UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 4

Problem 4.1. Show that the set $Q(\bar{D})$ of arbitrage-free prices of the contract with dividend process $\bar{D}$ is given by

$$Q(\bar{D}) = \{\mathbb{E}^{\pi}[\bar{D}|\cdot] \,:\, \pi \in \mathcal{M}\},$$

where $\mathcal{M}$ denotes the set of all present-value vectors. Conclude that a financial market is complete if and only if each dividend process admits exactly one arbitrage-free price.

Problem 4.2 (Prices as discounted expectations)**.** Let $\mathcal{F}$ be an arbitrage free financial market, and let π be a process of present-value prices. For each non-terminal node ξ, define $\delta(\xi;\pi) = \frac{1}{\pi(\xi)} \sum_{\xi' >_c \xi} \pi(\xi')$ (where the sum is taken over all children of ξ), and set $r(\xi;\pi) = 1/\delta(\xi;\pi) - 1$. The (not-necessarily-positive) quantity $r(\xi;\pi)$ is called the **π-implied short rate** at ξ.

(1) Suppose that there exists a short-lived bond for the node ξ with price $q(\xi)$. Then $\delta(\xi;\pi) = q(\xi)$, for all $\pi \in \mathcal{M}$.
(2) Give an example of a situation where $r(\xi;\pi)$ depends on π.
(3) A process X is called an **$\mathcal{N}$-martingale density**, if $X(\xi) > 0$ for all $\xi \in \mathcal{N}$ and

$$X(\xi) = \sum_{\xi' >_c \xi} X(\xi'),$$

for all non-terminal nodes ξ. An $\mathcal{N}$-martingale density X is said to be **normalized** if $X(\xi_0) = 1$. Show that for each normalized $\mathcal{N}$-martingale density there exists a unique probability - denoted by $\mathbb{Q}_X$ - on $(\Omega, \mathcal{A}_T)$ with the property that for $A \in \mathcal{A}_t$, $t = 0, \dots, T$, we have

$$\mathbb{Q}_X(A) = \sum X(\xi),$$

where the sum is taken over all nodes ξ such that $(t,\omega) \in \xi$ for some $\omega \in A$.
(4) A process Y is called **$\mathcal{N}$-predictable** if $Y(\xi) = X(\xi^-)$ for some (adapted) process X and all non-initial nodes ξ (where ξ^- denotes the parent of the non-terminal node ξ). Characterize all processes X which are both $\mathcal{N}$-predictable and $\mathcal{N}$-martingale densities.
(5) Show that, for each $\pi \in \mathcal{M}$, there exists a unique normalized martingale density X_π and a predictable process β with $\beta_\pi(\xi_0) = 1$ such that

$$\pi(\xi) = \beta(\xi) X_\pi(\xi).$$

The process β is called the **π-implied discount process**, and the measure $\mathbb{Q}_\pi = \mathbb{Q}_{X_\pi}$ is called the **martingale measure** (corresponding to π). What is the relation between β and r?
(6) For a dividend process D, let $\hat{D}$ be the **π-discounted** version of D, which is given by $\hat{D}(\xi) = \beta(\xi) D(\xi)$. Let the random variable C on $(\Omega, \mathcal{A}_T)$ be defined by $C(\omega) = \sum \hat{D}(\xi)$, where the sum is taken over all ξ such that $(t,\omega) \sim \xi$ for *some* $t = 0, \dots, T$. Show that

$$\mathbb{E}^{\pi}[D|\xi_0] = \mathbb{E}^{\mathbb{Q}_\pi}[C],$$

where the right-hand side is the (proper) expectation of the random variable C with respect to the measure $\mathbb{Q}_\pi$.

Problem 4.3 (The Cox-Ross-Rubinstein Model)**.** In this model $T \geq 1$ and $\Omega = \{(x_1, \dots, x_T) : x_i \in \{1-b, 1+a\}, i = 1, \dots, T\}$, where $a, b > 0$, $b < 1$. Set

$$X_0(\omega) = q_0, \text{ and } X_t(\omega) = \Pi_{i=1}^t x_i, \text{ where } \omega = (x_1, \dots, x_T),$$

for $t = 1, \dots, T$. Let $\mathcal{A}_t$ be the algebra generated by $X_0, X_1, \dots, X_t$ (since X is, by construction, adapted to $(\mathcal{A}_0, \dots, \mathcal{A}_T)$ we will interpret it as a function on the nodes without explicit mention). The financial market $\mathcal{F}$ (which defines the Cox-Ross-Rubinstein model) is based on Ω, the filtration $(\mathcal{A}_0, \dots, \mathcal{A}_T)$, and 2^T contracts:

- A **"stock":** with the dividend process D given by

$$D_{\text{stock}}(\xi) = \begin{cases} X(\xi), & \xi \text{ is terminal} \\ 0, & \text{otherwise,} \end{cases}$$

- A system of **short-lived bonds:** for each non-terminal node ξ, there is a contract $D_{\text{bond}-\xi}$, issued at ξ, with the dividend process

$$D_{\text{bond}-\xi}(\xi') = \begin{cases} 1, & \xi' >_c \xi, \\ 0, & \text{otherwise,} \end{cases}$$

The prices of these contracts are fixed and given by

-

$$q_{\text{stock}}(\xi) = \begin{cases} X(\xi), & \xi \text{ is } \textit{not} \text{ terminal} \\ 0, & \text{otherwise,} \end{cases}$$

- A system of **"short-lived bonds"**. For each non-terminal node ξ, there is a contract $D_{\text{bond}-\xi}$, issued at ξ, with the dividend process

$$q_{\text{bond}-\xi}(\xi') = \begin{cases} 1/(1+r), & \xi' = \xi, \\ 0, & \text{otherwise,} \end{cases}$$

  for some $r > 0$.

Test your understanding of the whole theory by answering the following questions:

(1) For which values of a, b and r is the market arbitrage-free?

(2) Assume from now on that a, b and r are such that the market is arbitrage-free. Show that there is a unique process of present-value prices, and that the market is complete.

(3) Decompose the (unique) π into a product of a normalized martingale density and the implied discount process. In particular, what is the value of $\mathbb{Q}_\pi(\omega)$ for $\omega \in \Omega$?

(4) Let $\bar{D}$ be a divided process such that (and we assume this only for simplicity) $\bar{D}(\xi) = 0$, unless ξ is terminal. Assume, further, that $\bar{D}_T$ is $\mathcal{A}_T$-measurable, i.e., that there exists a function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\bar{D}(\xi) = \phi(X(\xi))$, for non-terminal ξ. Show

that the (unique) arbitrage-free price of $\bar{D}$, given, as we know, by $\mathbb{E}^{\pi}[\bar{D}|\xi_0]$, can be written as

$$(1+r)^{-T} \sum_{k=0}^{T} \binom{T}{k} q_{up}^{k}\, q_{down}^{T-k}\ \phi\Big(q_0(1+a)^k(1-b)^{T-k}\Big),$$

where $q_{up} = \frac{r+b}{a+b}$ and $q_{down} = \frac{a-r}{1+b}$.

(5) (*) Assume now that $T_n = \tau n$, $a_n = \mu\frac{1}{n} + \sigma\frac{1}{\sqrt{n}}$, $b_n = -\mu\frac{1}{n} + \sigma\frac{1}{\sqrt{n}}$ and $r_n = \rho\frac{1}{n}$, so that the times between periods become smaller and smaller, but then so do the magnitudes of price changes between them. Assume, also, that ϕ is bounded and continuous. Pass to the limit in the formula above as $n \to \infty$ (use the Moivre-Laplace formula which is a special case of the Central Limit Theorem), and show that it converges towards an expectation of a function (related to ϕ) of a normal random variable. When $\phi(x) = (K-x)^+$, you should get the celebrated Black-Scholes formula for the arbitrage-free price of a put option with strike K.