Integrable systems and Special Kähler geometry

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Special Kähler Manifolds

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Abstract: We give an intrinsic definition of the special geometry which arises in global $N = 2$ supersymmetry in four dimensions. The base of an algebraic integrable system exhibits this geometry, and with an integrality hypothesis any special Kähler manifold is so related to an integrable system. The cotangent bundle of a special Kähler manifold carries a hyperkähler metric. We also define special geometry in supergravity in terms of the special geometry in global supersymmetry.

31. arXiv:dg-ga/9711002 [pdf, ps, other] math.DG

The moduli space of special Lagrangian submanifolds

Authors: Nigel Hitchin

Abstract: This paper considers the natural geometric structure on the moduli space of deformations of a compact special Lagrangian submanifold L^n of a Calabi-Yau manifold. From the work of McLean this is a smooth manifold with a natural L^2 metric. It is shown that the metric is induced from a local Lagrangian immersion into the product of cohomology groups $H^1(L) \times H^{n-1}(L)$. Using this appr...

▽ More

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INTEGRABLE SYSTEMS

- symplectic manifold (M^{2n}, ω)

- proper map $h : M^{2n} \to B^n$

- Lagrangian fibres: ω vanishes on $M_x = h^{-1}(x)$

- $x \in B^{\text{reg}} \Rightarrow$ (connected component of) a fibre is a torus

- tangent vector $X \in T_xB \Rightarrow$ section of normal bundle of M_x
- lift to vector field $\tilde{X}$ along M_x
- $i_{\tilde{X}}\omega$ well-defined and closed on M_x
- cohomology class: $T_xB \cong H^1(M_x, \mathbf{R})$

- $x \in U \subset B$ contractible, $H^1(h^{-1}(U), \mathbf{R}) \cong H^1(M_x, \mathbf{R})$

- $x, y \in U \Rightarrow T_xB \cong T_yB$ flat connection

- $x \in U \subset B$ contractible, $H^1(h^{-1}(U), \mathbf{R}) \cong H^1(M_x, \mathbf{R})$
- $x, y \in U \Rightarrow T_x B \cong T_y B$ flat connection
- $\omega = d\theta$ in $h^{-1}(U)$, $d\theta = 0$ on M_x
- flat coordinates $\quad x_i = \int_{C_i} \theta \qquad C_i \in H_1(M_x, \mathbf{Z})$

$\Rightarrow$ flat torsion-free affine connection

- suppose $h: M \to B$ is holomorphic

- M Kähler, $[\Omega] \in H^2(M, \mathbf{R}) \Rightarrow$ real symplectic form on B:

$$(X, Y) = \int_{M_x} i_{\tilde{X}}\omega \wedge i_{\tilde{Y}}\omega \wedge \Omega^{n-1}$$

- $\Rightarrow$ flat symplectic connection

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- $\Rightarrow$ flat symplectic connection

- complex structure?

1. Definition and Basic Properties

We introduce the following definition.

Definition 1.1. *Let M be a Kähler manifold with Kähler form ω. A* **special Kähler** *structure on M is a real flat torsionfree symplectic connection ∇ satisfying*

$$d_\nabla I = 0, \tag{1.2}$$

where I is the complex structure on M.

- $I : T \to T$ complex structure $I^2 = -1$
- $I \in \Omega^1(M, T) \qquad d_\nabla I = 0 \in \Omega^2(M, T)$
- locally $I = d_\nabla X$, X Hamiltonian vector field

HIGGS BUNDLES

- compact Riemann surface Σ, genus > 1

- holomorphic vector bundle V rank n

- Higgs field $\Phi \in H^0(\Sigma, \mathrm{End}\, V \otimes K)$

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- holomorphic vector bundle V rank n

- Higgs field $\Phi \in H^0(\Sigma, \mathrm{End}\, V \otimes K)$

- stability $\Rightarrow$ moduli space $\mathcal{M}$

- $\mathcal{M}$ is symplectic $((V, \Phi)$ "conjugate variables"$)$

- $\mathrm{det}(x-\Phi) = x^n + a_1 x^{n-1} + \ldots + a_n,\ a_j \in H^0(\Sigma, K^j)$

- $h : \mathcal{M} \to \mathcal{B} = H^0(\Sigma, K) \oplus \cdots \oplus H^0(\Sigma, K^n)$

- $\det(x - \Phi) = x^n + a_1 x^{n-1} + \ldots + a_n$, $a_j \in H^0(\Sigma, K^j)$

- $h : \mathcal{M} \to \mathcal{B} = H^0(\Sigma, K) \oplus \cdots \oplus H^0(\Sigma, K^n)$

- determinant line bundle $\Rightarrow [\Omega] \in H^2(\mathcal{M}, \mathbf{Z})$

- hyperkähler metric $\omega_1 = \Omega, \omega_2 + i\omega_3 =$ symplectic form

$\Rightarrow$ algebraic completely integrable Hamiltonian system

- det$(x - \Phi) = x^n + a_1 x^{n-1} + \ldots + a_n = 0$ algebraic curve S

- $\mathcal{B}^{\text{reg}}$ = smooth spectral curves

- fibre $h^{-1}(b) \cong$ Jac(S) abelian variety

- $S : x^n + a_1 x^{n-1} + \ldots + a_n = 0$

- $\pi : S \to \Sigma$ n-fold branched covering

- L line bundle on S

  Direct image: $U \subset \Sigma$ $H^0(U, \pi_* L) \overset{def}{=} H^0(\pi^{-1}(U), L)$

- $\pi_* L = V$ rank n vector bundle

- x single valued section of $\pi^* K$ on S

- $x : H^0(\pi^{-1}(U), L) \to H^0(\pi^{-1}(U), L \otimes \pi^* K)$

- $= \Phi : H^0(U, V) \to H^0(U, V \otimes K)$

Higgs field

- $(V, \Phi) \mapsto (V, \lambda\Phi)$ $\mathbf{C}^*$-action

- holomorphic symplectic form $\omega \mapsto \lambda\omega$

- $\Rightarrow$ vector field X, $\mathcal{L}_X\omega = \omega \Rightarrow d(i_X\omega) = \omega$

- $(V,\Phi) \mapsto (V,\lambda\Phi)$ $\mathbf{C}^*$-action

- holomorphic symplectic form $\omega \mapsto \lambda\omega$

- $\Rightarrow$ vector field X, $\mathcal{L}_X\omega = \omega \Rightarrow d(i_X\omega) = \omega$

- periods of $\mathrm{Re}(i_X\omega)$ on fibre $h^{-1}(b)$

  $\Rightarrow$ real flat coordinates on $\mathcal{B}^{\mathrm{reg}}$

- symplectic basis A_i, B_i for $H_1(\mathrm{Jac}(S), \mathbf{Z})$

  $\Rightarrow$ real coordinates x_i, y_i

  $\Rightarrow$ holomorphic coordinates z_i, $\mathrm{Re}\, z_i = x_i$ or w_i, $\mathrm{Re}\, w_i = y_i$

- Kähler potential for Special Kähler metric =

$$K = \frac{1}{2} \mathrm{Im} \sum_i w_i \bar{z}_i$$

- symplectic basis A_i, B_i for $H_1(\mathsf{Jac}(S), \mathbf{Z})$

  $\Rightarrow$ real coordinates x_i, y_i

  $\Rightarrow$ holomorphic coordinates z_i, $\mathrm{Re}\, z_i = x_i$ or w_i, $\mathrm{Re}\, w_i = y_i$

- Kähler potential for Special Kähler metric =

$$K = \frac{1}{2}\,\mathrm{Im} \sum_i w_i \bar{z}_i$$

- dual basis α_i, β_i of $H^1(\mathsf{Jac}(S), \mathbf{Z})$

- $$[i_X \omega] = \sum_i z_i \alpha_i + w_i \beta_i$$

Kähler potential $K = \mathrm{Im} \int_{\mathsf{Jac}(S)} i_X \omega \wedge i_X \bar{\omega} \wedge \Omega^{n-1}$

SPECTRAL CURVES

30. arXiv:math/9901069 [pdf, ps, other] math.DG

The moduli space of complex Lagrangian submanifolds

Authors: N J Hitchin

Abstract: Following an earlier paper on the differential-geometric structure of the moduli space of special Lagrangian submanifolds in a Calabi-Yau manifold, we follow an analogous approach for compact complex Lagrangian submanifolds of a (Kählerian) complex symplectic manifold. The natural geometric structure on the moduli space is a special Kähler metric, but we offer a different point of view on the lo… ▽ More

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- $x^n + a_1 x^{n-1} + \ldots + a_n = 0$, $x \in H^0(S, \pi^* K)$

- x embeds S in total space of K = cotangent bundle of Σ

- one-dimensional $\Rightarrow$ Lagrangian $\subset (T^*\Sigma, \varpi)$

- $\mathcal{B}^{\text{reg}}$ = moduli space of spectral curves

  = moduli space of Lagrangian fibres in $\mathcal{M}$

- tangent space $T_b\mathcal{B} \cong H^0(S,K) \cong H^0(\mathrm{Jac}(S),\Omega^1)$

- $\det(x-\lambda\Phi) = \lambda^n \det(\lambda^{-1}x - \Phi)$

  $\Rightarrow$ action of λ on $\mathcal{M} \Rightarrow \lambda^{-1}$ on $T^*\Sigma$

- $X \in T_b\mathcal{B} = i_X\omega|_{\mathrm{Jac}(S)} \in H^0(\mathrm{Jac}(S),\Omega^1)$

  $= -i_Y\varpi|_S = \theta =$ canonical 1-form on $T^*\Sigma$

- same special Kähler metric on $\mathcal{B}^{\mathrm{reg}}$

- Kähler potential

$$K = \mathrm{Im} \int_{\mathrm{Jac}(S)} i_X \omega \wedge i_X \bar{\omega} \wedge \Omega^{n-1} = \mathrm{Im} \frac{1}{4} \int_S \theta \wedge \bar{\theta}$$

(D.Baraglia & Z.Huang, *Special Kähler geometry of the Hitchin system and topological recursion*, arXiv:1707.04975v2)

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(D.Baraglia & Z.Huang, *Special Kähler geometry of the Hitchin system and topological recursion*, arXiv:1707.04975v2)

- $SL(2,\mathbf{C})$- Higgs bundles $S : x^2 - q = 0, q \in H^0(\Sigma, K^2)$

$$K = \int_\Sigma \sqrt{q\bar{q}}$$

QUESTION

$\int_S \theta \wedge \bar{\theta}$ is defined for any compact curve in K, singular or otherwise ...

.... what geometry does it define?

THE CRITICAL LOCUS

- $h : M^{2m} \to B^m$ functions $h_1, \ldots, h_m$

- **critical locus** = critical points for some $f = \sum_i c_i h_i$

  $= \{x \in M : \mathrm{rk}\, Dh_x \leq m\}$

- = singular locus of a degenerate torus

- Hamiltonian vector fields $X_1, \ldots, X_d$ vanish at $x \in M^{2m}$

- action on tangent space T_x preserving symplectic form ω

- $L \subset T_x$ span of all X_i, $\dim L = m - d$

- Hamiltonian vector fields $X_1, \ldots, X_d$ vanish at $x \in M^{2m}$
- action on tangent space T_x preserving symplectic form ω
- $L \subset T_x$ span of all X_i, $\dim L = m - d$
- $\omega(X_i, X_j) = 0 \Rightarrow L \subset L^\perp$
- $L^\perp / L$ symplectic, dimension $2d$

- $X_1, \ldots, X_d$ span a commutative subalgebra of $\mathfrak{sp}(2d, \mathbf{C})$.

- **Defn**: The point x is called *nondegenerate* if this is a Cartan subalgebra.

- $\mathfrak{sp}(2d, \mathbf{C}) \cong$ quadratic functions on $\mathbf{C}^{2d}$

- commutative subalgebra spanned by the Hessians of $h_1, \ldots, h_d$

SUR CERTAINS SYSTEMES DYNAMIQUES SEPARABLES.

By J. VEY.*

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1. Algèbres de Liouville. Soit X une variété symplectique, disons analytique réelle, de dimension $2m$; et soient $f_1, \ldots, f_m$ m fonctions analytiques réelles sur X, commutant deux à deux pour le crochet de Poisson. Si les fonctions f_i sont indépendantes, et si les variétés

$$f_1 = C_1, \ldots, f_m = C_m$$

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Action-angle coordinates at singularities for analytic integrable systems*

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- nondegeneracy + analytic $\Rightarrow$ local normal form

- $C_d = \{x \in M : \dim \ker Dh_x = d\}$ is a submanifold

- dimension $2(m-d)$ and symplectic

- ... and is a fibration by tori of dimension $(m-d)$

= subintegrable system

THE DERIVATIVE OF $h : \mathcal{M} \to \mathcal{B}$

- holomorphic vector bundle: $\bar{\partial}_A : \Omega^0(V) \to \Omega^{01}(V)$

- holomorphic Higgs field: $\bar{\partial}_A \Phi = 0$

- tangent vector to $\mathcal{M}$: $\bar{\partial}_A \dot{\Phi} + [\dot{A}, \Phi] = 0$

  modulo $(\dot{A}, \dot{\Phi}) = (\bar{\partial}_A \psi, -[\psi, \Phi])$

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  modulo $(\dot{A}, \dot{\Phi}) = (\bar{\partial}_A \psi, -[\psi, \Phi])$

- = hypercohomology $\mathbf{H}^1$ of complex of sheaves

$$\mathcal{O}(\mathrm{End}\, V) \stackrel{[\Phi,-]}{\to} \mathcal{O}(\mathrm{End}\, V \otimes K)$$

Two spectral sequences

- $\to H^0(\Sigma, \mathrm{End}\, V \otimes K) \to \mathbf{H}^1 \to H^1(\Sigma, \mathrm{End}\, V) \to$

  $\sim$ fibration $\mathcal{M} \supset T^*\mathcal{N} \to \mathcal{N}$

  $\mathcal{N}$ moduli space of stable bundles

- $0 \to H^1(\Sigma, \ker \mathrm{ad}(\Phi)) \to \mathbf{H}^1 \to H^0(\Sigma, \mathrm{coker}\, \mathrm{ad}(\Phi)) \to 0$

  $\sim$ integrable system $h : \mathcal{M} \to \mathcal{B}$

Two spectral sequences

- $\to H^0(\Sigma, \mathrm{End}\, V \otimes K) \to \mathbf{H}^1 \to H^1(\Sigma, \mathrm{End}\, V) \to$

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  $\sim$ integrable system $h : \mathcal{M} \to \mathcal{B}$

- Φ everywhere regular $\Rightarrow$ centralizer generated by $1, \Phi, \ldots, \Phi^{n-1}$

  $\Rightarrow \ker \mathrm{ad}(\Phi) \cong \mathcal{O} \oplus K^{-1} \oplus \cdots \oplus K^{-(n-1)}$

  $\Rightarrow \mathrm{coker}\, \mathrm{ad}(\Phi) \cong K \oplus K^2 \oplus \cdots \oplus K^n$

- $p : \mathrm{End}\, V \to \mathrm{coker}\, \mathrm{ad}(\Phi)$, $\mathrm{tr}(\Phi^k \dot{\Phi}) = \mathrm{tr}(\Phi^k p(\dot{\Phi}))$

  $\mathrm{tr}\, \Phi^k$ = universal polynomial in $a_1, \ldots, a_k$

- everywhere regular $\Rightarrow$ not on critical locus

- Suppose Φ regular except on a divisor $D \subset \Sigma$ (e.g. S is reduced)

$$0 \to \mathcal{O} \oplus K^{-1} \oplus \cdots \oplus K^{-(n-1)} \to \ker \mathrm{ad}(\Phi) \to \mathcal{C} \to 0$$

$\mathcal{C}$ supported on D

- $H^0(\mathcal{C}) \to H^1(\Sigma, \mathcal{O} \oplus K^{-1} \oplus \cdots \oplus K^{-(n-1)})$

  Serre duality $\Rightarrow$ linear functions on base $\mathcal{B}$

  $=$ functions critical at (V, Φ).

- $\Rightarrow$ Hessians localized around points $x_i \in D$

NODAL SPECTRAL CURVES

- spectral curve $S : x^n + a_1 x^{n-1} + \ldots + a_n = 0$

  singular at some point over $x_i \in D \subset \Sigma$

- suppose ordinary double point $(x, z) = (0, 0)$

  local form $x^2 = z^2$

- fibre in $\mathcal{M}$ over $S \in \mathcal{B} \sim$ rank one torsion-free sheaves

- locally free at singularity

$$\Rightarrow x(f_0(z) + xf_1(z)) = z^2 f_1(z) + xf_0(z)$$

$$\Rightarrow \text{Higgs field} \quad \Phi = \begin{pmatrix} 0 & z^2 \\ 1 & 0 \end{pmatrix}$$

- regular nilpotent at $z = 0$

- non-locally free at singularity

- $\Rightarrow$ direct image of locally free on normalization

- normalize $x^2 = z^2$ to $\{x = z\} \cup \{x = -z\}$

- $\Rightarrow$ Higgs field $\Phi = \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix}$

....vanishes at $z = 0$ and $\Phi'(0)$ semisimple

- if S has d nodes, then d functions h_i are critical

- symplectic vector space $L^{\perp}/L \cong \bigoplus_{i=1}^{d} L_i$

- $L_i \cong$ tangent space of coadjoint orbit of semisimple $a_i \in \mathfrak{sl}(2)$

$\Rightarrow$ nondegenerate critical locus

- $x^m = z^m$ normalize to

  $$\{x = z\} \cup \{x = \omega z\} \cup \cdots \cup \{x = \omega^{m-1} z\}$$

- $$\Phi \sim z \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \omega & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \omega^{m-1} \end{pmatrix}$$

- nondegeneracy $\Rightarrow$ integrable system

- base $\mathcal{B}_d$ = spectral curves with d nodes (Severi variety)

- nondegeneracy $\Rightarrow$ integrable system

- base $\mathcal{B}_d$ = spectral curves with d nodes (Severi variety)

- fibre $\cong$ Jac(S'), S' = normalization of S

  $p : S' \to S$, p_*L = torsion-free sheaf on S

- tangent space to $\mathcal{B}_d = H^0(S, \pi^* K^n \otimes \mathcal{I}) \cong H^0(S', K_{S'})$

$$H^0(S', K_{S'})^* \cong H^1(S', \mathcal{O})$$

REMARK

- $f : \mathcal{B} \to \mathbf{C}$ linear

  $= (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in H^1(\Sigma, \mathcal{O}) \oplus H^1(\Sigma, K^{-1}) \oplus \cdots$

- $x \in H^0(K, \pi^* K)$ $\alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1} \in H^1(K, \mathcal{O})$

- principal $\mathbf{C}$-bundle over K = Calabi-Yau 3-manifold Z

- normalization $S' \subset Z$

  $\Rightarrow$ critical points $\sim$ sheaves on a Calabi-Yau

SPECIAL KÄHLER METRIC ON $\mathcal{B}_d$

- subintegrable system $\mathcal{C}_d \subset \mathcal{M}$, $\mathbf{C}^*$-invariant

- restrict $i_X\omega \Rightarrow$ infinitesimal deformation of Jac$(S') \subset \mathcal{C}_d$

- $H^0(\text{Jac}(S'), \Omega^1) \cong H^0(S', K_{S'})$, $i_X\omega \sim p^*\theta$

  $p : S' \to S$

- subintegrable system $\mathcal{C}_d \subset \mathcal{M}$, $\mathbf{C}^*$-invariant

- restrict $i_X\omega \Rightarrow$ infinitesimal deformation of $\mathrm{Jac}(S') \subset \mathcal{C}_d$

- $H^0(\mathrm{Jac}(S'), \Omega^1) \cong H^0(S', K_{S'})$, $i_X\omega \sim p^*\theta$

  $p : S' \to S$

- Kähler potential $=$ $\mathrm{Im}\,\frac{1}{4}\int_{S'} p^*(\theta \wedge \bar{\theta}) = \mathrm{Im}\,\frac{1}{4}\int_S \theta \wedge \bar{\theta}$

- $\mathcal{B}_d \subset \mathcal{B}$

- complex structure on $\mathcal{B}^{\text{reg}}$ extends to $\mathcal{B}_d$

- Kähler potential extends

- flat connection has a logarithmic pole

  $\Rightarrow$ singular in normal direction, induces a flat connection along $\mathcal{B}_d$

EXAMPLE

- $SL(2,\mathbf{C})$ Higgs bundles $\mathrm{tr}\,\Phi = 0, \Lambda^2 V \cong \mathcal{O}$

- $\mathcal{B} = H^0(\Sigma, K^2)$ spectral curve $x^2 = q(z)$

- fibre = Prym variety $\mathrm{P}(S, \Sigma)$

- $\theta = x d\pi = \sqrt{q}$ $\qquad$ $\mathrm{Im} \int_S \theta \wedge \bar{\theta} = \int_\Sigma \sqrt{q\bar{q}}$

- Σ genus 2, $y^2 = f(z) = (z - z_1) \dots (z - z_6)$

- quadratic differential $\qquad q = (c_0 + c_1 z + c_2 z^2)\dfrac{dz^2}{f(z)}$

- one node: $\qquad q = (a_0 + a_1 z)(z - z_i)\dfrac{dz^2}{f(z)}$

  $\mathcal{B}_1 = 6$ two-dimensional subspaces in $\mathbf{C}^3$

- two nodes: $\qquad q = a(z - z_i)(z - z_j)\dfrac{dz^2}{f(z)}$

  $\mathcal{B}_2 = 15$ one-dimensional subspaces in $\mathbf{C}^3$

- $$\int_S \theta \wedge \bar{\theta} = \int_{\mathbf{C}} \frac{|c_0 + c_1 z + c_2 z^2|}{|f(z)|} dz d\bar{z}$$

- two nodes: $$\int_S \theta \wedge \bar{\theta} = |a| \int_{\mathbf{C}} \frac{|(z - z_i)(z - z_j)|}{|f(z)|} dz d\bar{z}$$

- $K = k|a|$

  $a = w^2 \Rightarrow$ flat metric on $a \neq 0$, singular at origin

  $$\nabla(dw) = 0 \Rightarrow \nabla = \frac{d}{da} + \frac{1}{2a}$$

MORE GENERALLY...

- suppose $d = 2g - 2$
- $\deg K^2 = 4g - 4$, $d = 2g - 2 \Rightarrow q = s^2$ for $s \in H^0(\Sigma, KU)$ where U^2 is trivial
- $\mathcal{B}_{2g-2} = H^0(\Sigma, KU)/\pm 1$, $\dim H^0(\Sigma, KU) = g - 1$
- Kähler potential $\sim \int_\Sigma s\bar{s}$

$\Rightarrow$ flat metric

- $U \in H^1(\Sigma, \mathbf{Z}_2)$ acts on $\mathcal{M}$ for $SL(2, \mathbf{C})$

- $(V, \Phi) \mapsto (V \otimes U, \Phi)$

- fixed point set $= \mathcal{C}_{2g-2} =$ hyperkähler submanifold

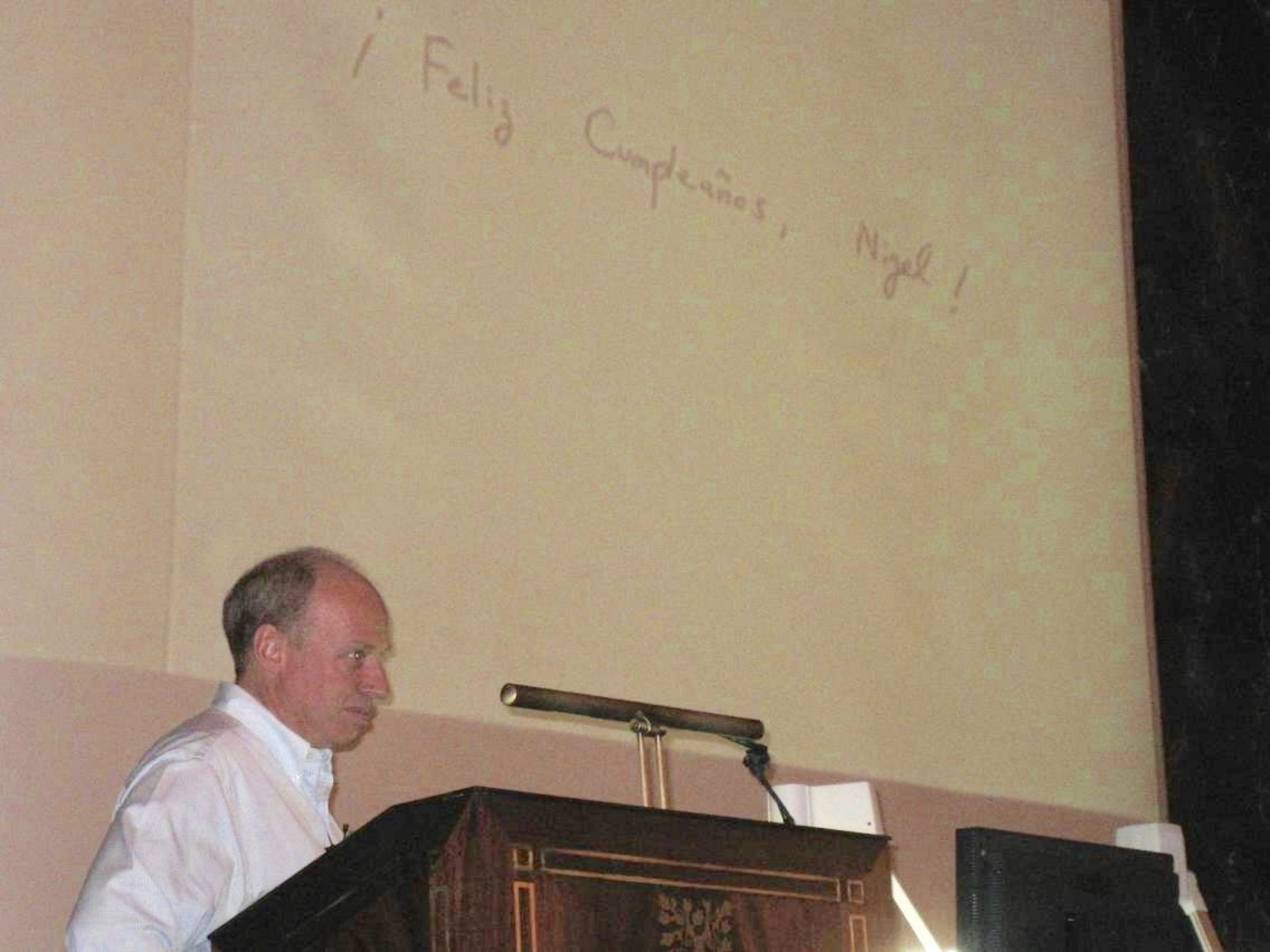
- $p : S' \to \Sigma$ unramified $\Rightarrow$

  L^2 hyperkähler metric = flat metric = semiflat metric

D.Gaiotto, G.Moore & A. Neitzke, *Wall crossing, Hitchin systems and the WKB approximation*, Adv. Math. **234** (2013) 239–403.

R.Mazzeo, J.Swoboda, H.Weiss, F.Witt, *Asymptotic geometry of the Hitchin metric*, arXiv:1709.03433

L.Fredrickson, *Exponential decay for the asymptotic geometry of the Hitchin metric*, Int. J. Mod. Phys. arXiv:1810.01554


¡ Feliz Cumpleaños, Nigel !

Feliz cumpleaños, Dan!