UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 2

Problem 2.1. Prove the fundamental theorem of asset pricing.

(*Hint.* Use the fact that $\langle W \rangle$ is a convex set which does not intersect the (convex and compact) unit simplex $\mathcal{S}^n = \{(c_1, \dots, c_n) \in \mathbb{R}^n_+ \, : \, \sum_{i=1}^n c_i = 1\}$.)

Problem 2.2. Show that

$$\dim \langle W \rangle = \sum_{\xi \in \mathcal{N}^-} \operatorname{rank}(D(\xi^+) + q(\xi^+)),$$

where $\mathcal{N}^-$ denotes the set of all non-terminal nodes and $D(\xi^+) + q(\xi^+)$ is a $b(\xi) \times J$-matrix, whose rows correspond to all children of ξ, and columns to all contracts.

(*Hint:* Construct the matrix $\tilde{W}$ be multiplying each row of W by $\pi(\xi)$, where ξ is the node corresponding to that row and π is a present-value price process. $\tilde{W}$ has the same rank as W and you can use the "martingale" property $\pi(\xi) q_j(\xi) = \sum_{\xi' >_c \xi} \pi(\xi')(D^j(\xi') + q_j(\xi'))$ to perform appropriate operations on its rows to make it block-diagonal.)

Problem 2.3. Consider a market where $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $T = 1$, $\mathcal{A}_0 = \{\Omega, \emptyset\}$ and $\mathcal{A}_1 = \mathcal{P}(\Omega)$ (where $\mathcal{P}(X)$ denotes the power set of X, i.e., the set of all subsets of X.) Moreover, there are two contracts, both issued at ξ_0, with dividend processes

$$D^1(\xi_1) = 10, \; D^1(\xi_2) = -20, \; D^1(\xi_3) = 60,$$

and

$$D^2(\xi_1) = 20, \; D^2(\xi_2) = 30, \; D^2(\xi_3) = 10.$$

(1) Characterize the set Q of all price-processes $q = (q^1, q^2)$, such that there is no arbitrage.
(2) For each price process $q \in P$, characterize the set of all vectors of present-value prices $\Pi(q)$.
(3) Pick a vector $q \notin P$, and construct an arbitrage portfolio, i.e., a portfolio process (a pair of numbers, really) z such that $Wz \geq 0$, $Wz \neq 0$.