

Higson: Baum-Cordes Conj

A families index construction (Lusztig)

$\Lambda =$ free ab. grp = lattice in v.sp. V .

$V/\Lambda =$ torus

$V^*/\Lambda^* =$ torus = $\Lambda^* = \{v^* \mid v^*[\Lambda] \in \mathbb{Z}\}$

$v^* \in V^*/\Lambda^* \iff$ char of $\Lambda: v^*(\lambda) = e^{2\pi i \langle v^*, \lambda \rangle}$

i.e. $v^*: \Lambda \rightarrow U(1)$

\iff flat unitary line bun on V/Λ

Define: $\boxed{K_*^{geo}(V/\Lambda) \xrightarrow{\mu} K^*(V^*/\Lambda^*)}$ ideally we want to take the zero modes of D (i.e. $\ker D$) at each v^* and view it as $v^* \in V^*/\Lambda^*$ on dual torus

$(M, E, f) \mapsto \text{Index } \{D_{M, E \otimes L_{v^*}}\}_{v^* \in V^*/\Lambda^*}$ family of Dirac ops on V^*/Λ^*

$f: M \rightarrow V/\Lambda$

Some facts about μ :

1) μ is an iso

2) suppose: M has spin str., $E =$ triv bun., M has positive scalar curvature. Then $\mu(M, E, f) = 0$. Each $D_{M, E \otimes L_{v^*}}$ is invertible.

3) (what Lusztig was interested in) Suppose (M, S, f) S^* dual of spinor bundle (forget $\mathbb{Z}/2$ grading)

Then $D_{M, S^*} =$ signature operator acting on diff forms.

$\ker D_{M, S^*} =$ de Rham cohomology of M .

$\mu(M, S, f)$ is an (oriented) hntpy inv.

note: V/Λ is spin. we have fund class = $\mu(V/\Lambda, \text{triv}, \text{id})$

$M \xrightarrow{f} V/\Lambda$ commutes up to hntpy

$M' \xrightarrow{f'} V/\Lambda$

then $\mu(M, S, f) = \mu(M', S', f')$

So... the torus V/Λ admits no metric of positive scalar ~~curvature~~ curvature. (in any dim!)

IV.2

and if $h: M \xrightarrow{\sim} M'$ as before then for any $\alpha \in H^*(V/\Lambda)$

$$\int_M L(M) \wedge f^* \alpha = \int_{M'} L(M') \wedge f'^* \alpha$$

Gromov-Lawson-Rosenberg: If M is spin and aspherical ($\pi_1 M \cong B\pi_1 M$) it can support no metric of pos. scalar curv.

~ x ~

Cont. fields of Hilb sp. VS Hilbert modules:

$H =$ cont. field over $(\text{cpt}) X$

$P(H) =$ cont sections

* that's a module over $C(X)$

* it has a $C(X)$ -valued inner prod: $\langle s_1, s_2 \rangle =$ pointwise inner prod

one has: $\langle s, s \rangle \geq 0$

$$\langle s_1, s_2 f \rangle = \langle s_1, s_2 \rangle f \quad (f \in C(X))$$

* $P(H)$ complete wrt $\|s\| = \|\langle s, s \rangle\|^{1/2} =$ sup norm of f_n in $C(X)$
i.e. $P(H)$ is a Hilbert module

Observe: we can recover H from this Hilbert $C(X)$ -mod str.

$$H_x = P(x) \otimes_{C(x)} \mathbb{C}_x$$

π a discrete grp

$C^*(\pi)$ = a completion of grp alg (there is not a unique choice)

$C^*(\pi)$ is a C^* -alg.

(choosing the "right" completion is important for Baum - Connes conj).

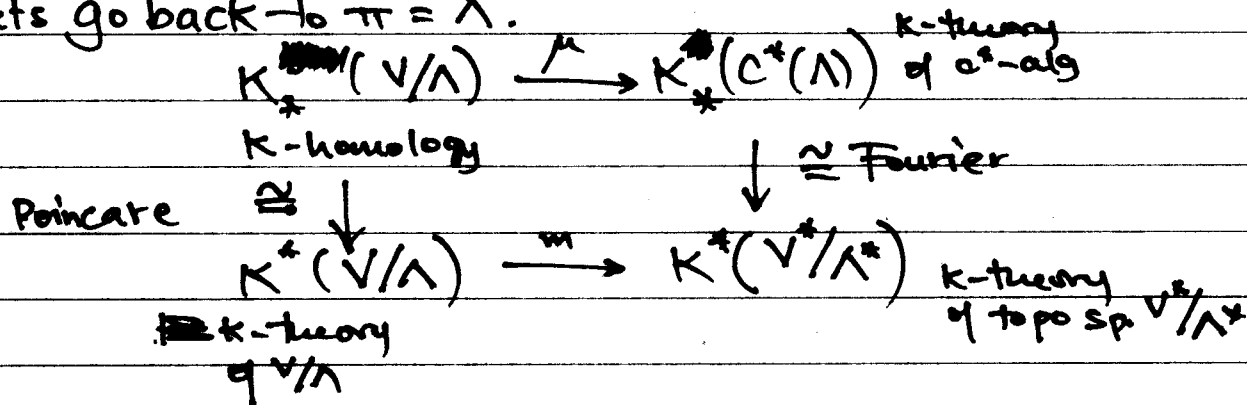
For π abelian, $C^*(\pi) = C(V/\Lambda^*)$ (due to Fourier)

Let $\mathcal{M} = C^*(\pi)$ viewed as a right module.

\mathcal{M} is a Hilbert module: $\langle m_1, m_2 \rangle = m_1 * m_2 \in C^*(\pi)$

Que: Why believe Baum-Connes Conj?

Example: Let's go back to $\pi = \Lambda$.

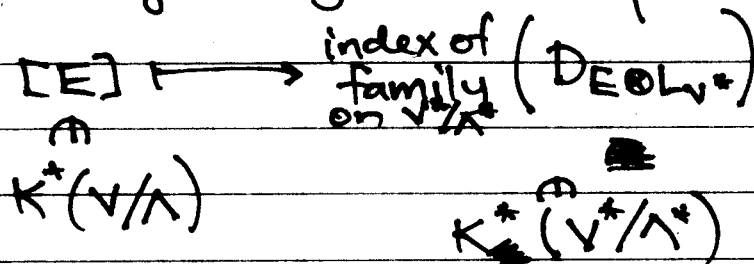


Fact: $m^2 = id$

So, in this case, the conj works (all due to Fourier + Poincaré).

Let's look at this example more closely.

The map m is given by the Dirac op on V/Λ .



The basic cts field here is

$$H_{V^*} = L^2(V/\Lambda, L_{V^*})$$

L_{V^*} unitary line bun
 \downarrow
 V/Λ

A section of L_{V^*} is a twist periodic fn on V :

$$s_{V^*}(v+k) = v^*(k)s(v) \quad k \in \Lambda.$$

The Fourier trans. is an elt of $\ell^2(\Lambda^* + v^*)$.

We get: smooth sections $\Gamma(H) = \mathcal{S}(V)$ Schwartz fns on V .

The basic field/module $(H/P(H))$ is a completion of $\mathcal{S}(V)$

H has this str:

- 1) $C^*(\Lambda)$ -module str. (after completion)
- 2) a repn of $C(V/\Lambda)$ as module maps
- 3) an operator D . ($C^*(\Lambda)$ -linear)

1) $f \cdot g$ action by translation.

$$\langle f_1, f_2 \rangle(g) = \langle f_1, f_2 \cdot g \rangle_{L_2}$$

- 2) ptwise multi
- 3) Dirac

This data allows us to construct μ .

There is a dual collection to this data used to construct μ^{-1} .

- 1) $C(V/\Lambda)$ - Hilb. module str.
- 2) a ~~repn~~ ^{repn} of $C^*(\Lambda)$
- 3) an operator (which is $C(V/\Lambda)$ -linear)
 $D = \text{"multi by } x \text{"} = \text{cliff. multi by } v \in V.$