Homework 3 - Solutions

Problem 1 (Self-financing in continuous-time)**.**

Suppose that we have a Brownian market model with two assets M and S, and let $\hat{X}_t$ denote the value of a portfolio which is self-financing when denominated in the terms of the numeraire M. That is to say, there exist processes H and K such that

$$\widehat{X}_t = H_t\,\widehat{M}_t + K_t\,\widehat{S}_t, \text{ and} \tag{1}$$

$$\mathrm{d}\widehat{X}_t = H_t\,\mathrm{d}\widehat{M}_t + K_t\,\mathrm{d}\widehat{S}_t. \tag{2}$$

where $\widehat{M} = M/M = 1$, and $\widehat{S} = S/M$. Using Itô's lemma, show that X is still self-financing when denominated in cash. That is, show that (1) and (2) imply

$$\begin{aligned} X_t &= H_t\,M_t + K_t\,S_t, \text{ and} \\ \mathrm{d}X_t &= H_t\,\mathrm{d}M_t + K_t\,\mathrm{d}S_t. \end{aligned}$$

Solution: We have:

$$\begin{aligned} \mathrm{d}X_t &= \mathrm{d}(\widehat{X}M)_t \\ &= M_t\,\mathrm{d}\widehat{X}_t + \widehat{X}_t\,\mathrm{d}M_t + \mathrm{d}[\widehat{X}, M]_t \\ &= M_t\big(K_t\,\mathrm{d}\widehat{M}_t + H_t\,\mathrm{d}\widehat{S}_t\big) + \big(K_t\,\widehat{M}_t + H_t\,\widehat{S}_t\big)\mathrm{d}M_t + H_t\,\mathrm{d}\big[\widehat{M}_t, M\big]_t + K_t\,\mathrm{d}\big[\widehat{S}_t, M\big]_t \\ &= K_t\big(M_t\,\mathrm{d}\widehat{M}_t + \widehat{M}_t\,\mathrm{d}M_t + \mathrm{d}[\widehat{M}, M]_t\big) + H_t\big(M_t\,\mathrm{d}\widehat{S}_t + \widehat{S}_t\,\mathrm{d}M_t + \mathrm{d}[\widehat{S}, M]_t\big) \\ &= K_t\,\mathrm{d}(\widehat{M}M)_t + H_t\,\mathrm{d}(\widehat{S}M)_t \\ &= K_t\,\mathrm{d}M_t + H_t\,\mathrm{d}S_t. \end{aligned}$$

Problem 2 (Implying the risk-neutral density for the stock)**.** Working is a complete model, suppose that the call prices are twice differentiable as a function of strike. Let $p(t, x)$ denote the density of S_t under the pricing measure $\mathbb{Q}$, and notice that

$$C(T, K) = \mathbb{E}^{\mathbb{Q}}[e^{-rT}(S_T - K)^+] = e^{-rT}\int_K^\infty (x - K)\,p(T, x)\,\mathrm{d}x$$

where $C(T, K)$ is the initial price for a call with maturity T and strike K. Differentiate this expression twice with respect to K and write the density of S_T under the pricing measure in terms of the call prices.
Solution: We have

$$\begin{aligned} C_K(T, K) &= -e^{-rT}\int_K^\infty p(T, x)\,\mathrm{d}x, \text{ and} \\ C_{KK}(T, K) &= e^{-rT}p(T, K), \text{ so} \\ p(T, K) &= e^{rT}C_{KK}(T, K). \end{aligned}$$

Problem 3 (Change of numeraire)**.** In this problem, we are working in the setting of the Black-Scholes model on a finite time interval $[0, T]$ with the assets M and S where M and S solve

$$\mathrm{d}M_t = rM_t\mathrm{d}t \tag{3}$$

$$\mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t. \tag{4}$$

In class, we showed that S/M is a martingale on the time interval $[0, T]$ under the pricing measure $\mathrm{d}\mathbb{Q}^M$ given by

$$\frac{\mathrm{d}\mathbb{Q}^M}{\mathrm{d}\mathbb{P}} \triangleq \exp\{-\lambda W_T - (\lambda^2/2)\,T\}$$

where $\lambda = (\mu - r)/\sigma$ is the "market price of risk". We would now like to compute the pricing measure if we choose S as the numeraire.

1. Using Itô's lemma, compute $\mathrm{d}(M/S)$.
2. Use Girsonov's Theorem to find a measure $\mathbb{Q}^S$ under which M/S is a martingale.
3. Set $Z_t^M \triangleq \mathbb{E}^{\mathbb{P}}[\mathrm{d}\mathbb{Q}^M/\mathrm{d}\mathbb{P} \mid \mathcal{F}_t]$ for $t \in [0, T]$. In the previous homework, we showed that $(M/M)Z^M$ and $(S/M)Z^M$ are $\mathbb{P}$-martingales when M/M and S/M are a $\mathbb{Q}^M$-martingales. Notice that we can write

$$(M/M)Z^M = (M/S)\,(S/M)Z^M, \text{ and}$$
$$(S/M)Z^M = (S/S)\,(S/M)Z^M.$$

   This immediately gives us a candidate for $Z^S = \mathrm{d}\mathbb{Q}^S/\mathrm{d}\mathbb{P}$. Compute Z^S explicitly and check that this agrees with your calculation above.

Solution:

1. We have

$$\begin{aligned}\mathrm{d}(M/S)_t &= \mathrm{d}M_t/S_t - (M_t/S_t^2)\mathrm{d}S_t + (M_t/S_t^3)\mathrm{d}\langle S\rangle_t \\ &= -\sigma(M_t/S_t)\big(\mathrm{d}W_t + (\sigma - \lambda)\mathrm{d}t\big).\end{aligned}$$

2. We take $\mathrm{d}\mathbb{Q}^S/\mathrm{d}\mathbb{P} = \exp\{-(\sigma - \lambda)W_t - (\sigma - \lambda)^2/2\,t\}$.
3. The candidate is

$$\begin{aligned}(S_T/M_T)Z_T^M &= \frac{\exp\{(\mu - \sigma^2/2)\,T + \sigma W_T\}}{\exp\{rT\}}\exp\{-\lambda W_T - (\lambda^2/2)\,T\} \\ &= \exp\{(\mu - r - \sigma^2/2 - \lambda^2/2)T + (\sigma - \lambda)W_T\} \\ &= \exp\{-(\sigma - \lambda)^2/2)\,T + (\sigma - \lambda)W_T\}\end{aligned}$$

   which agrees with the calculation above.

Problem 4 (Fokker-Planck or Kolmogorov Forward Equation)**.** Let X_t solve the SDE:

$$\mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t,$$

and suppose that X_t has a twice continuously differentiable density $p(t, x)$ for all $t > 0$. That is, $\mathbb{P}[x_0 < X_t < x_1] = \int_{x_0}^{x_1} p(t, x)\,\mathrm{d}x$. Finally let $f(x)$ be a twice continuously differentiable function which is zero outside of some interval $[a, b]$.

1. Using Itô's Lemma, write $f(X_t) - f(X_s)$ in the form

   $$f(X_t) - f(X_s) = \int_s^t g(t, X_s)\mathrm{d}s + \int_s^t h(t, X_s)\,\mathrm{d}W_s$$

   for appropriate g and h.

2. Take expectations on both sides and rewrite the result as integrals against the density p. You may interchange the order of time integrals and expectations.

3. Use integration by parts to move all the derivatives off of f, and differentiate both side with respect to t.

Conclude that, since the previous equation holds for all f with two derivatives and compact support, we must in fact have

$$p_t = -\frac{\partial}{\partial x}(\mu p_x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 p_{xx}).$$

This PDE is know to physicists as the Fokker-Planck equations and to mathematicians at the Kolmogorov forward equation.

Solutions:

1. We have

   $$\begin{aligned} f(X_t) - f(X_s) = &\int_0^t \mu(s, X_s) f_x(s, X_s) + \tfrac{1}{2}\sigma^2(s, X_s) f_{xx}(s, X_s)\,\mathrm{d}s \\ &+ \int_0^t \sigma(s, X_s) f_x(s, X_s)\mathrm{d}W_s \end{aligned}$$

2. We have

   $$\begin{aligned} &\int_a^b f(x)\big(p(t, x) - p(s, x)\big)\,\mathrm{d}x \\ &\quad = \int_a^b \int_s^t \big(\mu(u, x) f_x(u, x) + \tfrac{1}{2}\sigma^2(u, x) f_{xx}(u, x)\big)\, p(u, x)\,\mathrm{d}u\,\mathrm{d}x \end{aligned}$$

3. We then have

   $$\begin{aligned} \int_a^b f(x)\, p_t(t, x)\,\mathrm{d}x = &-\int_a^b f(t, x)\frac{\partial}{\partial x}\big[\mu(t, x) p_x(t, x)\big]\,\mathrm{d}x \\ &+ \frac{1}{2}\int_a^b f(x)\,\frac{\partial^2}{\partial x^2}\big[\sigma^2(t, x) p_{xx}(t, x)\big]\Big)\,\mathrm{d}x \end{aligned}$$