RTG Mini-Course
Perspectives in Geometry Series*

Jacob Lurie†

Lecture II: Higher Category Theory (1/22/2009)

Recall that we want to extend a TQFT to include data about manifolds of dimension n, $n-1$ and $n-2$, etc.. In order to include this extra data it is natural to put this into a higher categorical framework.

Definition 1. *A category (or 1-category) $\mathcal{C}$ consists of*

1. *A collection of objects,* $\mathrm{Obj}(\mathcal{C})$
2. *for every* $X, Y \in \mathrm{Obj}(\mathcal{C})$*, a set* $\mathrm{Hom}_{\mathcal{C}}(X,Y)$
3. *A composition law* $\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$
4. *An associativity requirement and units*

Then we can naively try to extend this to a structure such that we have morphisms between morphisms:

Definition 2. *A strict 2-category $\mathcal{C}$ consists of*

1. *A collection of objects,* $\mathrm{Obj}(\mathcal{C})$
2. *for every* $X, Y \in \mathrm{Obj}(\mathcal{C})$*, a category* $\mathrm{Hom}_{\mathcal{C}}(X,Y)$
3. *A functor composition law* $\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$

*These notes are by no means a substitute for the lectures, found at http://www.ma.utexas.edu/users/plowrey/dev/rtg/perspectives.html

†Scribes: Braxton Collier, Parker Lowrey, Michael B. Williams.

4. *Associativity (on the nose):* $H \circ (G \circ F) = (H \circ G) \circ F$ *as elements in* $\mathrm{Fun}(\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z), \mathrm{Hom}_{\mathcal{C}}(X,Z))$ *and units*

Example 1. Example of a 1-category *Given X, a topological space, and a point $x_0 \in X$, we can form the fundamental group $\pi_1(X, x_0)$. If we do not choose a base point, we can form a category $\pi_{\leq 1}(X)$ with $\mathrm{Obj}(\pi_{\leq 1}(X)) = \{x | x \in X\}$ and given two points x_0, x_1, we define $\mathrm{Hom}(x_0, x_1) = \{\{paths\ between\ x_0\ and\ x_1\}/homotopy\ of\ paths\}$. Composition is given by concatenation of paths. You can check the details to see that this is indeed a category.*

Now, this example understands aspects of the zero and first homotopy groups of X. If we wanted to form something that captures higher homotopy groups, we can use higher category theory.

Definition 3 (sketch)**.** *The fundamental 2-groupoid $\pi_{\leq 2}(X)$ is a 2-category defined roughly as:*

- $\mathrm{Obj}(\pi_{\leq 2}(X)) =$ *points of X.*
- *Morph*$(\pi_{\leq 2}(X))$ =*paths in X.*
- 2-*Morph*$(\pi_{\leq 2}(X)) = \{\{$*homotopies of paths*$\}/$*homotopies of homotopies of paths*$\}$.

Is this a strict 2-category? Yes. But it is not as straightforward as one would think: concatenation of paths is only associative up to homotopy. In particular, if we define a path as a map from the unit interval, and composition through the following: A path $p : [0,1] \to X$ such that $p(0) = x$, $p(1) = y$. Suppose given another path $q : [0.1] \to X$ with $q(0) = y$ and $q(1) = z$. Define

$$q \circ p : [0,1] \to X := \begin{cases} p(2t) & 0 \leq t \leq \frac{1}{2} \\ q(2t-1) & \frac{1}{2} < t \leq 1 \end{cases}$$

then this is associative up to homotopy, not on the nose. We could get around this in two ways. First we could redefine a path as a map of from intervals of any length. Then composition would be strict, and this would form a strict 2-category. Or we could redefine a 2-category in an equivalent manner. In particular, if we define

Definition 4. *A non-strict 2-category, or bicategory, is defined by the same data as above for the strict 2-category, except for associativity we only require $H \circ (G \circ F) \cong (H \circ G) \circ F$ where the congruence is by a specified natural isomorphism (part of data) that has some sort of coherence property.*

It turns out that strict 2-categories and bicategories are equivalent i.e. we can "strictify" a bicategory.

We will now define this for a general n:

Definition 5. *A strict n-category $\mathcal{C}$ consists of*

1. *A collection of objects,* $\mathrm{Obj}(\mathcal{C})$
2. *for every* $X, Y \in \mathrm{Obj}(\mathcal{C})$*, a* $n-1$ *category* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$
3. *A composition law* $\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$ *of* $n-1$ *functors*
4. *associativity (with units) on the nose*

This is a precise definition through recursion. Also have a parallel idea of non-strict n-category. It is easier to think of these through examples (to many coherence diagrams to write down):

Example 2. Fundamental n-groupoid $\pi_{\leq n}X$ *for a topological space* X *is defined by the following data:*

1. $\mathrm{Obj}(\pi_{\leq n}X)$ *= points of* X*.*
2. *for* $0 < i < n$*, i-morphisms are i-fold homotopies of paths.*
3. *n-morphisms are n-fold homotopies of paths, up to the equivalence relation of (n+1)-fold homotopies of paths.*

In particular, 1-morphisms are paths, 2-morphisms are homotopies of paths, etc. This "n-category" does not fit into the description above for $n \geq 3$. In other words, you cannot always strictify a non-strict n-category.

As an example of such behavior, for a groupoid $\mathcal{C}$, and an object $x \in \mathcal{C}$ then $\mathrm{Hom}(x,x)$ is a group. If we are in a 2-category, this is a category. But the replacement for the group structure is a monodial (tensor) category. If this is a strict 2-category, then we would in fact have a strict tensor category. If we are in a 3-category then $\mathrm{Hom}_{\mathrm{Hom}_{\mathcal{C}(x,x)}}(\mathrm{id}_x, \mathrm{id}_x)$ has 2 monodial structures, 1 coming from the fact that $\mathrm{Hom}_{\mathcal{C}(x,x)}$ is monodial, and the other as the set of morphisms to itself. This implies that we have a braided monodial category. If our category is a strict 3-category then we have a symmetric monodial category.

Now, for $\pi_{\leq 3}(S^2)$, then $\mathrm{Hom}_{\pi_{\leq 3}(S^2)}(id_x, id_x)$ is the category whose objects are maps of S^2 to S^2. There are non-trivial automorphisms of these maps coming from $\pi_3(S^2)$. This category has a braided monodial structure that is not symmetric (coming from Hopf invariant).

Definition 6. *An n-groupoid is an n-category such that all morphisms $1 \leq k \leq n$ are invertible (up to a higher invertible morphism).*

Thesis 1. *By taking a limit of our example for an n-fundamental groupoid of a topological space we can define for a space X, $\pi_{\leq\infty}X$. This is the characteristic/only example of such an object. The construction $X \to \pi_{\leq\infty}X$ determines a bijection between $\{$topological spaces$\}$ up to weak homotopy equivalence and ∞-groupoids up to equivalence. This should be true of any reasonable definition of higher category theory.*

In other words, we should rig the definition so that this will be true i.e. we could define an ∞-groupoid is a topological space, for which by "topological space" we mean anything we can do homotopy theory on. Even better, a Kan complex.

As an informal definition, an (∞, n) category is a higher category where all k-morphisms are invertible for all $k > n$.

Example 3. $(\infty, 0)$*-category* $\Longleftrightarrow$ ∞*-groupoid* $\Longleftrightarrow$ *topological space*

Let's try to implement this into an inductive definition where rather than 0-categories being sets, they are ∞-groupoids.

Definition 7. *An (∞, n)-category $\mathcal{C}$ has the following data*

1. *A set (or class) of objects,* $\mathrm{Obj}(\mathcal{C})$
2. *For each X, Y in* $\mathrm{Obj}(\mathcal{X})$, $\mathrm{Hom}(X, Y)$ *is a* $(\infty, n-1)$*-category*
3. *Composition is given by composition of $n-1$ functors*
4. *Associativity (with units) (up to isomorphism)*

This is not a precise definition yet, we would need to give the associativity.

Lets implement this for $n = 1$.

Example 4. *For a $(\infty, 1)$-cat $\mathcal{C}$, then for $X, Y \in \mathcal{C}$* $\mathrm{Hom}(X, Y)$ *is a* $(\infty, 0)$ *category, and thus, by above, equivalent to a topological space.*

We can require "strict" topological category and "non-strict" topological category, where the associativity is up to homotopy. These two ideas are the same, much like $n = 2$ for normal 2-categories.

Different implementations give equivalent answers: quasi-categories, Segal categories, complete Segal spaces. They are all attempts to implement the definition above.

What motivates this? First, it is probably easier to define (∞, n)-categories and then specialize to regular n-categories. Second, it will help us out in these TQFT problems.

Almost examples of TQFT's (d = 2)

Example 5. The B-model *Given* A*, an associative algebra over* $\mathbb{C}$*, define* $\mathrm{Hoch}^*(A) := \mathrm{Ext}^*_{A\otimes A}(A, A)$*. Now, if* $X = \mathrm{Spec}A$ *is an affine variety then we define* $\mathrm{Hoch}^*(\mathrm{Spec}A) := \mathrm{Hoch}^*(A)$*, this globalizes to non-affine cases.*

Given a smooth n-dimensional variety $\mathrm{Hoch}^{2n}(X) \cong H^n(X, \wedge^n TX$*. If* X *is Calabi-Yau, then* $\wedge^n TX$ *is trivial, thus we replace* $\wedge^n TX$ *by* $\wedge^n T^*X$ *and take trace to* $\mathbb{C}$*. Serre duality the trace pairing is non-degenerate, and therefore that* $\mathrm{Hoch}^*(X)$ *is a frobenious algebra (commutative in the category of* $\mathbb{Z}_2$ *graded vector spaces). By the folk theorem, we have a TQFT whose value on the circle is Hochschild cohomology.*

Example 6. Twisted K-theory of Freed-Hopkins-Teleman. *G a compact Lie group,* $l \in H^4(BG, \mathbb{Z})$*. We can use* l *to define a central extension* $S^1 \to \widetilde{LG} \to LG$*. Now* $\mathrm{Rep}(\widetilde{LG})$ *has a tensor product given by fusion. The Grothendieck group has a ring structure (determined by fusion product), called the Verlinde algebra. By F-H-T,* $K(\mathrm{Rep}(\widetilde{LG})) \cong K^\tau_G(G)$*, where* τ *depends on* l*. This latter object is a frobenious algebra over* K*-theory. This the value on* S^1 *of a TQFT.*

Example 7. String topology *Let* M *be a compact manifold. Then* $H_*(LM; \mathbb{C})$ *has an algebra structure (Chas-Sullivan). In fact, there are more operations on* $H_*(LM; \mathbb{C})$ *coming from Riemann surfaces with boundary (the above product is from the pair of pants). No trace though since the disc* D^2*, thought of as a morphism from* S^1 *to a point, is not allowed. This is an example of what people call a noncompact TQFT.*

All of these say TQFT $\rightleftarrows$ Homology. Can we lift this to the chain complex level?

Question: What should it mean to have a chain complex valued TQFT? In other words, given a $d-1$ dimensional closed manifold, and $B : M \to M$, $\mathcal{Z}(M)$ is a chain complex of vector spaces, and $\mathcal{Z} : \mathcal{Z}(M) \to \mathcal{Z}(M)$ a chain morphism. Naively, in this translation, if we had two bordisms B, B' that are diffeomorphic, then we should we have $Z(B) = Z(B')$. This requires to many choices and is too strong. Therefore, we should only require them to be chain homotopic via a homotopy that depends on a choice of diffeomorphism.

We will now try to axiomatize this. What do we want?

Closed $(n-1)$ manifolds	$\rightsquigarrow$	chain complexes
Bordisms	$\rightsquigarrow$	chain maps
Diffeomorphisms	$\rightsquigarrow$	chain homotopies
Isotopies	$\rightsquigarrow$	chain homotopies between chain homotopies

and so on. In other words, $\mathcal{Z}$ should be a functor between $(\infty, 1)$ categories where the domain category is $\widetilde{\mathrm{Cob}}(n)$ has as objects, $n-1$ manifolds. morphisms: bordisms. 2-morphisms: diffemorphisms. 3-morphisms: isotopies.

For the remainder, we will go over a more precise description of an $(\infty, 1)$-category. It will not be used in future lectures. We will model $(\infty, 1)$-categories by Complete Segal Spaces (Rezk).

Complete Segal Spaces

Let $\mathcal{C}$ be an $(\infty, 1)$ category. We don't know what this means yet, but we know what an $(\infty, 0)$ category is. Let's build on this.

Step 1. Take $\mathcal{C}$, throw out non-invertible morphisms to get a topological space X_0. This is an okay invariant of our category, but it loses some information. How should we remember this information?

Step 2. Take $\mathrm{Fun}([1], \mathcal{C})$. We let X_1 be the classifying space for the objects of this category.

Step 3. To remember composition, we notice that a composable morphisms is a functor $[2] \to \mathcal{X}$. So, let X_2 classify $\mathrm{Fun}([2], \mathcal{C})$.

$$\vdots$$

Step n. Let X_n classify $\mathrm{Fun}([n], \mathcal{C})$.

Thesis 2. *The collection of spaces $\{X_n\}_{n \geq 0}$ remembers everything about $\mathcal{C}$.*

What structures exist on $\{X_n\}_{n \geq 0}$? First note: every monotone map $f : [m] \to [n]$ induces a map $f^* : \mathrm{Fun}([n], \mathcal{C}) \to \mathrm{Fun}([m], \mathcal{C})$ and therefore a map $X_n \to X_m$. This makes our collection into a simplicial space. This simplicial space satisfies the "Segal condition": The diagram

$$\begin{tikzcd} X_{n+m} \arrow[r] \arrow[d] & X_n \arrow[d] \\ X_m \arrow[r] & X_0 \end{tikzcd}$$

is homotopy Cartesian.

Definition 8. *A Segal space is a simplicial topological space satisfying the Segal condition (above).*

Now, let $X.$ be a Segal space. Can define its homotopy category, hX: $\mathrm{Obj}(hX)$: Points of X_0. For $x_0, x_1 \in \mathrm{Obj}(hX)$, $\mathrm{Hom}(X, Y) = \pi_0(\{X\} \times^h_{X_0} X_1 \times^h_{X_0} \{Y\})$. For composition, we use the Segal condition.

In particular, $\phi \in X_1$ it decides a morphism in hX. We'll say its invertible if this morphism in hX is invertible.

If $X_\bullet$ is a Segal space, let $\widetilde{X}_1$ be the subspace consisting of invertible elements.

Definition 9. $X_\bullet$ *is complete if* $X_0 \to \widetilde{X}_1$ *is a weak homotopy equivalence.*

Thesis 3. *Complete Segal space* $\Leftrightarrow$ $(\infty, 1)$*-category.*