Symmetries, patterns and orbifolds

Tim Perutz

Saturday Morning Math Group,
UT Austin, September 15, 2012

2-dimensional patterns with symmetry

Wallpaper patterns

These patterns are all called ‘wallpaper patterns’ by mathematicians, though only one of them is actually wallpaper. Here’s what distinguishes them from other patterns:

- They all have **translational symmetry**: you can shift the whole pattern by a certain amount, in a certain direction, and it will look the same.
  For the brickwork pattern we can shift it upwards by a whole number of bricks.
- You can translate in (at least) two **different** directions.
  For the bricks we can shift up and down or sideways. There are also combinations—first move up, then sideways.
- They **don’t have too many** translations: in any given direction, there’s a smallest unit of translation.
  Vertical black and white stripes have too many translations.

Different kinds of symmetries

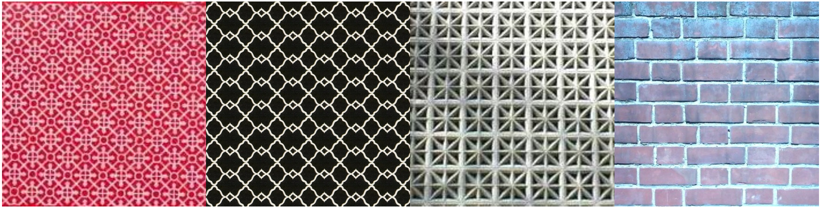
Besides translations, wallpaper patterns have other kinds of symmetry—operations that leave the pattern unchanged:

- **Rotations:** rotate by a certain angle about a certain center.
- **Reflections:** mirror symmetry—reflect across a straight line.
- **Glides**: reflect across a line, then translate in the direction of that line. Think: footprints!

Combinations of translations, rotations, reflections and glides are again translations, rotations, reflections and glides.
For example, a combination of two reflections is a rotation. Think: periscope!

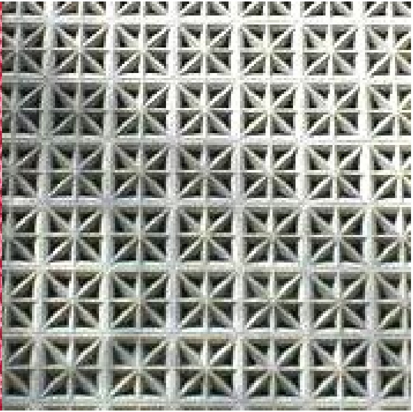
Symmetry types

All four of these patterns have a rectangular look to them, but only two of them have the same symmetries. Which ones?



Symmetry types

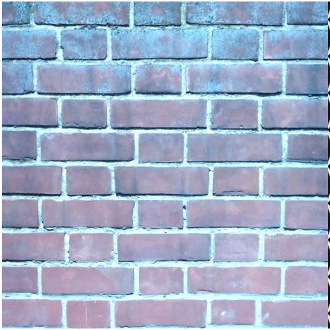
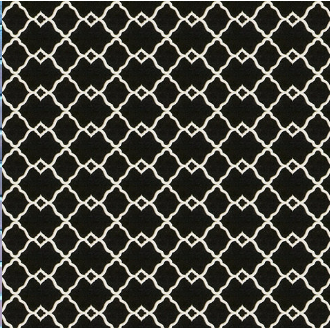
These two have the same symmetries:



Both have mirror lines which are horizontal, vertical or diagonal. The only centers of rotation are on the places where mirrors meet.

Symmetry types

Unlike the playing card and the wallpaper, these two have horizontal and vertical mirrors, but no diagonal mirrors.

In the brickwork pattern, there's a half-turn (180° rotation) not centered on a mirror. So these two belong to different symmetry types.

Invariants

In mathematics, when we're trying to decide whether two complicated things are **alike** in some definite sense, it's a great strategy to look for **invariants**.

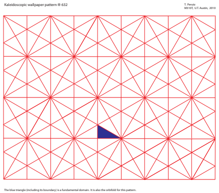
- The invariants should be simpler sorts of things than the original things (e.g. numbers instead of geometric objects).
- When two things are alike, their invariants must be the same.

We'll apply this strategy to help us distinguish and classify wallpaper patterns.
Patterns are considered alike if they have the same symmetries.

Invariants for patterns

$$\text{patterns} \xrightarrow{\text{invariant of a pattern}} \text{orbifolds} \xrightarrow{\text{invariant of an orbifold}} \text{signatures}$$

An orbifold is a geometric gadget—simpler than a pattern—while a signature is a code of numbers and symbols.



The little blue triangle is the orbifold for this pattern.

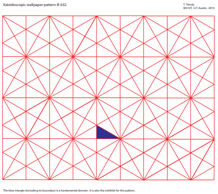
The signature of the orbifold is $*632$.

Overall, we say that the signature of this pattern is $*632$.

Invariants for patterns

$$\text{patterns} \xrightarrow{\text{invariant of a pattern}} \text{orbifolds} \xrightarrow{\text{invariant of an orbifold}} \text{signatures}$$

An orbifold is a geometric gadget—simpler than a pattern—while a signature is a code of numbers and symbols.



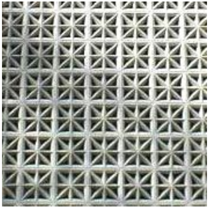
The little blue triangle is the orbifold for this pattern.

The signature of the orbifold is $*632$.

Overall, we say that the signature of this pattern is $*632$.

Alike patterns have the same signature

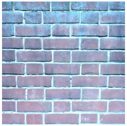
Signature $*442$.



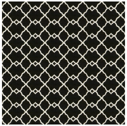
Signature $*442$.

If patterns have different signatures, they must have different symmetries

Signature $*442$.



Signature $2*22$.



Signature $*2222$.

Perfect invariants

Could it be that two patterns have the same invariants, yet different symmetries?

- For some invariants, the answer is yes. For instance, I could define a numerical invariant of patterns which takes the value 0 if the pattern has a mirror symmetry, and 1 if it does not. This is an invariant, but it only groups patterns into two broad classes.
- For the signature, the answer is no! The signature determines the orbifold, and the orbifold determines the symmetries.
- The signature is a **perfect invariant**.Two patterns with with the same signature have the same symmetries.

What is the orbifold of a pattern?

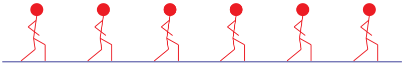
Definition

The **orbifold** of a pattern is the geometric object we get from the plane when we regard two points to be equal if they are related by a symmetry of the pattern.

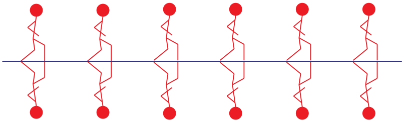
This is hard to understand until you try some examples.

Frieze patterns and their orbifolds

Wallpaper patterns are complicated, so we'll start with something simpler: **frieze patterns**. These have translational symmetries, but only in one direction. Moreover, there's a smallest unit of translation in that direction.



Orbifold = cylinder with two infinite ends



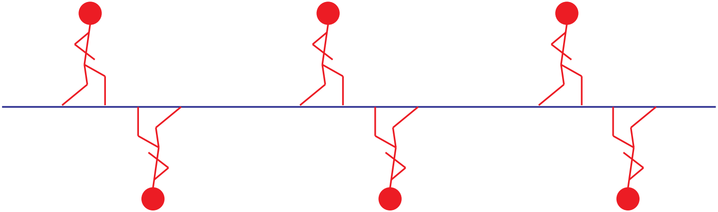
Orbifold = cylinder with one boundary circle, one infinite end

Frieze patterns and their orbifolds

What's the orbifold here?

Hint 1: You need scissors and tape.

Hint 2: centers of rotation give sharp points like cones.

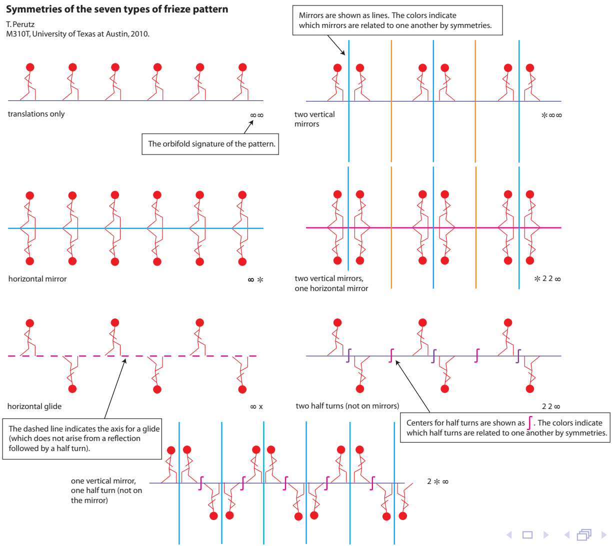


Signatures

The signature of an orbifold describes the features you would need to add to make it (strictly, you should start from a sphere).

- $*$ means you punch a hole, so as to make a boundary
- $*632$. Numbers after $*$ mean that you add corners to that boundary. If the number is N, the corner makes an angle of $180^\circ/N$.
- 333. Numbers before (or without) $*$ mean sharp cone points. If you see a number N, take a pie-slice of angle $360^\circ/N$, pull it around, and tape together the two edges adjacent to the angle so to make a cone.
- ∞ before (or without) $*$ means punch a hole and stretch it out to make an infinite cylindrical end.
- ∞ after $*$ means an infinite strip-like end.
- $\circ$ means add a 'handle'
- $\times$ means add a 'cross cap' (weird)

Signatures of frieze patterns



Price list

Orbifold features (as listed in the signatures) aren't free. Each has a price.

Feature	symbol	price ($)
boundary	$*$	1
corner	$(*)n$	$(n-1)/n$
cone-point	n	$(n-1)/(2n)$
cylindrical end	∞	1
strip-like end	$(*)\infty$	$1/2$
handle	$\circ$	2
cross-cap	$\times$	1

For instance, $*632$ costs $1+\frac{5}{12}+\frac{2}{6}+\frac{1}{4}=2$.

The orbifold theorem

Theorem

The price of an orbifold that comes from a wallpaper pattern or a frieze pattern is exactly \$2.

Exercise: check that these signatures do cost exactly \$2

- the 17 wallpaper signatures (no ∞ symbols)

$$*2222,\quad *333,\quad *442,\quad *632,\quad 2*22,\quad 3*3,\quad 4*2,\quad 22*,$$
$$2222,\quad 333,\quad 442,\quad 632,\quad **,\quad *\times,\quad \times\times,\quad 22\times,\quad \circ$$

- the 7 frieze signatures (which include ∞ symbols)

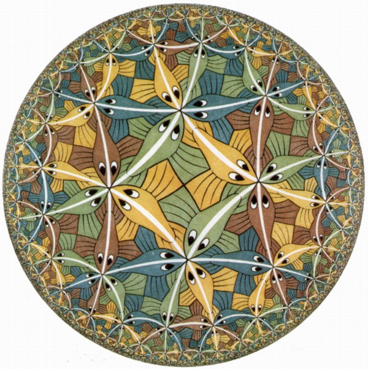
$$\infty\infty,\quad \infty*,\quad *\infty\infty,\quad 22\infty,\quad *22\infty,\quad \infty\times,\quad *\infty\infty.$$

17 wallpaper patterns and 7 frieze patterns

- In the last slide, I listed 17 ways you can spend exactly \$2 on orbifold parts (not ∞) and 7 ways you can spend exactly \$2 on orbifold parts (including ∞).
- **Challenge 1**: prove that these are the only signatures that cost exactly \$2.
- **Challenge 2**: For each of these signatures, identify the orbifold, and find a pattern that has this orbifold.
- Because the signature is a perfect invariant of wallpaper or frieze patterns, we conclude:
  there are exactly 17 symmetry-types of wallpaper pattern, and 7 symmetry-types of frieze pattern.

An expensive orbifold

Not a wallpaper pattern, but a **hyperbolic tiling**.



It has an orbifold, and a signature, but it costs more than $2.

Thanks for listening

Next time you see some wallpaper, bathroom tiles, bricks, or something else with a wallpaper or frieze pattern, try to work out its signature.
It takes some practice.

Happy orbifolding!

Further reading: J. Conway, H. Burgiel, C. Goodman-Strauss, *The Symmetries of Things.*