Topological Holography and the $\mathcal{N}=4$ chiral algebra

(Joint with Davide Gaiotto)

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Happy $59\,^{3}/_{4}$ birthday Dan!

Infinite Chiral Symmetry in Four Dimensions

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Abstract: We describe a new correspondence between four-dimensional conformal field theories with extended supersymmetry and two-dimensional chiral algebras. The meromorphic correlators of the chiral algebra compute correlators in a protected sector of the four-dimensional theory. Infinite chiral symmetry has far-reaching consequences for the spectral data, correlation functions, and central charges of any four-dimensional theory with $\mathcal{N} = 2$ superconformal symmetry.

$\mathcal{N} = 2$ superconformal theory

$\Rightarrow$ Vertex Algebra

$\mathcal{N} = 4$ with gauge group $U(N) \Rightarrow$ chiral algebra $\mathcal{C}_N$

GOAL: Relate $\mathcal{C}_N$, $N \to \infty$ with Holographic Dual

Holographic Dual is the
B-model on $SL_2(\mathbb{C})$

We will show that

Large N VOA $\cong$ A VOA built
from "Witten diagrams"
on $SL_2(\mathbb{C})$

(Standard holographic dictionary,
Witten 98)

$\mathcal{N}=4$ chiral algebra:

- Z_1, Z_2 bosonic spin $1/2$ fields in adjoint rep. of GL_N

$$Z_1(0) \cdot Z_2(z) \sim z^{-1}$$

- Perform BRST Reduction by GL_N

Equivalently:

$\mathcal{N}=4$ chiral algebra $\simeq$

Chiral Differential Operators

on $\mathfrak{gl}_N / GL_N$

(adjoint quotient stack)

Mode Algebra:

Quantum Hamiltonian Reduction

of $\mathrm{Diff}(\mathfrak{gl}_N((z)))$ by $GL_N((z))$

Classical Version (phase space)

$$(T^*L\mathfrak{gl}_N)/\!/LGL_N = (L\mathfrak{gl}_N \oplus L\mathfrak{gl}_N^*)/\!/LGL_N$$

$$= \left\{ Z_1, Z_2 \in \mathfrak{gl}_N((z)),\ [Z_1, Z_2] = 0 \right\} / GL_N((z))$$

Variant: Include matter

$J^\beta \in \mathrm{Hom}(\mathbb{C}^{K|K}, \mathbb{C}^N)$

$I_\alpha \in \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^{K|K})$

$Z_1, Z_2 \in \mathfrak{gl}_N$

$I_\alpha \cdot J^\beta \sim \delta_\alpha^\beta \, 1/z$

$Z_i \cdot Z_2 \sim 1/z$

BRST reduce by GL_N

Phase Space:

$Z_i \in \mathfrak{gl}_N((z)),\ I_\alpha \in \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^{K|K})((z)),$

$J^\beta \in \mathrm{Hom}(\mathbb{C}^{K|K}, \mathbb{C}^N)((z))$

$[Z_1, Z_2] + IJ = 0$, modulo $GL_N((z))$

$\mathcal{N}=4$ vertex algebra $\cong$ algebra of operators for the open-string B-model for a B-brane $\mathcal{O}_{\mathbb{C}}^{\oplus N} \in \mathrm{Coh}(\mathbb{C}^3)$

Phase Space $\left\{\text{Moduli of sheaves on } \mathbb{C}^\times \times \mathbb{C}^2 \text{ deforming } \mathcal{O}_{\mathbb{C}^\times}^{\oplus N}\right\}$

$$\cong \left\{ Z_1, Z_2 \in gl_N[z, z^{-1}],\ [Z_1, Z_2] = 0 \right\} \Big/ GL_N[z, z^{-1}]$$

sheaf $\mathcal{Y} \rightsquigarrow \pi_* \mathcal{Y}$ vector bundle on $\mathbb{C}^\times$, coordinates z_1, z_2 on $\mathbb{C}^2 \rightsquigarrow$

$$Z_1, Z_2 \in \mathrm{End}(\pi_* \mathcal{Y}),\ [Z_1, Z_2] = 0$$

With matter: also include space-filling brane $\mathcal{O}_{\mathbb{C}^3} \otimes \mathbb{C}^{K|K}$

Phase Space (roughly)

$$\left\{ \begin{array}{l} \text{Torsion free rank } (K|K) \text{ sheaves on } \mathbb{C}^\times \times \mathbb{C}^2 \\ \text{flat over } \mathbb{C}^\times, \text{ second Chern class } N[\mathbb{C}^\times], \text{framed} \end{array} \right\}$$

I, J, Z_i : ADHM data

Goal: Relate large N vertex algebra (with matter) to closed string B-model (with space-filling brane)

Closed string fields: (X a $CY3$)

- $\alpha \in \Omega^{2,1}(X) = \Omega^{0,1}(X, TX)$

  Beltrami differential: controls deformation of complex structure

- $A \in \Omega^{0,1}(X) \otimes \Pi\mathbb{C}^2$

  Holomorphic Chern-Simons gauge field for $\Pi\mathbb{C}^2$

Equations of motion:

α defines integrable deformation of complex str

A holomorphic $\Pi\mathbb{C}^2$ bundle

X CY3 $\qquad C \subseteq X$ Riemann surface

What is the closed string field sourced by a brane on C?

Answer: $\alpha \in \Omega^{2,1}(X)$ such that

$$\bar\partial \alpha = N \delta_C \in \Omega^{2,2}(X)$$

Example: $\mathbb{C} \subseteq \mathbb{C}^3$

$z \qquad z, w_1, w_2$

$$\alpha = \frac{dw_1 dw_2 (\bar w_1 d\bar w_2 - \bar w_2 d\bar w_1)}{|w|^4} \sim \frac{(\bar w_1 d\bar w_2 - \bar w_2 d\bar w_1)}{|w|^4} \partial_z$$

(Using $\partial_z \vee (dw_1 dw_2 dz) = dw_1 dw_2$)

Holographic Dual to theory on $\mathbb{C} \subseteq \mathbb{C}^3$:

Complex manifold $\mathbb{C}^3 \setminus \mathbb{C}$ where holomorphic functions are $F(z, \bar{z}, \omega_i, \bar{\omega}_i)$ satisfying

$$\boxed{\bar{\partial} F + N \frac{\varepsilon_{ij}\, \bar{\omega}_i\, d\bar{\omega}_j}{|\omega|^4} \partial_z F = 0}$$

Solutions to this equation: ω_1, ω_2 and

$$u_1 = \omega_1 z - N \frac{\bar{\omega}_2}{|\omega|^2}$$

$$u_2 = \omega_2 z + N \frac{\bar{\omega}_1}{|\omega|^2}$$

$$\boxed{u_2 \omega_1 - u_1 \omega_2 = N}$$

Conclusion: Holographic dual to the chiral algebra is topological B-model on

$$SL_2(\mathbb{C}) = \mathbb{H}^3 \times S^3$$

Goal: Compute large N vertex algebra from B-model on $SL_2(\mathbb{C})$

Two approaches:

1) Define a holographic vertex algebra using Witten diagrams

2) Use Global Symmetry Algebras

Large N global symmetry algebra

$\mathcal{O}$ a single trace operator

Modes $\oint z^k dz\, \mathcal{O}(z)$ form a Lie algebra

The global symmetry algebra consists of modes preserving vacuum at 0 and ∞

If $\mathcal{O}$ is of spin r, global symmetries are

$$\oint z^k dz\, \mathcal{O}(z), \qquad 0 \leq k \leq 2r-2$$

E.g.: Kac-Moody vertex algebra

J_a copy of $\mathfrak{g}$ in spin 1 fields

All modes: $\oint z^k J_a dz$ form $\mathfrak{g}((z))$

Global symmetries: $\mathfrak{g}$, $\oint J_a dz$

Virasoro vertex algebra: T spin 2

All modes: $L_n = \oint z^{n+1} T dz$

Global symmetries: $L_{-1}, L_0, L_1 \cong sl_2$

Global symmetries of large N VOA $=: \mathfrak{a}_\infty^{CFT}$

$$\left.\begin{array}{l} \mathrm{Tr}\, Z_1^2,\ \mathrm{Tr}\, Z_2^2 \\ \mathrm{Tr}\, Z_1 Z_2 \end{array}\right\}$$ Kac-Moody for $sl_2(\mathbb{C})_R$

$$\Longrightarrow sl_2(\mathbb{C})_R \subseteq \mathfrak{a}_\infty^{CFT}$$

$T = \varepsilon_{ij} \mathrm{Tr}\, Z_i \partial Z_j$ Virasoro generator

$$\Longrightarrow sl_2(\mathbb{C})_c \subseteq \mathfrak{a}_\infty^{CFT}$$

$\mathfrak{a}_\infty^{CFT}$ is an infinite dimensional Lie alg.

e.g. $\displaystyle\oint z^k \,\mathrm{Tr}(Z_1^n)\, dz \in \mathfrak{a}_\infty^{CFT}$

$k \leq n-2$

Holographic Global Symmetry Algebra

$\subseteq$ modes of holographic VOA

is Lie algebra of closed string states

Bosonic Part: $\mathfrak{a}_\infty^{hol} = \mathrm{Vect}_0(SL_2(\mathbb{C}))$

divergence free holomorphic vector fields

With matter: also include open string states

$\mathcal{O}(SL_2(\mathbb{C})) \otimes \mathfrak{gl}(K|K)$

$$sl_2(\mathbb{C})_L \oplus sl_2(\mathbb{C})_R \subseteq a_\infty^{hol}$$

$sl_2(\mathbb{C})_L$: identify with conformal symmetry $\quad sl_2(\mathbb{C})_C \subseteq a_\infty^{CFT}$

Theorem

$$a_\infty^{CFT} \cong a_\infty^{hol}$$

as Lie algebras (also holds when we include matter)

This provides very strong constraints on the VOA:

$$\left[\oint z^k \mathcal{O}(z)dz, \oint z^l \mathcal{O}'(z)dz\right]$$

expressed in terms of OPE $\mathcal{O}(0)\cdot\mathcal{O}'(z)$

VOA determined up to 1 parameter: N

Corollary: Large N VOA

$\cong$ Holographic VOA

Large N VOA with matter:

$J^\beta \in \mathrm{Hom}(\mathbb{C}^{K|K}, \mathbb{C}^N)$ $\quad Z_1, Z_2 \in \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N)$

$I_\alpha \in \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^{K|K})$ $\quad SU(2)_R$

Gauge-invariant open-string operators:

$I_\alpha Z_{i_1} \cdots Z_{i_n} J^\beta$

- Adjoint of $gl(K|K)$
- Spin $n/2$ rep. of $SU(2)_R$
- Spin $n/2+1$ under Lorentz $SO(2)$

Corresponding global symmetries:

$$\oint z^k dz \, I_\alpha Z_{i_1} \cdots Z_{i_n} J^\beta$$

$$0 \leq k \leq n$$

- Spin $n/2$ under both $SU(2)_R$ and SL_2 conformal symmetry

All open string global symmetries

$$\in gl(K|K) \otimes \left(\bigoplus_j V_j^L \otimes V_j^R \right) \cong gl(K|K) \otimes \mathcal{O}(SL_2)$$

SL_2 conformal symmetry

$SU(2)_R$

Open string global symmetries

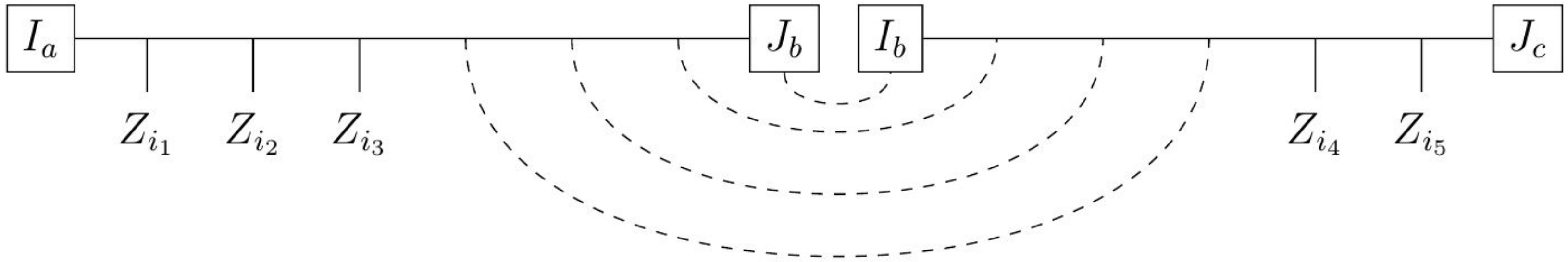
$$E_\alpha^\beta(F), \qquad F \in \mathcal{O}(SL_2(\mathbb{C}))$$

Our Theorem says

$$[E_\alpha^\beta(F), E_\beta^\gamma(G)] = E_\alpha^\gamma(F \cdot G)$$

(multiplication in $\mathcal{O}(SL_2(\mathbb{C}))$)

Reconstruct $SL_2(\mathbb{C})$ from the global symmetry algebra!



Wick contraction of open-string operators

= multiplication in $\mathcal{O}(SL_2(\mathbb{C}))$

Coordinates u_1, u_2, w_1, w_2, on $SL_2(\mathbb{C})$

$$u_1 w_2 - u_2 w_1 = N$$

$$E_\alpha^\beta(u_i) = \oint I_\alpha Z_i J^\beta dz$$

$$E_\alpha^\beta(w_i) = \oint I_\alpha Z_i J^\beta z dz$$

$$E_\alpha^\beta(1) = \oint I_\alpha J^\beta dz$$

Easy calculation:

$$[E_\alpha^\beta(u_1), E_\beta^\gamma(w_2)] - [E_\alpha^\beta(u_2), E_\beta^\gamma(w_1)]$$

$$= N E_\alpha^\gamma(1)$$

Closed string operators:

$\mathrm{Tr}\, Z_{[i_1} \cdots Z_{i_k]}$ spin $k/2$ rep of $SU(2)_R$

spin $k/2$ under Lorentz $SO(2)$

$\mathrm{Tr}\, Z_{j_1} \partial_z Z_{j_2} Z_{[i_1} \cdots Z_{i_k]} \varepsilon_{j_1 j_2}$

spin $k/2$ rep of $SU(2)_R$

spin $k/2+2$ under $SO(2)$

Corresponding Global Symmetries:

$$\oint z^l dz \, \mathrm{Tr}\, Z_{[i_1} \cdots Z_{i_k]} \quad \text{for } l \leq k-2$$

$$\in \bigoplus \left(V^L_{k/2-1} \otimes V^R_{k/2} \right)$$

$$\oint z^l dz \, \mathrm{Tr}(Z \partial Z) Z_{[i_1} \cdots Z_{i_k]} \quad \text{for } l \leq k+2$$

$$\in \bigoplus \left(V^L_{k/2+1} \otimes V^R_{k/2} \right)$$

Together these are Fourier modes of

$$\mathrm{Vect}_0(SL_2(\mathbb{C})) \cong \bigoplus \left(V^L_{j \pm 1} \otimes V^R_j \right)$$

Theorem Bosonic closed string global symmetries

$\cong$ $\mathrm{Vect}_0(SL_2(\mathbb{C}))$ as a Lie algebra

Many more examples to be worked out:

- affine quiver theories $\leftrightarrow$ $SL_2(\mathbb{C})/\Gamma$
- class S chiral algebras (Gaiotto, Maldacena)

Related work:

- Ishtiaque, Moosavian, Zhou

Conclusion: Holographic dual to the chiral algebra is topological B-model on

$$SL_2(\mathbb{C}) = \mathbb{H}^3 \times S^3$$

How does this relate to ordinary holography?

C., Si Li (2016): conjecture that localization of type IIB on $AdS_5 \times S^5$ is top'l B-model on $(\underset{w_i}{\mathbb{C}^3} \setminus 0) \times \underset{z_j}{\mathbb{C}^2}$ with background field

$$N \, \partial_{z_1} \partial_{z_2} \frac{\varepsilon_{ijk} \bar{w}_i d\bar{w}_j d\bar{w}_k}{|w|^6}$$

This top' string has $psl(3|3)$ symmetry (residue of $psl(4|4)$ on $AdS_5 \times S^5$).

Rastelli et al: Localize using $Q \in psl(3|3)$

Corresponding super-ghost is

$$w_1 z_1 + N \frac{\bar{w}_2 d\bar{w}_3 - \bar{w}_3 d\bar{w}_2}{|w|^4} \partial_{z_2} + \dots$$

Localizes top' string to $w_1 = 0,\ z_1 = 0$

Conclusion: Holographic dual to the chiral algebra is topological B-model on

$$SL_2(\mathbb{C}) = \mathbb{H}^3 \times S^3$$

How to recover the large N chiral algebra?

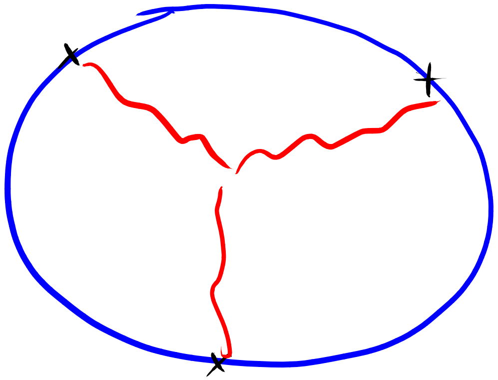
Effective theory on $\mathbb{H}^3$ is "higher spin gravity"

Usual holographic dictionary:

- Modifications of boundary conditions on $\partial \mathbb{H}^3 = \mathbb{CP}^1$

$\longleftrightarrow$ Single trace operators at large N

CFT correlators $\longleftrightarrow$ Witten diagram computations

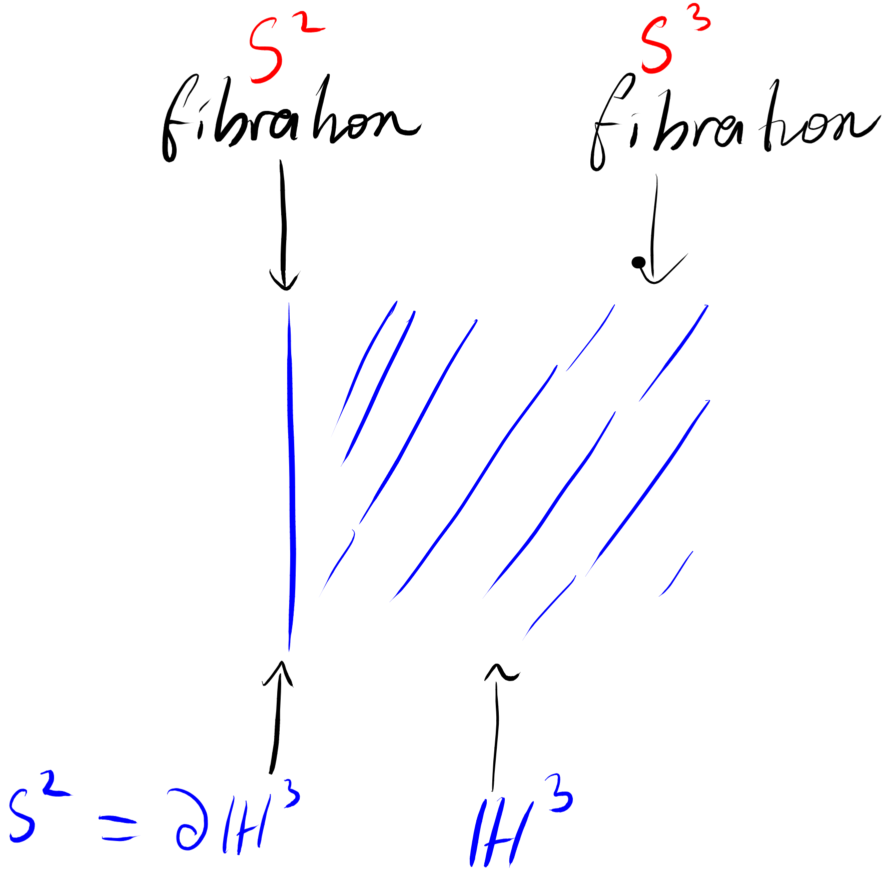


Holomorphic Picture:

$SL_2(\mathbb{C})$ compactifies to

$$u_1 w_2 - w_1 u_2 = N Y^2 \subseteq \mathbb{CP}^4$$

$Y=0$: $\mathbb{CP}^1 \times \mathbb{CP}^1$; holomorphic boundary



Holomorphic boundary is $\mathbb{CP}^1 \times \mathbb{CP}^1$, fibred over $\partial \mathbb{H}^3 = \mathbb{CP}^1$

Boundary conditions in holomorphic language:

$\alpha \in \Omega^{2,1}(SL_2(\mathbb{C}))$ has order 1 pole at $Y=0$

$A \in \Omega^{0,1}(SL_2(\mathbb{C})) \otimes psl(1|1)$ has order 1 zero at $Y=0$

Near boundary of $SL_2(\mathbb{C})$ have coordinates

w on $\mathbb{CP}^1_R$ (rotated by $SU(2)_R$)

z on $\mathbb{CP}^1_C$ (chiral algebra plane)

n normal direction (pole at $z=\infty, w=\infty$)

Modifications of boundary conditions at

$z=0, \; n=0$

$\longleftrightarrow$ Local operators of dual gauge theory

Possible Modifications: allow

$$A \in \Omega^{0,1}(SL_2(\mathbb{C})) \otimes psl(1|1)$$

to have a pole of order $K \geq 0$ at $z = 0$:

$$A \sim z^{-K} \omega^{l} \delta_{\bar{z}=0}, \quad l \leq K$$

Choices transform in representation

$[K+1, K/2+1]$ of

$$SU(2)_R \times SO(2)$$

(Dim^n $K+1$, spin $K/2$ under $SO(2)$ Lorentz transformations)

$\alpha \in \Omega^{2,1}(SL_2(\mathbb{C}))$ modified boundary conditions:

$$\alpha \sim n^{-k} dn\, \delta_{z=0} \quad \in \left[k, \frac{k-1}{2}\right]$$

Or:

$$\alpha \sim n^{-k} dw\, \delta'_{z=0} \quad + n^{-1-k} dn dw\, \iota_{\partial_z} \delta_{z=0}$$

$$\in [k-1, k/2+1]$$

Possible deformations of boundary conditions:

Bosonic: $[k+1, k/2]_{k\geq 1}$ and $[k+1, k/2+2]_{k\geq 0}$

Fermionic: $[k+1, k/2+1]_{k\geq 0}$ (two copies)

(+ z derivatives)

Single Trace operators of large N chiral algebra:

Bosonic: $\mathrm{Tr}\, X_1^k \in [k+1, k/2] \quad k \geq 1$

$\mathrm{Tr}\, X_1^k (X_i \partial_z X_j)\varepsilon_{ij} \in [k+1, k/2+2] \quad k \geq 0$

Fermionic: $\mathrm{Tr}\, X_1^k b \in [k+1, k/2+1] \quad k \geq 0$

$\mathrm{Tr}\, X_1^k \partial c \in [k+1, k/2+1] \quad k \geq 0$

Perfect match! ($\Rightarrow$ indices are the same but also no cancellation)

Conjecture: $\mathcal{N}=4$ chiral algebra correlation functions

$$\langle O_1(z_1) \ldots, O_k(z_k)\rangle = \left(\frac{\partial}{\partial b_1} \cdots \frac{\partial}{\partial b_k} Z_{KS}[b]\right)_{b=0}$$

b = Boundary condition

b_i = Variations of b.c. corresponding to O_i

Can make explicit computations of 2 and 3 point functions from the "gravity" side.

These match the chiral algebra computations

[No complete proof yet, verified for certain simple operators]

Example computation

$$\langle \mathrm{Tr}(b X_1^k), \mathrm{Tr}(\partial c X_2^k) \rangle \sim N^{k+1} z^{-k-2}$$

How to compute from gravity side?

$\mathrm{Tr}(\partial c X_2^k) \longleftrightarrow$ boundary operator

$$\int_{\mathbb{CP}^1 \times \mathbb{Z}} \omega^k d\omega \, \partial_n^{k+2} A^{0,1}$$

$\mathrm{Tr}(b X_1^k)$ corresponds to a boundary field with leading singularity $n^{-k} \delta_{z=0}$

Want to find $A^{0,1}$ with this leading singularity satisfying $\bar\partial A^{0,1} = 0$ (EOM)

$\bar\partial$ operator deformed by

$$N \frac{n^2 d\bar{w}}{(1+|w|^2)^2} \partial_z$$

Explicit Solution:

$$A = n^{-k} \delta_{z=0} + N n^{2-k} \frac{\bar{w}}{(1+|w|^2)} \delta'_{z=0} + \frac{N^2 n^{4-k}}{2} \left(\frac{\bar{w}}{(1+|w|^2)} \right)^2 \delta^{(2)}_{z=0}$$

$$+ \cdots + \frac{1}{k!} n^k N^k \left(\frac{\bar{w}}{1+|w|^2} \right)^k \delta^{(k)}_{z=0}$$

$$+ \frac{1}{k!} N^{k+1} n^{k+2} \frac{\bar{w}^k d\bar{w}}{(1+|w|^2)^{k+2}} \quad \frac{1}{z^{k+2}}$$

Only term present at $z \neq 0$ is

$$\frac{1}{k!} N^{k+1} n^{k+2} \frac{\bar{\omega}^k d\bar{\omega}}{(1+|w|^2)^{k+2}} \frac{1}{z^{k+2}}$$

Apply operator

$$\int_{\mathbb{C}P^1 \times \mathbb{Z}} \omega^k d\omega \, \partial_n^{k+2} A$$

corresponding to $\mathrm{Tr}\, \partial c X_2^k$

find gravitational two point function

$$N^{k+1} \frac{1}{z^{k+2}} \int_{\omega} \frac{|w|^{2k} dw d\bar{w}}{(1+|w|^2)^{k+2}}$$

Outlook

- Further Examples:

  D3 probing ADE singularity $\longleftrightarrow SL_2(\mathbb{C})/\Gamma$

- "Giant gravitons"

  Local operators are sourced by B-branes in $SL_2(\mathbb{C})$ living on complex curves

- Other classes of examples:

  M2 branes / matrix vector models (C., 2017)

  Line defects in 2d YM (Ishtiaque, Moosavian, Zhou: 2018)