RTG Mini-Course
Perspectives in Geometry Series

Jacob Lurie*

Lecture I: Topological Field Theories (1/20/2009)

All manifolds are smooth, compact, oriented, and possibly with boundary. A manifold is closed if it has empty boundary. We define a category $\mathbf{Cob}(d)$ as follows:

objects:	close $(d-1)$-manifolds
morphisms:	bordisms, modulo diffeomorphism
composition:	gluing of bordisms

Definition (Atiyah)**.** A *topological quantum field theory* (TQFT) of dimension d is a $\otimes$-functor

$$Z : \mathbf{Cob}(d) \longrightarrow \mathbf{Vect}_{\mathbb{C}}.$$

That Z is a $\otimes$-functor (or more precisely, a *symmetric monoidal functor*) means that tensor products are respected, with the tensor product in $\mathbf{Cob}(d)$ being disjoint union. Thus

$$Z(M \coprod N) \cong Z(M) \otimes Z(N), \quad Z(\varnothing) \cong \mathbb{C}.$$

Example (d=2)**.** The only closed, connected 1-manifold is S^1. Suppose we know that

$$Z(S^1) = A,$$

for some vector space A. Because Z is a $\otimes$-functor, we can evaluate Z on all objects in $\mathbf{Cob}(2)$ (which are disjoint unions of copies of S^1). For example,

$$Z(S^1 \coprod S^1 \coprod S^1) = A \otimes A \otimes A.$$

What about morphisms? There are several interesting examples.

*Scribes: Braxton Collier, Parker Lowrey, Michael B. Williams

- If B_1 is the bordism from two circles to one circle (i.e., the pair of pants), then

$$Z(B_1) = m : A \otimes A \longrightarrow A.$$

We can think of this a multiplication on A, as it is associative, commutative, and has a unit element.

- If B_2 is the bordism from the empty set to one circle, then

$$Z(B_2) : Z(\varnothing) \cong \mathbb{C} \longrightarrow A,$$

We identify this map with an element $1 \in A$, which turns out to be a unit for m.

- If B_3 is the bordism from the circle to the empty set, then

$$Z(B_3) = \mathrm{tr} : A \longrightarrow \mathbb{C},$$

which we call the *trace map.*

Composing these maps gives a nondegenerate pairing, i.e., provides an isomorphism $A \cong A^\vee$:

$$A \otimes A \xrightarrow{m} A \xrightarrow{\mathrm{tr}} \mathbb{C},$$

which is also the value of the bordism from two circles to the empty set. Note that this implies that A must be finite dimensional.

Definition. A vector space A with multiplication as above is a *commutative Frobenius algebra* (CFA).

Theorem. *Two-dimensional* TQFT*s are equivalent to finite-dimensional* CFA*s. That is, conversely to the above example, a* CFA *A gives rise to a* TQFT *Z such that*

$$Z(S^1) = A, \quad Z(B_2) = 1, \quad Z(B_3) = \mathrm{tr}\,.$$

¿From this prespective, a TQFT in dimension d emphasizes $(d-1)$-manifolds. Another point of view is to consider d-manifolds, which we can think of as bordisms from the empty set to itself, thought of as a $(d-1)$-manifold. Since $Z(\varnothing) = \mathbb{C}$, this gives rise to a map

$$Z(M) : \mathbb{C} \longrightarrow \mathbb{C},$$

which is just multiplication by some scalar $\lambda \in \mathbb{C}$. Identifying $Z(M) = \lambda$, we obtain a diffeomorphism invariant for the manifold M.

Example. We saw that in dimension 2, a CFA A gives rise to a TQFT Z, and we would like to be able to compute $Z(\Sigma_g)$ for a genus g surface Σ_g. For this we can chop up Σ_g into smaller pieces, namely discs and pairs of pants, on which the value of Z is easy to compute. For example, $Z(\Sigma_0) = \mathrm{tr}(1)$, and $Z(\Sigma_1) = \dim A$.

For large d, it is not always possible to simplify calculations by cutting a manifold along submanifolds. Another idea is to use a triangulation, but this requires more general gluing menthods involving manifolds with corners, and this is not handled by our original definition of TQFT. There are more elaborate definitions that have been proposed for an *extended* TQFT, which (roughly!) takes the form of a rule that make the following associations:

- closed d-manifold $\rightsquigarrow$ complex number
- closed $(d-1)$-manifold $\rightsquigarrow$ complex vector space
- bordism of $(d-1)$-manifolds $\rightsquigarrow$ linear map of complex vector spaces

(So far this agrees with the origial definition.)

- closed $(d-2)$-manifold $\rightsquigarrow$ $\mathbb{C}$-linear category, e.g., $\mathbf{Vect}_{\mathbb{C}}$
- bordism of $(d-2)$-manifolds $\rightsquigarrow$ $\mathbb{C}$-linear functors of $\mathbb{C}$-linear categories
- $\cdots$

We also need a number of compatibility relations at each step.

We want an extended TQFT to still be some sort of functor, but to make this precise we need higher category theory. Then a TQFT is a $\otimes$-functor of d-categories. More on higher categories will be explained in future lectures.

Very informally, we now state the *cobordism hypothesis* (CH) of Baez and Dolan:

Extended TQFT*s are "easy" to describe, construct, and classify.*

The idea is similar to that described above: specify a TQFT on simple pieces, such as Euclidean space or simplices, and let the categorical machinery do the rest.

Example (d=1)**.** Objects in $\mathbf{Cob}(1)$ are oriented 0-manifolds, that is, collections of points labelled "+" or "−". A TQFT is a functor

$$Z : \mathbf{Cob}(1) \longrightarrow \mathbf{Vect}_{\mathbb{C}}$$

so suppose

$$Z(+) = X, \quad Z(-) = Y.$$

How are these vector spaces related? The bordism from the 0-manifold $\{+,-\}$ to the empty set gives a map

$$X \otimes Y \longrightarrow \mathbb{C},$$

and the bordism from the empty set to $\{+,-\}$ gives a map

$$\mathbb{C} \longrightarrow X \otimes Y.$$

These maps exhibit X and Y as duals, and both are finite dimensional. Therefore to specify Z on objects, we only need $Z(X) = X$, with $\dim X < \infty$. For example,

$$Z(+,+,-) = X \otimes X \otimes X^{\vee}.$$

For morphisms, we look at connected bordisms B, of which there are 5.

- B_1 is a bordism from $+$ to $+$ means $Z(B_1) = \mathrm{id}_X$
- B_2 is a bordism from $-$ to $-$ means $Z(B_2) = \mathrm{id}_{X^\vee}$
- B_3 is a bordism from $\{+,-\}$ to the empty set means $Z(B_3)$ is a map

$$\mathrm{ev}_X : X \otimes X^\vee \longrightarrow \mathbb{C} \qquad (\textit{evaluation})$$

- B_4 is a bordism from the empty set to $\{+,-\}$ means $Z(B_4)$ is a map

$$\mathrm{coev}_X : \mathbb{C} \longrightarrow X \otimes X^\vee \qquad (\textit{coevaluation})$$

- B_5 is a bordism from the empty set to itself (i.e., $B_5 = S^1$) means we can compose the prevous two cases:

$$\mathbb{C} \xrightarrow{Z(B_4)} Z(+,-) \xrightarrow{Z(B_3)} \mathbb{C}.$$

  The middle term is $X \otimes X^\vee \cong \mathrm{End}(X)$, and thinking of the first map as inclusion of the unit, we have

$$Z(S^1) = \mathrm{tr}(\mathrm{id}_X) = \dim X.$$

Note that in the last case, we recovered from the only closed 1-manifold the only invariant of a complex vector space (the dimension).

What we would like to say is that an extended TQFT Z is determined in all dimensions by its action $Z(*)$ on a point. There are, of course, several obstructions to this being true.

Obstruction 1. There is no canonical identification of a neighborhood of a manifold with Euclidean space, or in other words, many manifolds have nontrivial tangent bundles.

To address this, we must modify the definition of a TQFT.

Definition. If $m \leq d$ and M is an m-manifold, a *d-framing* of M is a trivialization of the bundle $TM \oplus \underline{\mathbb{R}}^{d-m}$ of rank d, where $\underline{\mathbb{R}}^{d-m} \cong M \times \mathbb{R}^{d-m}$ is the trivial bundle of rank $d-m$ over M.

Definition (Sketch)**.** The d-category $\mathbf{Cob}(d)^{\mathrm{fr}}_{\mathrm{ext}}$ has the following data:

objects:	d-framed 0-manifolds
1-morphisms:	d-framed bordisms between d-framed 0-manifolds (i.e., d-framed 1-manifolds)
2-morphisms:	d-framed bordisms between d-framed 1-manifolds (i.e., d-framed 2-manifolds)
$\vdots$	$\vdots$
d-morphisms:	d-framed d-manifolds with corners, modulo diffeomorphism relative to boundaries

We also want a tensor operation to be given by disjoint union of manifolds.

We can now state a slightly more precise version of CH:

Theorem. *Let $\mathcal{C}$ be a d-category with product $\otimes$. Then the following are equivalent:*

(1) $\otimes$-functors $\boldsymbol{Cob}(d)^{\mathrm{fr}}_{\mathrm{ext}} \to \mathcal{C}$

(2) objects $X \in \mathcal{C}$

The relationship is given by $X = Z(*)$.

There is a problem with part (2) in this statement. Recall that in the case $d = 1$, we required that vector space $X = Z(*)$ be finite-dimensional. This means that not every object in $\mathbf{Vect}_{\mathbb{C}}$ is realizable as $Z(*)$.

Obstruction 2. Not every object $X \in \mathcal{C}$ can appear as $Z(*)$. Specifically, we need some "finiteness" condition. Categorically, we need objects to be "fully dualizable."

Therefore we substitute

(2') fully dualizable objects $X \in \mathcal{C}$

into the hypothesis.

Further goals in this series of talks are

- Explain the statement of CH in more detail
- Sketch a proof of CH
- Give examples and sketch some applications