UNIVERSITY OF TEXAS AT AUSTIN

HW Assignment 5

Problem 5.1. Show that (see slides for notation)

$$B(e) = \{c : \mathcal{N} \to \mathbb{R} \,:\, \forall \pi \in \mathcal{M},\ \pi \cdot c \leq \pi \cdot e\}.$$

(*Hint.* One direction is easy - just use positivity of π. For the other direction, it is clearly enough to consider the case $e = 0$, and suppose that there exists c^* such that $\pi \cdot c^* \leq 0$, for all $\pi \in \mathcal{M}$, but c^* is not dominated by any vector of the form Wz. That means that the compact and convex set $\{c^*\}$ and the closed and convex set $B(0)$ do not intersect. Therefore, there exists a linear functional $\tilde{\pi}$ which separates them. Show that $\tilde{\pi}$ can be normalized so that $\tilde{\pi} \in \mathcal{M}$ and reach a contradiction.

Problem 5.2. Prove that there exists a compact subset K of $B(e)$ such that $U(c) < U(e)$ for $c \in B(e) \setminus K$. (*Hint.* If one of the coordinates of c is "large positive ", some other will have to be "large negative". The fact that $U'(x) \to \infty$ as $x \to -\infty$ implies that a "large negative" coordinate will destroy the utility to the extent that no other "large positive" coordinate can repair. Now translate this into mathematics.)

Problem 5.3. Let $\mathcal{F}$ be an incomplete financial market. Given $\pi^* \in \mathcal{M}$, show that there exist a finite number $\bar{D}^{J+1}, \dots, \bar{D}^{J+m}$ of contracts with the property that the market $\mathcal{F}' = \mathcal{F} \cup \bar{D}^{J+1} \cup \dots \cup \bar{D}^{J+m}$ is complete and its unique present-value price process is π^*.

Problem 5.4. Let $\bar{q}$ be a MUBP for the contract $\bar{D}$. Then

(1) $\bar{q}$ is an arbitrage-free price process for $\bar{D}$.
(2) $\bar{q}$ is unique.

(*Hint.* For the first part, note that an arbitrage opportunity would certainly not be overseen by a utility-maximizing investor. She would pump money into it. In order to prove the second part, try to deduce it from the proof of the existence theorem. Alternatively, try to construct a purely financial argument, but that might be quite difficult.)