

# RTG Mini-Course

## Perspectives in Geometry Series\*

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### Lecture III: The Cobordism Hypothesis (1/27/2009)

A quick review of ideas from previous lectures:

- The original definition of an  $n$ -dimensional TQFT provided a diffeomorphism invariant for a closed  $n$ -manifold, which we could compute by cutting the manifold into pieces. This cutting was restricted, however, to closed submanifolds of codimension 1. We would like cut along a triangulation of a manifold, but this requires the notion of an *extended* TQFT.
- We discussed the notion of *higher categories*, namely  $(\infty, n)$ -categories, where all  $k$ -morphisms are invertible, for  $k > n$ . There were several motivations. First was the need to handle more complicated invariants obtained from a TQFT, e.g. cohomology classes. Second is that some diffeomorphism invariants are quite subtle, e.g., we need more refined information regarding cohomology classes that are  $\text{Diff}(M)$ -equivariant.

An example of an  $(\infty, n)$ -category is  $\mathbf{Bord}_n$ , which is comprised of the following data:

objects:	0-manifolds
1-morphisms:	bordisms of 0-manifolds (i.e., 1-manifolds with boundary)
2-morphisms:	bordisms of bordisms (i.e., 2-manifolds with corners)
$\vdots$	$\vdots$
$n$ -morphisms:	bordisms of bordisms of $\dots$ of bordisms (i.e., $n$ -manifolds with corners)
$(n + 1)$ -morphisms:	diffeomorphisms of $n$ -manifolds
$(n + 2)$ -morphisms:	isotopies of diffeomorphisms of $n$ -manifolds
$\vdots$	$\vdots$

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\*These notes are by no means a substitute for the lectures, found at <http://www.ma.utexas.edu/users/plowrey/dev/rtg/perspectives.html>

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In an  $(\infty, n)$ -category, we can think of the collection of  $n$ -morphisms as forming a topological space. For example, in the case of  $\mathbf{Bord}_n$ , the collection of  $n$ -morphisms is the classifying space for  $n$ -manifolds with corners.

Note that the category  $\mathbf{Bord}_n$  is a symmetric monoidal category (i.e., a  $\otimes$ -category) with  $\otimes = \coprod$ . Also, there are variants of this category depending on the type of manifolds under considerations: manifolds without orientation, with a framing, with a spin structure, etc. We will be interested in  $\mathbf{Bord}_n^{\text{fr}}$  where we consider  $n$ -framed manifolds.

Recall the Cobordism Hypothesis (CH):

**Theorem.** *Let  $\mathcal{C}$  be a symmetric monoidal  $(\infty, n)$ -category. Then there is a bijection*

$$\text{Fun}^{\otimes}(\mathbf{Bord}_n^{\text{fr}}, \mathcal{C}) \cong \{\text{fully dualizable objects of } \mathcal{C}\},$$

where  $\text{Fun}^{\otimes}(\mathbf{Bord}_n^{\text{fr}}, \mathcal{C})$  is the collection of symmetric monoidal functors  $\mathbf{Bord}_n^{\text{fr}} \rightarrow \mathcal{C}$ . The correspondence is given by  $Z \leftrightarrow Z(*)$ .

We will address the following questions in this lecture:

- What does it mean to be *fully dualizable*?
- What happens in the unframed case?
- What is the relation of CH to the Mumford conjecture?

Note that the statement of CH is nontrivial, even when  $n = 1$ , when we consider an extended TQFT (although it is easy if we consider an ordinary TQFT).

**Example.** Consider  $\text{Hom}_{\mathbf{Bord}_1}(\emptyset, \emptyset)$ , the classifying space for closed 1-manifolds (which are disjoint unions of copies of  $S^1$ ). This contains as a component the classifying space for one copy of  $S^1$ . This component classifies oriented circle bundles, giving a natural inclusion  $\mathbb{C}\mathbb{P}^{\infty} \subset \text{Hom}_{\mathbf{Bord}_1}(\emptyset, \emptyset)$ . Given an extended TQFT

$$Z : \mathbf{Bord}_1 \longrightarrow \mathcal{C},$$

we'd have

$$\begin{array}{ccc} \text{Hom}_{\mathbf{Bord}_1}(\emptyset, \emptyset) & \longrightarrow & \text{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}}) \\ \uparrow & \nearrow \psi & \\ \mathbb{C}\mathbb{P}^{\infty} & & \end{array}$$

In the case of a normal 1-dimensional TQFT, we had  $Z(*) = X \in \mathcal{C}$ , and previous computation showed that  $Z(S^1) = \dim X$  via

$$1_{\mathcal{C}} \xrightarrow{\text{coev}} X \otimes X^{\vee} \xrightarrow{\text{ev}} 1_{\mathcal{C}}.$$

In the extended TQFT case, we have more information. Namely, the map  $\psi$  gives us an object  $\dim X$  and an action on  $\dim X$  by the symmetry group of the circle. Our calculation of  $Z(S^1)$  in the non-extended case broke the symmetry group action by cutting apart the circle. Breaking symmetry is akin to picking a point of  $\mathbb{C}\mathbb{P}^\infty$ , thus our previous method gave the value of our extended TQFT on that point. This method is insufficient in this new world.

**Example.** Let  $\mathcal{C}$  be the  $(\infty, 1)$ -category whose objects are associative  $\mathbb{C}$ -algebras, and whose morphisms  $\text{Hom}_{\mathcal{C}}(A, B)$  are chain complexes of  $A, B$ -bimodules (details on how to make this an  $(\infty, 1)$  category are omitted). Here, every object is fully dualizable (defined below). For now it suffices to say that each has a dual, namely the opposite algebra. If we believe the Cobordism Hypothesis, then to each algebra  $A \in \mathcal{C}$  we can assign a field theory  $Z$ . Since our tensor product is the normal tensor product, then  $\mathbb{C}$  is the unit object and  $Z(S^1)$  is a chain complex of  $\mathbb{C}, \mathbb{C}$ -bimodules, which are  $\mathbb{C}$ -vector spaces. This is the Hochschild chain complex  $CH_*(A)$ . It is a classical fact that  $CH_*(A)$  carries an action by a circle.

We now turn to the notion of full dualizability.

**Definition.** If  $\mathcal{C} = \mathbf{Vect}_{\mathbb{C}}$ , then an object  $V \in \mathcal{C}$  is *fully dualizable* iff  $\dim V < \infty$ .

We'd like to generalize this notion to arbitrary categories, but what does "finite dimensional" mean? To answer this, we recall the notion of dual objects.

**Definition.** If  $V \in \mathcal{C} = \mathbf{Vect}_{\mathbb{C}}$ , we can define the *dual*  $V^\vee = \text{Hom}(V, \mathbb{C})$ , and there exists a pairing, called *evaluation*:

$$V \otimes V^\vee \xrightarrow{\text{ev}} \mathbb{C},$$

which defines a universal property. This relationship is symmetric iff  $\dim V < \infty$ . There is also a canonical map, called *coevaluation*:

$$\mathbb{C} \xrightarrow{\text{coev}} V \otimes V^\vee.$$

These two maps are compatible, in that the following compositions are the identity:

$$\begin{cases} V \xrightarrow{\text{id} \otimes \text{coev}} V \otimes V^\vee \otimes V \xrightarrow{\text{ev} \otimes \text{id}} V, \\ V^\vee \xrightarrow{\text{coev} \otimes \text{id}} V^\vee \otimes V \otimes V^\vee \xrightarrow{\text{id} \otimes \text{ev}} V^\vee. \end{cases} \quad (1)$$

This formalism makes sense in any symmetric monoidal category. We extend this naively to any symmetric monoidal  $(\infty, n)$ -category (or  $\otimes$ - $(\infty, n)$ -category for short):

**Definition.** Let  $\mathcal{C}$  be a  $\otimes$ - $(\infty, n)$ -category. An object  $X \in \mathcal{C}$  has a dual if it is dualizable in its homotopy category, i.e. if there is another object (called its *dual object*)  $X^\vee$  and 1-maps  $\text{ev}$  and  $\text{coev}$  satisfying the above relations up to homotopy.

**Definition.** Let  $\mathcal{C}$  be a  $\otimes$ - $(\infty, 1)$ -category. An object  $X \in \mathcal{C}$  is *fully dualizable* if it is dualizable.

What about for higher  $n$ ? Dualizability is not enough; we need the stricter notion of an object being “fully” dualizable. We approach this from a different perspective in the following digression.

We start by analyzing 2-categories. The prototypical example of a (strict) 2-category is that of categories: objects are categories, morphisms are functors, and 2-morphisms are natural transformations. We want to understand adjoint functors.

**Definition.** If  $\mathcal{C}$  and  $\mathcal{D}$  are categories, then two functors

$$F : \mathcal{C} \longrightarrow \mathcal{D}, \quad G : \mathcal{D} \longrightarrow \mathcal{C}$$

are *adjoint* if there are natural bijections

$$\mathrm{Hom}_{\mathcal{D}}(F(c), d) \cong \mathrm{Hom}_{\mathcal{C}}(c, G(d)),$$

for all  $c \in \mathcal{C}$  and  $d \in \mathcal{D}$ . We say that  $F$  is a *left adjoint* and that  $G$  is a *right adjoint*.

Adjointness works in any 2-category: taking  $d = F(c)$ , we have

$$\mathrm{id}_d \in \mathrm{Hom}_{\mathcal{D}}(d, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, G(d)) \cong \mathrm{Hom}_{\mathcal{C}}(c, G \circ F(c)),$$

which gives a map

$$c \longrightarrow (G \circ F)(c),$$

i.e., a natural transformation

$$u : \mathrm{id}_{\mathcal{C}} \longrightarrow G \circ F.$$

We can use  $u$  to recover the adjunction maps

$$\mathrm{Hom}_{\mathcal{D}}(d, d) \cong \mathrm{Hom}_{\mathcal{C}}((G \circ F)(c), G(d)) \xrightarrow{u} \mathrm{Hom}_{\mathcal{C}}(c, G(d)).$$

To guarantee that this is a bijection, we want “inverses”, that is,

$$v : F \circ G \longrightarrow \mathrm{id}_{\mathcal{D}}$$

inducing inverse maps. How might  $u$  and  $v$  be related? Ideally, the following compositions would be the identity:

$$\begin{cases} G \xrightarrow{\mathrm{id} \times v} G \circ F \circ G \xrightarrow{\mathrm{id} \times u} G, \\ F \xrightarrow{u \times \mathrm{id}} F \circ G \circ F \xrightarrow{v \times \mathrm{id}} F. \end{cases} \quad (2)$$

Note the similarity to the duality requirement in (1). The moral is that “adjoints are like duals”. We use the following example to clarify this relationship.

**Example.** Let  $\mathcal{C}$  be a  $\otimes$ -category. There exists a 2-category  $BC$  whose consisting of the following data:

$$\begin{array}{ll} \text{objects:} & \text{one object: } * \\ \text{1-morphisms:} & \mathrm{Hom}_{BC}(*, *) = \mathcal{C} \\ \text{composition:} & \otimes \text{ from } \mathcal{C} \end{array}$$

Then we have a correspondence

$$\{\text{duals in } \mathcal{C}\} \longleftrightarrow \{\text{adjoints in } BC\}.$$

A pair of 1-morphisms  $F, G : * \rightarrow *$  are adjoint in  $BC$  iff  $F$  and  $G$  are dual in  $\mathcal{C}$ .

We now end our digression and make a definition.

**Definition.** If  $\mathcal{C}$  is an  $(\infty, n)$ -category, we say that 1-morphisms

$$F : X \longrightarrow Y, \quad G : Y \longrightarrow X$$

are *adjoint* if there exist 2-morphisms

$$u : \text{id}_X \longrightarrow G \circ F, \quad v : F \circ G \longrightarrow \text{id}_Y$$

such that the relations (2) hold up to isomorphism.

We would like all our morphisms at all levels to have adjoints. This turns out to be too strong since in an  $(\infty, n)$ -category, this would imply that all  $n$ -morphisms are invertible. So we define inductively

**Definition.** An  $(\infty, n)$ -category *has adjoints* if

- when  $n = 1$ , no requirements;
- if  $n \geq 2$ , all morphisms have left and right adjoints;
- if  $n > 2$ , the  $(\infty, n - 1)$ -category  $\text{Hom}_{\mathcal{C}}(X, Y)$  has adjoints.

**Definition.** Let  $\mathcal{C}$  be a  $\otimes$ - $(\infty, n)$ -category. We say that  $\mathcal{C}$  *has duals* if

- $\mathcal{C}$  has adjoints;
- each object has a dual as defined above.

Equivalently, we say that  $BC$  has adjoints.

A modified version of CH now reads:

**Theorem.** *If  $\mathcal{C}$  had duals, then there is a bijection*

$$\text{Fun}^{\otimes}(\mathbf{Bord}_n^{\text{fr}}, \mathcal{C}) \cong \{\text{objects of } \mathcal{C}\}.$$

We note that the category  $\mathbf{Bord}_n^{\text{fr}}$  has duals, the dual of a manifold is the same manifold with opposite orientation and read backwards.

For any  $\otimes$ - $(\infty, n)$ -category  $\mathcal{C}$ , there is a largest “subcategory”  $\mathcal{C}^{\text{fd}} \subseteq \mathcal{C}$  such that  $\mathcal{C}^{\text{fd}}$  has duals. (The “fd” stands for *fully dualizable*.) It is obtained by throwing out all items without adjoints. In this way we obtain an equivalence of  $\infty$ -groupoids from CH:

$$\text{Fun}^{\otimes}(\mathbf{Bord}_n^{\text{fr}}, \mathcal{C}) \cong \text{Fun}^{\otimes}(\mathbf{Bord}_n^{\text{fr}}, \mathcal{C}^{\text{fd}}) \cong \{\text{objects of } \mathcal{C}\}.$$

**Definition.** An object  $X \in \mathcal{C}$  is *fully dualizable* if  $X \in \mathcal{C}^{\text{fd}}$ .

What does fully dualizable mean? Let  $\mathcal{C}$  be an  $(\infty, n)$ -category.

- $n = 1$ : An object  $X \in \mathcal{C}$  is fully dualizable if  $\mathcal{C}$  has dualizable;
- $n = 2$ : An object  $X \in \mathcal{C}$  is fully dualizable if and only if  $X$  is dualizable and  $\text{ev}$  and  $\text{coev}$  have left adjoints;
- $n > 2$ : This is more complicated. An object  $X \in \mathcal{C}$  is fully dualizable if
  - $X$  is dualizable;
  - $\text{ev}$  and  $\text{coev}$  have left adjoints;
  - the adjoints have adjoints
  - etc. . .

**Definition.** Let  $M^d$  be a manifold with  $d \leq n$ . An  $n$ -*framing* of  $M$  is an isomorphism

$$TM \oplus \underline{\mathbb{R}}^{n-d} \cong \mathbb{R}^n.$$

Observe that the group  $O(n)$  acts on the set of  $n$ -frames of any manifold. In this way we get an action of  $O(n)$  on  $\mathbf{Bord}_n^{\text{fr}}$ . In the statement of CH,  $O(n)$  acts on  $\mathbf{Bord}_n^{\text{fr}}$  and, by the Cobordism Hypothesis, on the objects of  $\mathcal{C}$ .

**Example.** If  $\mathcal{C}$  has duals, then each  $X \in \mathcal{C}$  has a corresponding  $X^\vee \in \mathcal{C}$ . There is a map

$$X \longmapsto X^\vee$$

which can be thought of as a contravariant functor  $\mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$ , an involution of  $\mathcal{C}$ , or an action of  $O(1)$  on the objects of  $\mathcal{C}$ .

**Example.** Let  $\mathcal{C}$  be an  $\infty$ -groupoid. For example,  $\mathcal{C} \cong \pi_{\leq \infty} X$  for some topological space  $X$ . That  $\mathcal{C}$  is a symmetric category means that  $X$  is an  $E_\infty$ -space, meaning that  $X$  has a multiplication that is commutative and associative up to (any possible) homotopy. In particular,  $\pi_0 X$  is a commutative monoid. Then  $\mathcal{C}$  has duals iff  $X$  is “grouplike”, which means  $\pi_0 X$  is an abelian group, iff  $X$  is an infinite loop space, in which case

$$X \cong \Omega^n Y = \text{Map}((D^n, S^{n-1}), (Y, *)).$$

Here,  $O(n)$  acts on  $D^n$  in a natural way that preserves the boundary  $S^{n-1}$ , and this induces an action on  $X$ .

In more sophisticated language, if  $\mathcal{C}$  is an  $\infty$ -groupoid, then  $O(n)$  acts via the  $J$ -homomorphism.

Suppose  $G$  is a topological group with a representation  $G \rightarrow O(n)$ . Then

- we can define the notion of  $G$ -structure on a manifold  $M^d$  (that is, a principle  $G$ -bundle  $P \rightarrow M$  such that the associated vector bundle is  $(P \times \mathbb{R}^n)/G \cong TM \oplus \underline{\mathbb{R}}^{n-d}$  in the usual way);
- using these manifolds, the category  $\mathbf{Bord}_n^G$  makes sense.

For example,

- $\mathbf{Bord}_n^{\mathrm{SO}(n)} \cong \mathbf{Bord}_n$ ;
- $\mathbf{Bord}_n^{\{1\}} \cong \mathbf{Bord}_n^{\mathrm{fr}}$ ;
- $\mathbf{Bord}_n^{\mathrm{Spin}(n)} \cong$  spin manifolds;
- $\mathbf{Bord}_n^{\mathrm{O}(n)} \cong$  bordism theory for non-oriented manifolds;
- $G$  acts on objects of  $\mathcal{C}$  if  $\mathcal{C}$  has duals.

We can restate CH for  $G$ -manifolds: If  $\mathcal{C}$  has duals, then there is a bijection

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^G, \mathcal{C}) \cong \{\text{objects of } \mathcal{C}\}^{hG} \cong \mathrm{Map}_G(EG, \{\text{objects of } \mathcal{C}\}).$$

The  $hG$  refers to homotopy fixed points under the  $G$ -action,  $\mathrm{Map}_G$  means  $G$ -equivariant maps, and  $EG$  is a contractible space with free  $G$ -action.

**Example.** Let  $n = 1$  and  $G = \mathrm{O}(1)$ . Then

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}_1^{\mathrm{O}(1)}, \mathcal{C}) \cong \{\text{objects of } \mathcal{C}\}^{h\mathrm{O}(1)},$$

which is the space of “symmetrically self-dual” objects of  $\mathcal{C}$ , that is, objects such that  $X = X^{\vee}$ . There is a nondegenerate pairing

$$X \otimes X^{\vee} \longrightarrow 1.$$

If  $\mathcal{C} = \mathbf{Vect}_{\mathbb{C}}$  and  $V \in \mathcal{C}$ , this says that  $\dim V < \infty$  with a nondegenerate symmetric bilinear form.