Jan 28^{th} '10

Higgson: Baum–Connes Conj

A families index construction (Lusztig)

Λ = free ab. grp = lattice in v.sp. V.

V/Λ = torus

V^*/Λ^* = torus = $\Lambda^* = \{v^* \mid v^*[\Lambda] \subseteq \mathbb{Z}\}$

$v^* \in V^*/\Lambda^* \iff$ char of Λ: $v^*(\lambda) = e^{2\pi i \langle v^*, \lambda\rangle}$

i.e. $v^*: \Lambda \to U(1)$

$\iff$ flat unitary line bun L_{v^*} on V/Λ

Define: $\boxed{K^{geo}_*(V/\Lambda) \xrightarrow{\mu} K^*(V^*/\Lambda^*)}$

$(M, E, f) \longmapsto \mathrm{Index}\{D_{M, E\otimes L_{v^*}}\}_{v^* \in V^*/\Lambda^*}$

$f: M \to V/\Lambda$ family of Dirac ops on V^*/Λ^*

ideally we want to take the zero modes of D (ie Ker D) at each v^* and view it as vbun on dual torus

Some facts about μ:

1) μ is an iso

2) suppose: M has spin str., E = triv bun., M has positive scalar curvature. Then $\mu(M,E,f) = 0$. Each $D_{M,E\otimes L_{v^*}}$ is invertible.

3) (what Lusztig was interested in) Suppose (M, S^*, f)
S^* dual of spinor bundle (forget $\mathbb{Z}/2$ grading)
Then D_{M,S^*} = signature operator acting on diff forms.
$\mathrm{Ker}\, D_{M,S^*}$ = de Rahm cohomology of M.
$\mu(M, S^*, f)$ is an (oriented) hmtpy inv.

note: V/Λ is spin. we have fund class $= \mu(V/\Lambda, \text{triv}, id)$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "h"'] & V/\Lambda \\ M' \arrow[ur, "f'"'] \end{tikzcd}$$ commutes up to hmtpy

then $\mu(M, S^*, f) = \mu(M', S'^*, f')$

So... the torus V/Λ admits no metric of positive scalar curvature. (in any dim!)

and if $h: M \xrightarrow{\sim} M'$ as before then for any $\alpha \in H^*(V/\Lambda)$

$$\int_M L(M) \wedge f^* \alpha = \int_{M'} L(M') \wedge f'^* \alpha$$

Gromov-Lawson-Rozenberg: If M is spin and aspherical ($M \cong B\pi_1 M$) it can support no metric of pos. scalar curv.

~ x ~

cont. fields of Hilb sp. VS Hilbert modules:

H = cont. field over (cpt) X

$\Gamma(H)$ = cont sections

* ~~thats~~ a module over $C(X)$
* it has a $C(X)$-valued inner prod: $\langle s_1, s_2 \rangle$ = pointwise inner prod

one has: $\langle s, s \rangle \geq 0$

$$\langle s_1, s_2 f \rangle = \langle s_1, s_2 \rangle f \quad (f \in C(X))$$

* $\Gamma(H)$ complete wrt $\|s\| = \|\langle s, s \rangle\|^{1/2}$ = sup norm of fn in $C^*(X)$

i.e. $\Gamma(H)$ is a Hilbert module

Observe: we can recover H from this Hilbert $C(X)$-mod str.

$$H_x = \Gamma(X) \otimes_{C(X)} \mathbb{C}_x$$

π a discrete grp

$\boxed{C^*(\pi)}$ = a completion of grp alg (there is not a unique choice)

$C^*(\pi)$ is a C^*-alg.

(choosing the "right" completion is important for Baum-Connes conj).

For $\underset{\Lambda}{\pi}$ abelian, $C^*(\pi) = C(V^*/\Lambda^*)$ (due to Fourier)

Let $\mathcal{M} = C^*(\pi)$ viewed as a right module.

$\mathcal{M}$ is a Hilbert module: $\langle m_1, m_2 \rangle = m_1^* m_2 \in C^*(\pi)$

Que: Why believe Baum-Connes Conj?

Example: Let's go back to $\pi = \Lambda$.

$$\begin{tikzcd}
K_*(V/\Lambda) \arrow[r, "\mu"] \arrow[d, "\cong"', "\text{Poincare}"] & K_*(C^*(\Lambda)) \arrow[d, "\cong \text{ Fourier}"] \\
K^*(V/\Lambda) \arrow[r, "m"] & K^*(V^*/\Lambda^*)
\end{tikzcd}$$

$K_*(V/\Lambda)$: K-homology

$K_*(C^*(\Lambda))$: K-theory of C^*-alg

$K^*(V/\Lambda)$: K-theory of V/Λ

$K^*(V^*/\Lambda^*)$: K-theory of topo sp. V^*/Λ^*

Fact: $m^2 = id$

So, in this case, the conj works (all due to Fourier + Poincare).
Let's look at this example more closely.
The map m is given by the Dirac op on V/Λ.

$$\underset{K^*(V/\Lambda)}{\overset{[E]}{\cap}} \longmapsto \underset{K^*(V^*/\Lambda^*)}{\overset{\text{index of family on } V^*/\Lambda^* \,(D_E \otimes L_{v^*})}{\cap}}$$

The basic cts field here is

$$H_{v^*} = L^2(V/\Lambda, L_{v^*})$$

$$\begin{array}{c} L_{v^*} \text{ unitary line bun} \\ \downarrow \\ V/\Lambda \end{array}$$

A section of L_{v^*} is a twist periodic fn on V:

$$s_{v^*}(v+k) = v^*(k)s(v) \quad k \in \Lambda.$$

The Fourier trans. is an elt of $\ell^2(\Lambda^* + v^*)$.

We get: smooth sections $\Gamma(H) = \mathcal{S}(V)$ Schwartz fns on V.

The basic field/module $(H/\Gamma(H))$ is a completion of $\mathcal{S}(V)$

H has this str:

1) $C^*(\Lambda)$-module str. (after completion)
2) a repn of $C(V/\Lambda)$ as module maps
3) an operator D. ($C^*(\Lambda)$-linear)

1) $f \cdot g$ action by translation.
   $\langle f_1, f_2 \rangle (g) = \langle f_1, f_2 \cdot g \rangle_{L_2}$
2) ptwise multi
3) Dirac

This data allows us to construct μ.
There is a dual collection to this data used to construct μ^{-1}.

1) $C(V/\Lambda)$ - Hilb. module str.
2) a repn of $C^*(\Lambda)$
3) an operator (which is $C(V/\Lambda)$-linear)
   $D =$ "multi by x" $=$ Cliff. multi by $v \in V$.