UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 1

Problem 1.1. Let Ω be a nonempty finite set, and let $\mathbb{A}$ be the family of all algebras on Ω, and let Π be the family of all partitions of Ω. Construct a mapping $F : \Pi \to \mathbb{A}$ as follows: for a partition $\mathcal{P} = \{A_1, A_2, \dots, A_k\}$ of Ω, let $F(\mathcal{P}) = \mathcal{A}$ be the family consisting of the empty set and all possible unions of elements of $\mathcal{P}$.

(1) Show that so defined family $\mathcal{A}$ is an algebra, and
(2) Show that the mapping F is one-to-one and onto.

Problem 1.2. For $n \in \mathbb{N}$, let a_n be the number of different algebras on Ω when Ω has exactly n elements. Show that

(1) $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, and that the following recursion holds

$$a_{n+1} = \sum_{k=0}^{n} \binom{n}{k} a_k,$$

where $a_0 = 1$ by definition, and

(2) the exponential generating function for the sequence $\{a_n\}_{n\in\mathbb{N}}$ is $f(x) = e^{e^x-1}$, i.e., that

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = e^{e^x-1}.$$

(*Note:* The numbers $a_0, a_1, \dots$ are called **Bell numbers**.)

Problem 1.3. Show the following equivalence: $\xi_2 \in \mathcal{N}$ is a successor of $\xi_1 \in \mathcal{N}$ if and only if there exists representatives (t_1, ω_1) and (t_2, ω_2) of ξ_1 and ξ_2 such that $t_1 \leq t_2$ and $\omega_2 = \omega_2$.