above and compute $\mathrm{d}f(t, W_t)$ to find H.

Solution:

1. We have $A_t = 3W_t$ and $B_t = 3W_t^2$.

2. We notice that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[W_T^3 \mid \mathcal{F}_t] &= W_t^3 + \mathbb{E}^{\mathbb{P}}\Big[\int_t^T 3W_u \mathrm{d}u + \int_t^T 3W_u^3 \mathrm{d}W_u \;\Big|\; \mathcal{F}_t\Big] \\ &= W_t^3 + \int_t^T 3\, \mathbb{E}^{\mathbb{P}}[W_u | \mathcal{F}_t] \mathrm{d}u \\ &= W_t^3 + \int_t^T 3W_t \mathrm{d}u = W_t^3 + 3(T-t)W_t.\end{aligned}$$

   This means that $C = \mathbb{E}^{\mathbb{P}}[W_T^3] = 0$.

3. With $f(t, x) = x^3 + 3(T-t)x$, we have $\mathrm{d}f(t, W_t) = 3\big(W_t^2 + (T-t)\big)\mathrm{d}W_t$, so $H_t = 3(W_t^2 + (T-t))$.