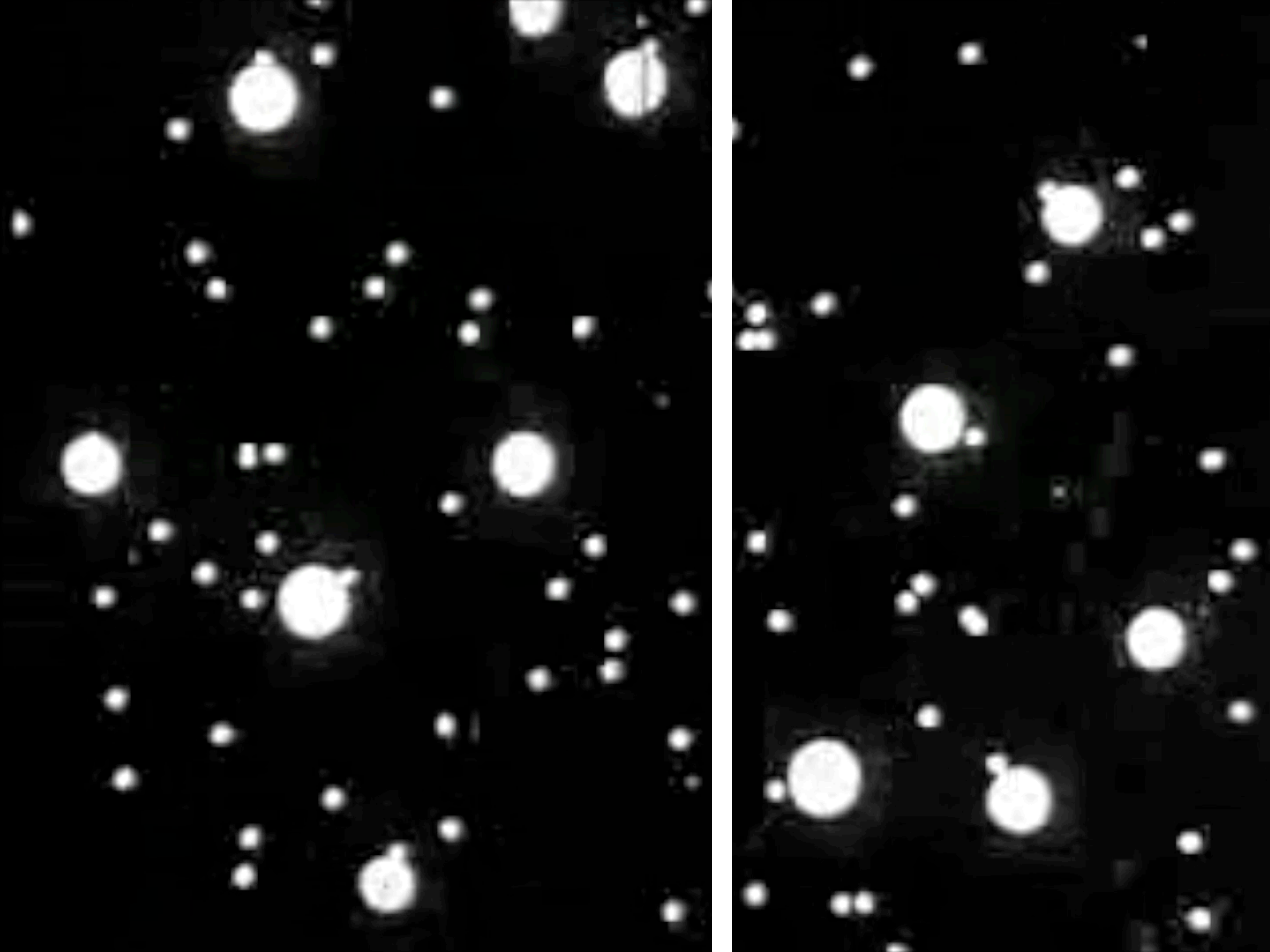
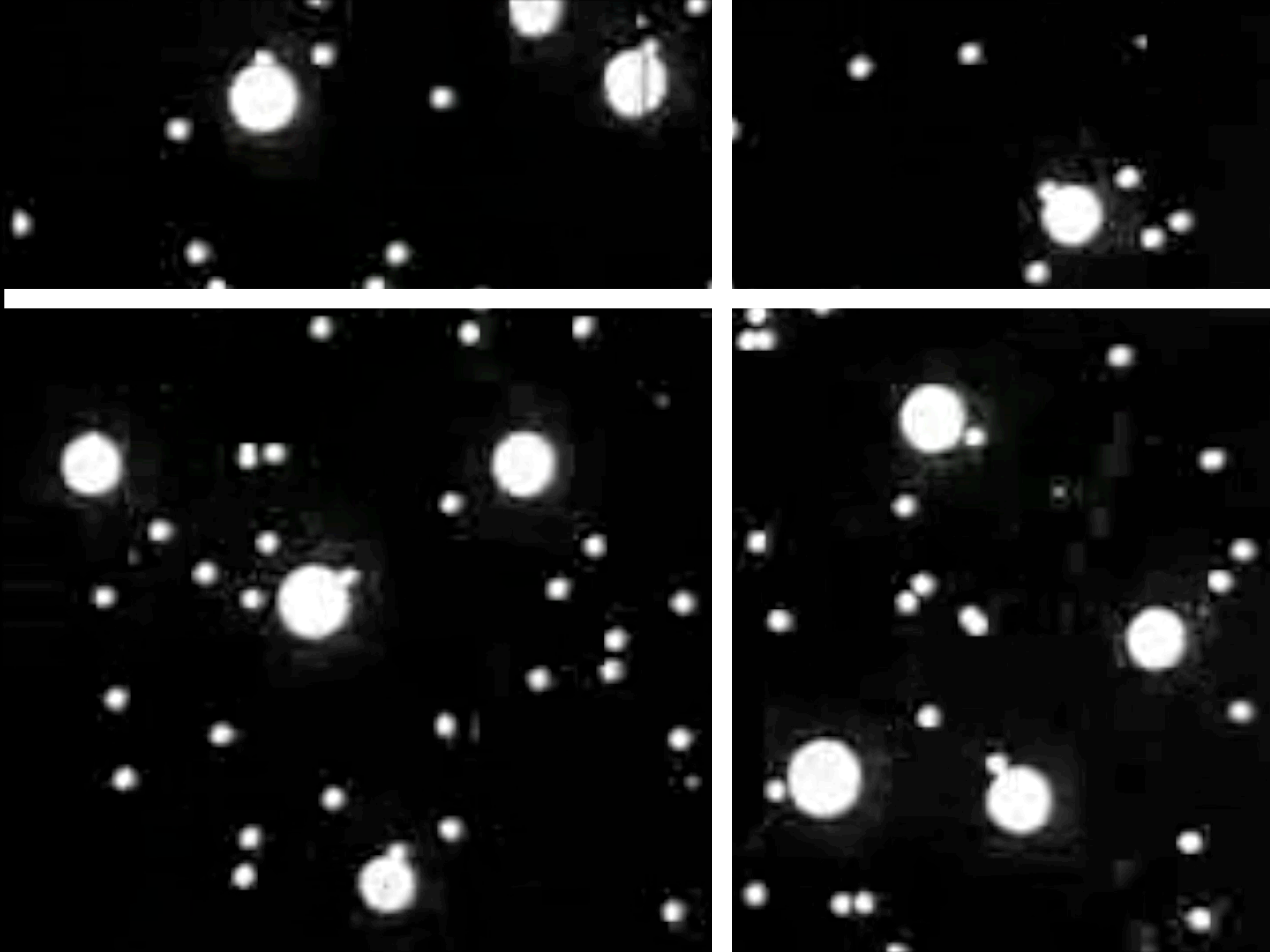
Moduli spaces of field theories and condensed matter physics

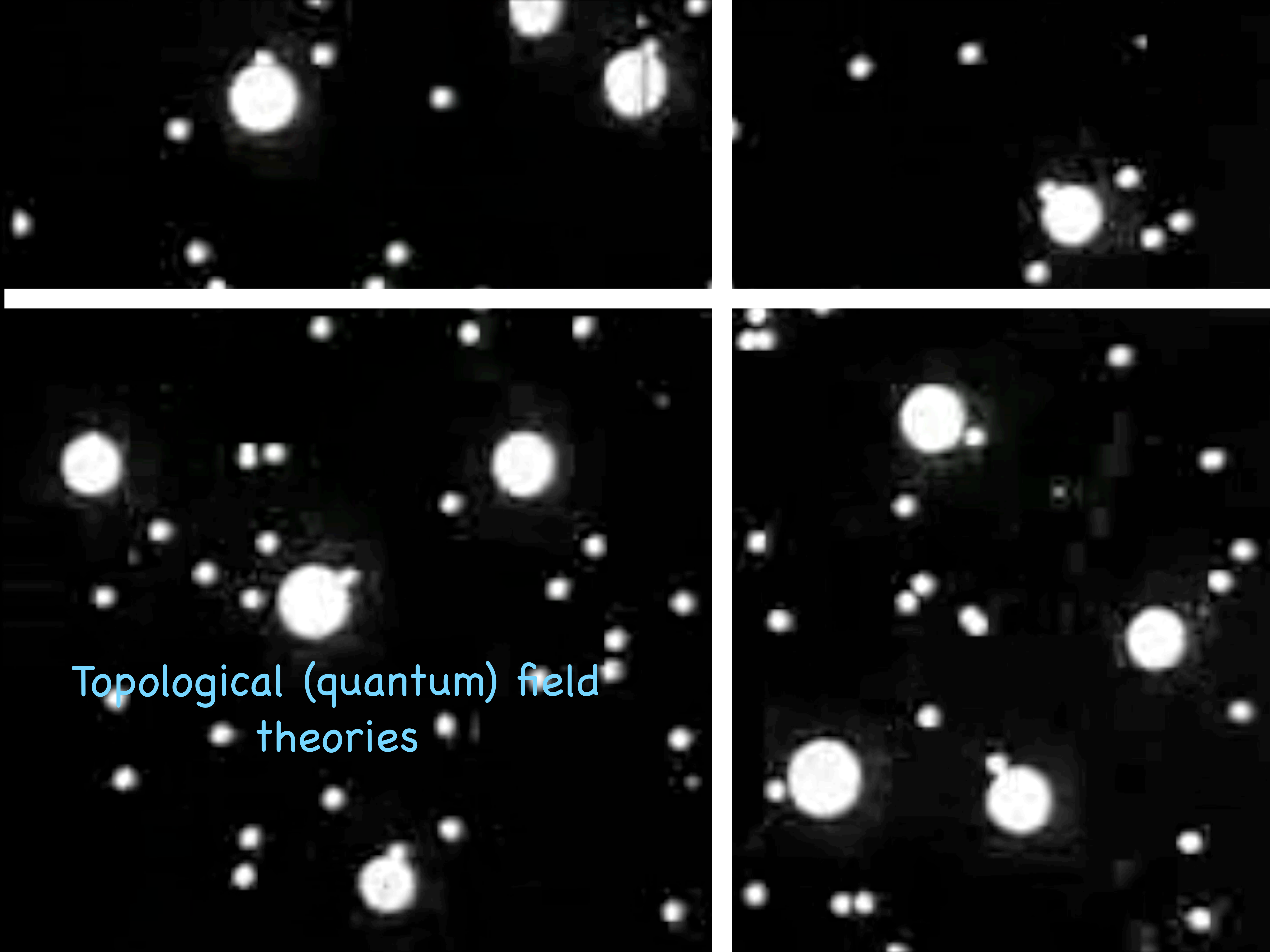
joint with Dan Freed

Reflection positivity and invertible topological phases

arXiv:1604.06527






Topological (quantum) field
theories

Symmetry protected topological phases

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Physical situation: zero-temperature quantum-mechanical states of matter that have a fixed (internal) symmetry and a finite energy gap.

Symmetry protected topological phases

Physical situation: zero-temperature quantum-mechanical states of matter that have a fixed (internal) symmetry and a finite energy gap.

Principle: the deformation class of a quantum system is determined by its low energy behavior.

Principle: the low energy physics of a gapped system is classified by a topological field theory.

Topological quantum field theories

Topological quantum field theories

(Atiyah, Segal, Witten)

An n-dimensional topological quantum field theory is...

$Z(M^n)$	complex number
$Z(M^{n-1})$	complex vector space
Z(cobordism)	linear transformation
Z(disjoint union)	tensor product

Fully extended topological quantum field theories

(...,Freed, ...)

A (fully extended) n-dimensional topological quantum field theory is a symmetric monoidal functor

$$Z : \mathrm{Bord}_n^X \to \mathcal{C}$$

Bord_n^X some symmetric monoidal infinity n-category of manifolds

$\mathcal{C}$ some symmetric monoidal infinity n-category of "values"

Topological quantum field theories

Fix a space X equipped with a vector bundle V of dimension n

Bord_n^X : The (∞, n)-category whose k-morphisms are k-dimensional manifolds M with corners, equipped with a map

$$f : M \to X$$

and an isomorphism

$$TM + R^{n-k} \approx f^* V$$

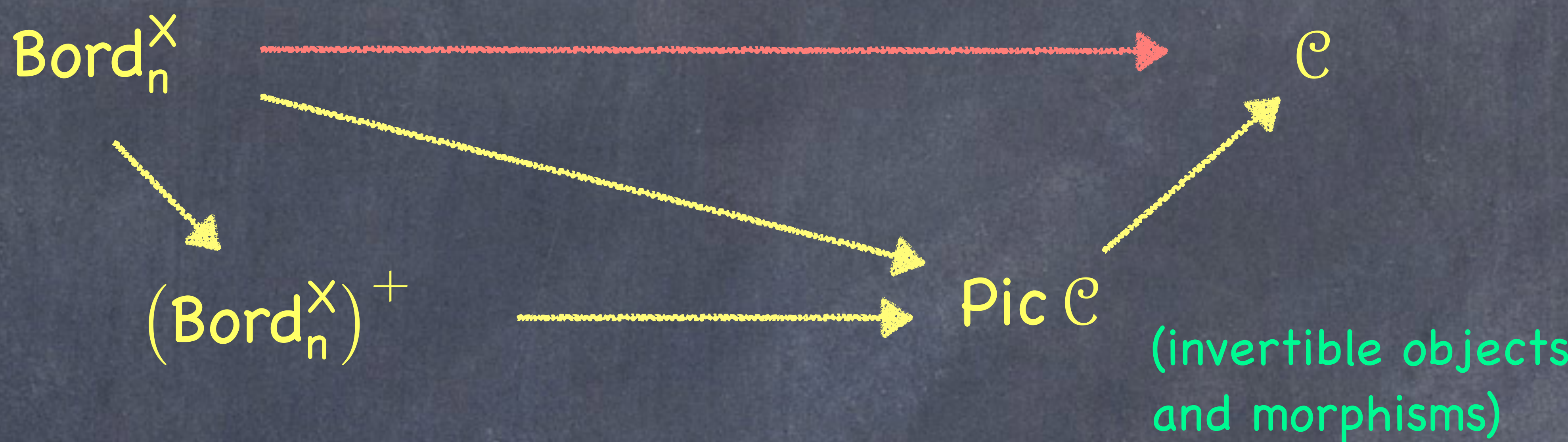
$\otimes$: disjoint union

Invertible theories

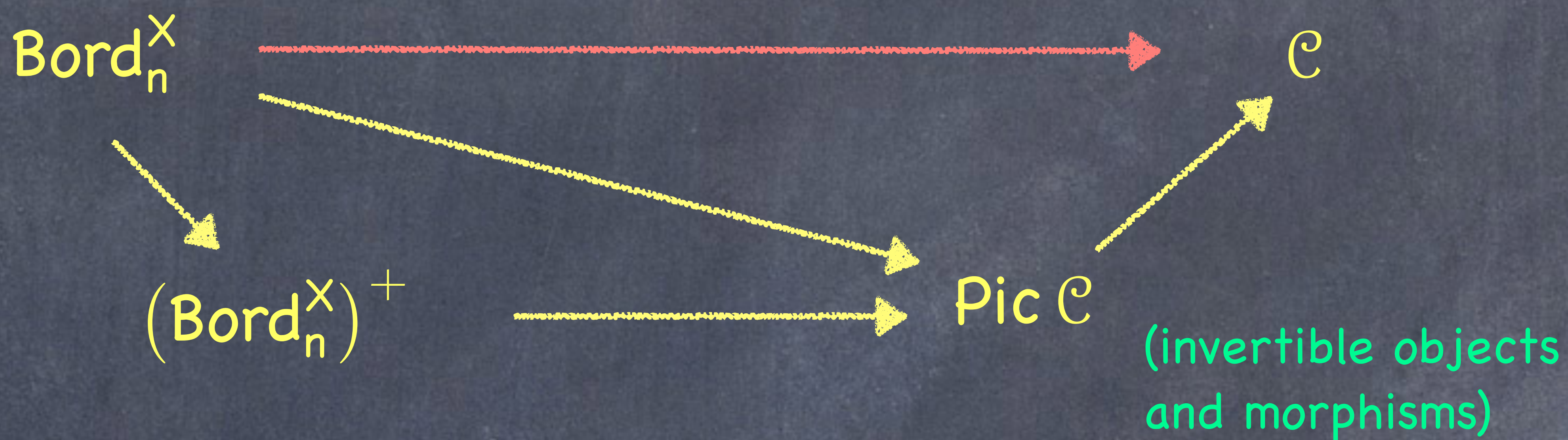
$$\mathrm{Bord}_n^X \longrightarrow \mathcal{C}$$

Invertible theories

Invertible theories

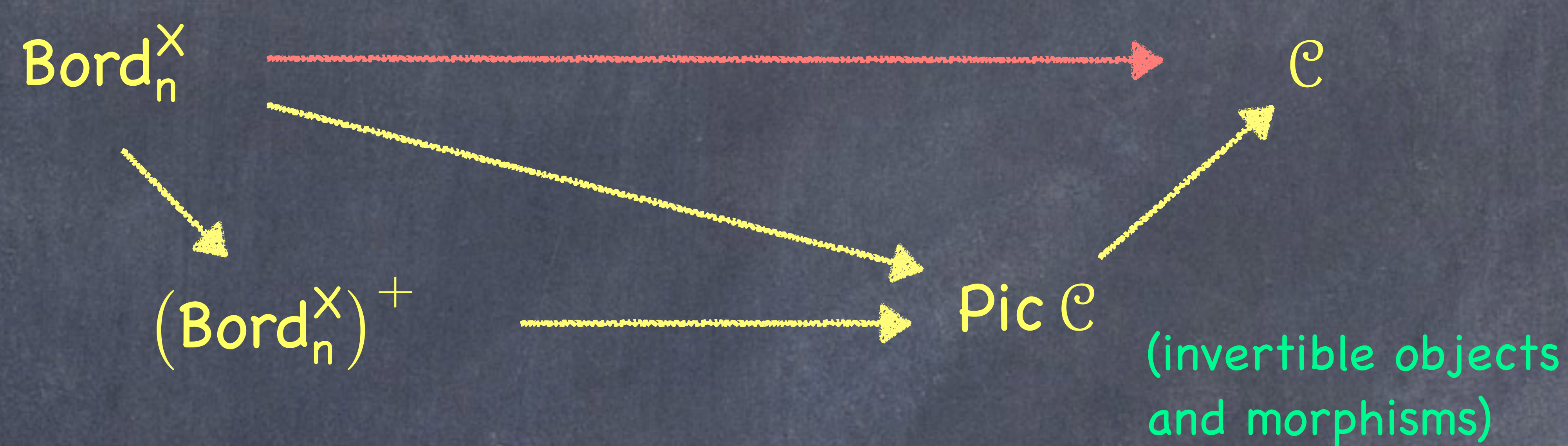


Invertible theories



Theorem (GMTW): $(\mathrm{Bord}_n^X)^+ = \Omega^\infty(\mathrm{Thom}(X; \mathbb{R}^n - V))$

Invertible theories



Theorem (GMTW): $(\mathrm{Bord}_n^X)^+ = \Omega^\infty(\mathrm{Thom}(X; \mathbb{R}^n - V))$

$\mathrm{Pic}\,\mathcal{C}$: some other spectrum (infinite loop space)

The basic model

The basic model

A map of spectra

$$\mathrm{Thom}(X; R^n - V) = S^n \wedge \mathrm{Thom}(X; -V) \to E$$

for some spectrum E.

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Madsen-Tillmann spectrum

General classification

Note: The space of maps

$$\mathrm{Thom}(X; R^n - V) \to E$$

is the space of sections of the associated bundle

$$\Omega^\infty E \to \Omega^{\infty + R^n - V} E \to X$$

Theorem (GMTW) The space of invertible field theories

$$\mathrm{Bord}_n^X \longrightarrow \mathcal{C}$$

is the space of sections of an associated bundle over X with fiber $\mathrm{Pic}\,\mathcal{C}$

General classification

Theorem (Lurie) The space of field theories

$$\mathrm{Bord}_n^X \longrightarrow \mathcal{C}$$

is the space of sections of an associated bundle over X with fiber the space $\mathcal{C}^{fd}$ of fully dualizable objects of $\mathcal{C}$.

General classification

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Baez-Dolan cobordism hypothesis

Field theory model

So far our field theory model of a "material" is just an arbitrary map of spectra (in the invertible case)

$$\mathcal{B} \to \mathcal{C}$$

The value category

The value category

$$\mathcal{B} \to \boxed{\mathcal{C}}$$

The value category

Let $\mathcal{C}_n$ be the "value" category of physical n-dimensional topological field theories.

$\mathcal{C}_0 = \mathbb{C}$ (complex numbers)

$\mathcal{C}_{n-1} = \Omega\mathcal{C}_n := \mathcal{C}_n(1, 1)$

The value category

Let $\mathcal{C}_n$ be the "value" category of physical n-dimensional topological field theories.

$$\mathcal{C}_0 = \mathbb{C} \quad \text{(complex numbers)}$$

$$\mathcal{C}_{n-1} = \Omega\mathcal{C}_n := \mathcal{C}_n(1, 1)$$

Examples:

$\mathcal{C}_1$ has something to do with the category Vect of complex vector spaces

$\mathcal{C}_2$ has something to do with the Morita category Alg of algebras over the complex numbers.

The value category (invertible case)

Since $\mathcal{C}_{n-1} = \Omega\mathcal{C}_n := \mathcal{C}_n(1,1)$ one has

$$\mathrm{Pic}\,\mathcal{C}_n = \Omega\,\mathrm{Pic}\,\mathcal{C}_{n+1}$$

and the spaces $\mathrm{Pic}\,\mathcal{C}_n$ form a spectrum $\mathrm{Pic}\,\mathcal{C}$:

$$\mathrm{Pic}\,\mathcal{C} = \{\mathrm{Pic}\,\mathcal{C}_n\}$$

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$$\mathrm{Pic}\,\mathcal{C} = \{\mathrm{Pic}\,\mathcal{C}_n\}$$

Since $\mathcal{C}_0 = \mathbb{C}$ one has

$$\pi_0\,\mathrm{Pic}\,\mathcal{C} = \mathbb{C}^\times$$

$$\pi_k\,\mathrm{Pic}\,\mathcal{C} = 0 \qquad k>0$$

The value category (invertible case)

$\pi_0 \, \mathrm{Pic}\, \mathcal{C} = C^\times$
$\pi_k \, \mathrm{Pic}\, \mathcal{C} = 0 \quad k>0$

There is a universal spectrum $IC^\times$ with this property

$$[E, IC^\times] = \hom(\pi_0 E, C^\times)$$

$$\pi_0 IC^\times = C^\times$$

$$\pi_n IC^\times = 0$$

$$\pi_{-n} IC^\times = \hom(\pi_n S^0, C^\times)$$

Brown-Comenetz dual of the sphere

The value category (invertible case)

Ansatz: $\mathrm{Pic}\,\mathcal{C}_n = (\mathbb{C}^\times)_n$

The value category (invertible case)

Ansatz: $\text{Pic}\,\mathcal{C}_n = (IC^\times)_n$

The universal property

$$[E, IC^\times] = \text{hom}(\pi_0 E, C^\times)$$

corresponds to the idea that the partition function determines the field theory.

The value category (invertible case)

Ansatz: $\mathrm{Pic}\,\mathcal{C}_n = (IC^\times)_n$

Reality check: $(IC^\times)_0 = C^\times$

$(IC^\times)_1 = \mathrm{Pic}(\mathrm{sVect})$

$(IC^\times)_2 = \mathrm{Pic}(\mathrm{sAlg})$

The value category (invertible case)

Ansatz: $\mathrm{Pic}\,\mathcal{C}_n = (\mathrm{I}C^\times)_n$

Reality check: $(\mathrm{I}C^\times)_0 = C^\times$

$(\mathrm{I}C^\times)_1 = \mathrm{Pic}(\mathrm{sVect})$

$(\mathrm{I}C^\times)_2 = \mathrm{Pic}(\mathrm{sAlg})$

suggests $\mathcal{C}_1 = \mathrm{sVect}$ compatible with "statistics"

$\mathcal{C}_2 = \mathrm{sAlg}$

The value category (invertible case)

To classify "phases" (deformations) we want to regard R as contractible

$$IZ(1) \to IC \to IC^{\times}$$

$$Z(1) = 2\pi i\, Z$$

$$IC \to IC^{\times} \to S^1 \wedge IZ(1)$$

Field theory model

Now our field theory model of a "material" is a map of spectra

$$\mathcal{B} \to \Sigma^n I\mathbb{C}^\times$$

or

$$\mathcal{B} \to \Sigma^{n+1} I\mathbb{Z}(1)$$

But the domain is still pretty arbitrary. More input is needed from the physical situation being modeled.

The bordism categories

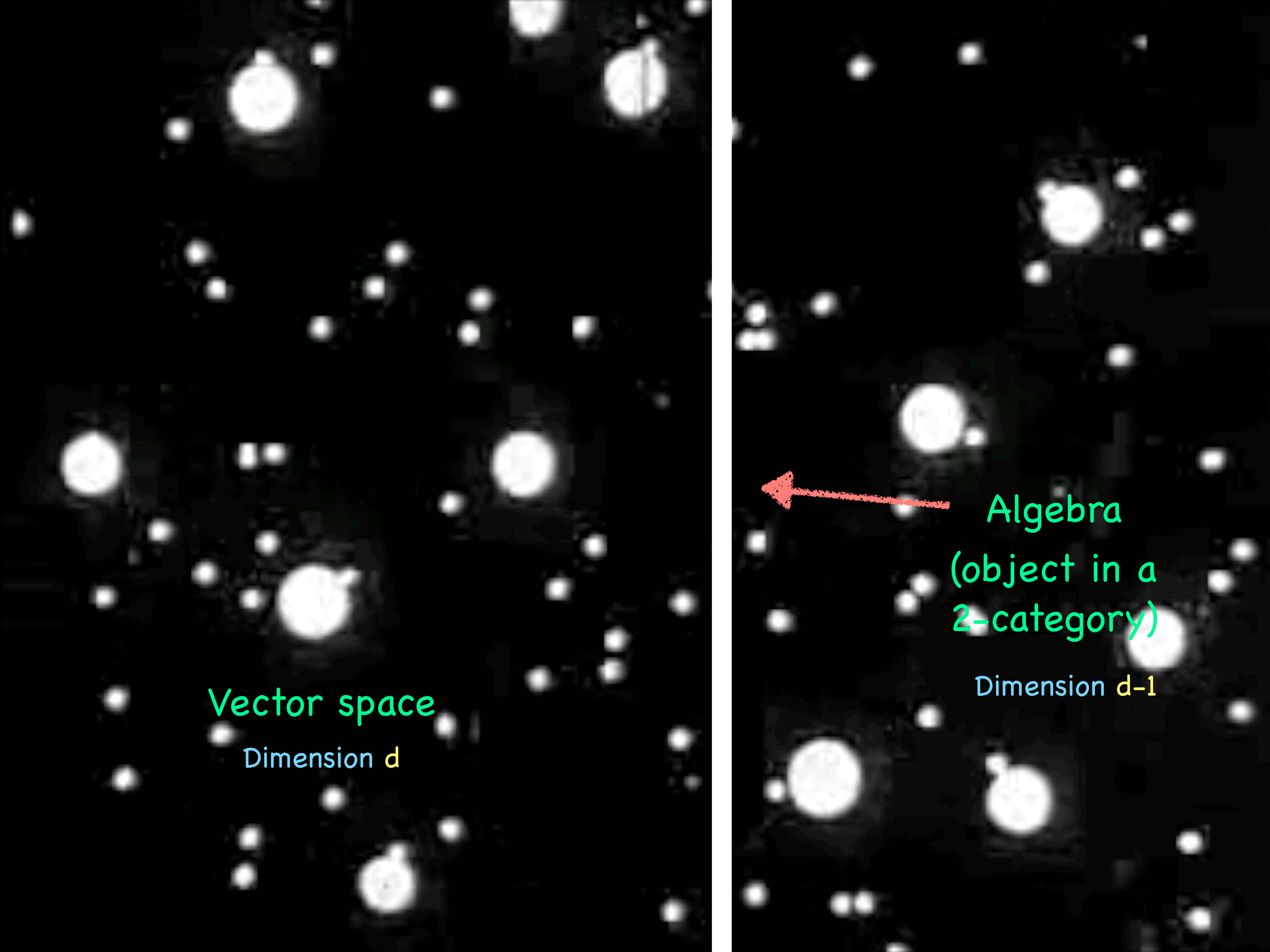
The bordism categories

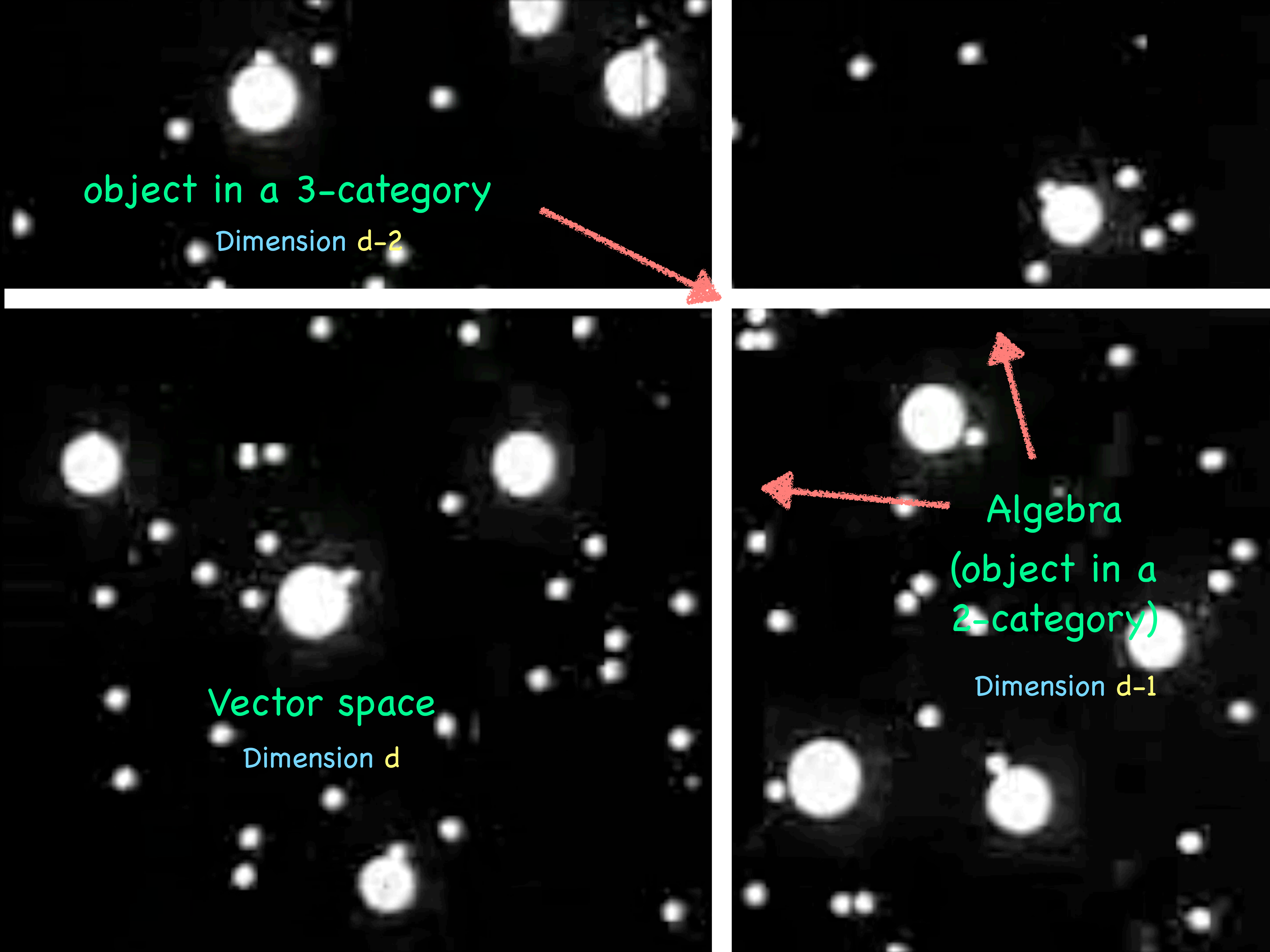
$$\boxed{\mathcal{B}} \longrightarrow \mathcal{C}$$


Dimension d


Dimension d


Vector space
Dimension d


Vector space
Dimension d
Algebra
(object in a
2-category)
Dimension d-1


object in a 3-category
Dimension d-2
Algebra
(object in a
2-category)
Dimension d-1
Vector space
Dimension d

Field theory model

Ansatz: The emergent field theory is a (d+1)-dimensional relativistic invariant field theory.

After Wick rotation it becomes (d+1)-dimensional topological quantum field theory for Spin manifolds.

So n=(d+1) and the model has something to do with maps

$$S^n MTSpin_n \to S^{n+1} I\mathbb{Z}(1)$$

Field theory model

In condensed matter physics there is an "internal" symmetry group I specified. It acts trivially on space, and comes equipped with a homomorphism

$$\phi : I \to \{\pm 1\}$$

specifying how I implements time reversal, and a homomorphism

$$k_0 : \{\pm 1\} \to K = \ker \phi$$

indicating the presence (or absence) of fermions.

Field theory model

Associated to this data is a family of group homomorphisms

$$H_n \to O_n$$

wtih kernel K, and fitting into homotopy pullback squares

$$\begin{array}{ccc} BH_{n-1} & \longrightarrow & BH_n \\ \downarrow{\scriptstyle \rho_{n-1}} & & \downarrow{\scriptstyle \rho_n} \\ BO_{n-1} & \longrightarrow & BO_n \end{array}$$

The bordism categories

Write

$$\mathsf{MTH}_n = \mathsf{Thom}(\mathsf{BH}_n; -\rho_n)$$

$$\mathsf{BH} = \lim \mathsf{BH}_n$$

$$\mathsf{MTH} = \lim \mathsf{S}^n\, \mathsf{MTH}_n$$

$$\mathsf{MTH} = \mathsf{Thom}(\mathsf{BH}; -\rho)$$

So n=(d+1) and, incorporating the (internal) symmetry I, the model has something to do with maps

$$\mathsf{S}^n \mathsf{MTH}_n \to \mathsf{S}^{n+1} I\mathbb{Z}(1)$$

The symmetry groups

s	H^c	I		Cartan
0	Spin^c	$\mathbb{T}$	(Spin^c_1)	A
1	Pin^c	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{T}$	(Pin^c_1)	AIII

s	H	I		Cartan
0	Spin	$\{\pm 1\}$	(Spin_1)	D
-1	Pin^+	$\mathbb{Z}/2\mathbb{Z} \times \{\pm 1\}$	(Pin^+_1)	DIII
-2	$\mathrm{Pin}^+ \ltimes_{\{\pm 1\}} \mathbb{T}$	$\mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{T}$	(Pin^+_2)	AII
-3	$\mathrm{Pin}^- \times_{\{\pm 1\}} SU_2$	$\mathbb{Z}/4\mathbb{Z} \times_{\{\pm 1\}} SU_2$	(Pin^+_3)	CII
4	$\mathrm{Spin} \times_{\{\pm 1\}} SU_2$	SU_2	(Spin_3)	C
3	$\mathrm{Pin}^+ \times_{\{\pm 1\}} SU_2$	$\mathbb{Z}/2\mathbb{Z} \times SU_2$	(Pin^-_3)	CI
2	$\mathrm{Pin}^- \ltimes_{\{\pm 1\}} \mathbb{T}$	$\mathbb{Z}/4\mathbb{Z} \ltimes_{\{\pm 1\}} \mathbb{T}$	(Pin^-_2)	AI
1	Pin^-	$\mathbb{Z}/4\mathbb{Z}$	(Pin^-_1)	BDI

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Model: classification requires a model incorporating the symmetry group, and expressions of locality and unitarity of time evolution.

Unitarity

(reflection positivity)

Reflection positivity

Wick rotated unitarity = reflection positivity

Reflection positivity

Wick rotated unitarity = reflection positivity

$M = M^{n-1}$ (spacelike slice)

$\overline{M} = M$ with time reversed

given $f : M \to X$ then $\overline{M}$ is M, equipped with

$$TM + R \xrightarrow{\text{Id}+(-1)} TM + R \approx f^* V_n$$

Reflection positivity

Wick rotated unitarity = reflection positivity

$M = M^{n-1}$ (spacelike slice)

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given $f : M \to X$ then $\overline{M}$ is M, equipped with

$$TM + \mathbb{R} \xrightarrow{\text{Id}+(-1)} TM + \mathbb{R} \approx f^* V_n$$

Reflection: $Z(\overline{M})$ is the complex conjugate of $Z(M)$

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$$Z(\overline{M}) = \overline{(Z(M))}$$

Duality: Since

$$\partial(M \times [0,1]) = M \amalg \overline{M}$$

the space $Z(\overline{M})$ is the dual of $Z(M)$, so $Z(M)$ acquires a Hermitian inner product:

$$Z(M) \otimes \overline{(Z(M))} = Z(M) \otimes Z(\overline{M}) \xrightarrow{Z(M \times [0,1])} \mathbb{C}$$

Reflection positivity

Wick rotated unitarity = reflection positivity

$$Z(M) \otimes \overline{(Z(M))} = Z(M) \otimes Z(\overline{M}) \xrightarrow{Z(M \times [0,1])} C$$

Positivity: This Hermitian inner product is positive definite.

Doubles

Suppose that $M = \partial N$. Then N gives a vector

$$Z(N) \in Z(M)$$

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In a reflection positive theory, we must have

$$\langle Z(N), Z(N) \rangle > 0$$

This number can be computed as

$$Z(\Delta)$$

in which

$$\Delta = N \bigcup_M N$$

Reflection

$$S^n MTH_n \longrightarrow S^{n+1} IZ(1)$$

Reflection

There are natural Z/2 actions on

$$S^n MTH_n \longrightarrow S^{n+1} IZ(1)$$

corresponding to reflection and complex conjugation.

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Model: The space of "reflection theories" is the space of Z/2 equivariant maps

$$S^n MTH_n \longrightarrow S^{n+1} IZ(1)$$

Complex Conjugation

Conjugation

There are many ways of extending the action of complex conjugation on $\mathbb{C}^\times$ to $\mathbb{I}\mathbb{C}^\times$

$$\mathbb{I}\mathbb{Z}(1) \to \mathbb{I}\mathbb{C} \to \mathbb{I}\mathbb{C}^\times$$

Conjugation

There are many ways of extending the action of complex conjugation on $C^\times$ to $\mathbb{C}^\times$

$$\mathbb{Z}(1) \to \mathbb{C} \to \mathbb{C}^\times$$

many choices here

no problem here

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We begin by modeling duality.

Duality

The dual of V is uniquely determined by V

$$V \otimes DV \to 1 \qquad \ldots$$

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The fiber of

$$E \vee E \to E$$

is the set of pairs (V,W) equipped with an isomorphism

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The fiber of

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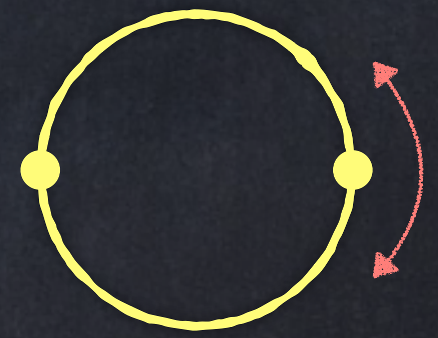
So it is equivalent to E, and the Z/2-action corresponds to the duality involution.

Duality

The map $E \vee E \to E$ is the smash product of the identity map of E with $Z/2_+ \to S^0$.

Cofibration sequence

$$Z/2_+ \to S^0 \to S^\sigma$$



S^σ

The duality involution is $E \wedge S^{\sigma-1}$.

Conjugation

The duality involution is $E \wedge S^{\sigma-1}$.

On the groupoid of complex vector spaces, the action of "conjugate dual" is homotopically trivial: U_n is homotopy equivalent to $GL_n(C)$.

Ansatz: The conjugation action on $IZ(1)$ is given by

$$IZ(1) \wedge S^{1-\sigma}$$

Conjugation

This checks out in the three cases we can interpret:

$$((IC^{\times})^{hZ/2})_0 = R^{\times}$$

$$((IC^{\times})^{hZ/2})_1 = \mathrm{Pic}(\mathrm{sVect}_R)$$

$$((IC^{\times})^{hZ/2})_2 = \mathrm{Pic}(\mathrm{sAlg}_R)$$

Reflection

Definition: The space of "reflection theories" is the space of Z/2 equivariant maps

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A reflection theory associates to each manifold an object equipped with a Hermitian inner product: a "higher Hermitian line."

Question: What does it mean for a higher Hermitian line to be "positive definite?"

Positivity

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It is a choice of deformation from a certain (n-1)-dimensional topological field theory to the trivial theory.

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It is a choice of deformation from a certain (n-1)-dimensional topological field theory to the trivial theory.

Definition: A reflection positive map

$$S^n MTH_n \to S^{n+1} I\mathbb{Z}(1)$$

is a reflection map, equipped with a positivity structure.

Reflection positivity

Theorem: The space of reflection positive maps

$$S^n MTH_n \to S^n I\mathbb{C}^\times$$

$$(\text{resp. } S^n MTH_n \to S^{n+1} I\mathbb{Z}(1))$$

is naturally weakly equivalent to the space of maps

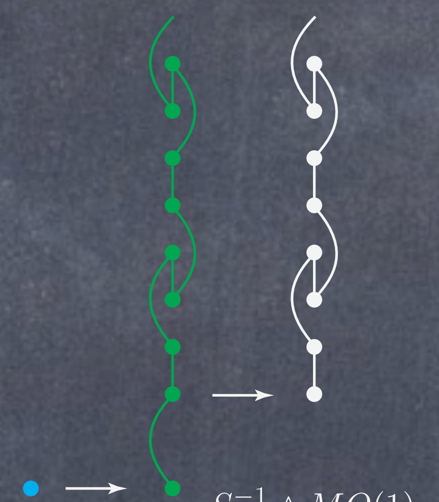
$$MTH \to S^n I\mathbb{C}^\times$$

$$(\text{resp. } MTH \to S^{n+1} I\mathbb{Z}(1))$$

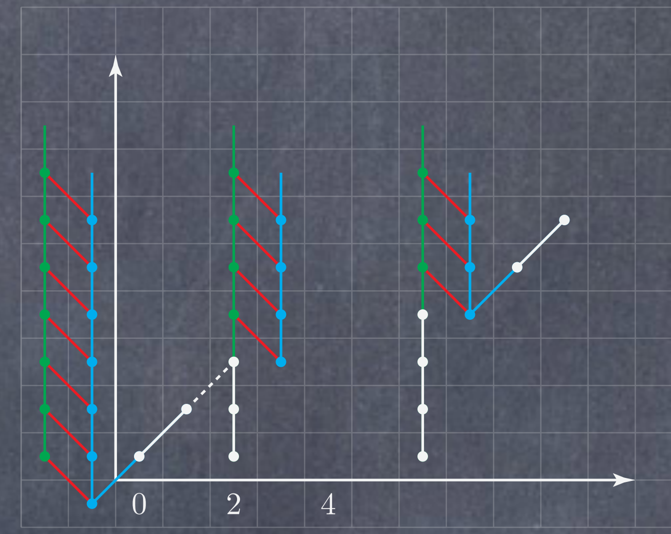
Computations

$MT\,\mathrm{Pin}^-$

$s = 1$



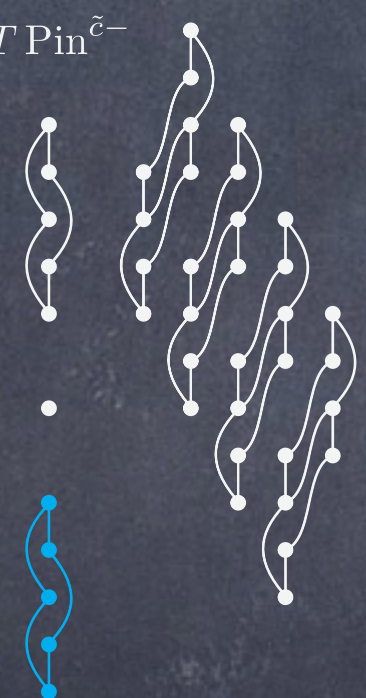
$S^{-1} \wedge MO(1)$



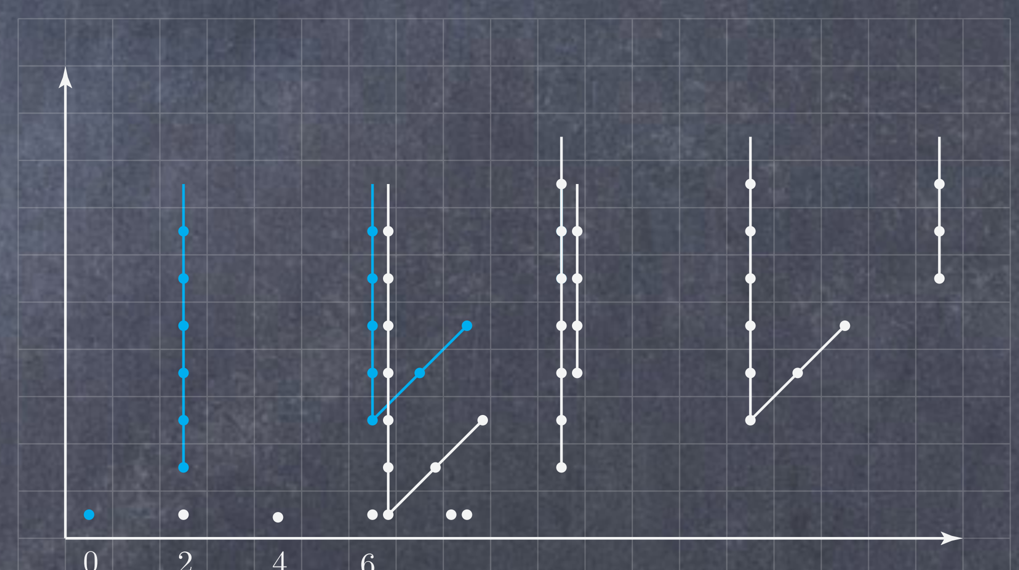
$\mathrm{Ext}_{A_1}(H^* S^{-1} \wedge MO(1), \mathbb{Z}/2)$

$MT\,\mathrm{Pin}^{\tilde{c}-}$

$s = 2$

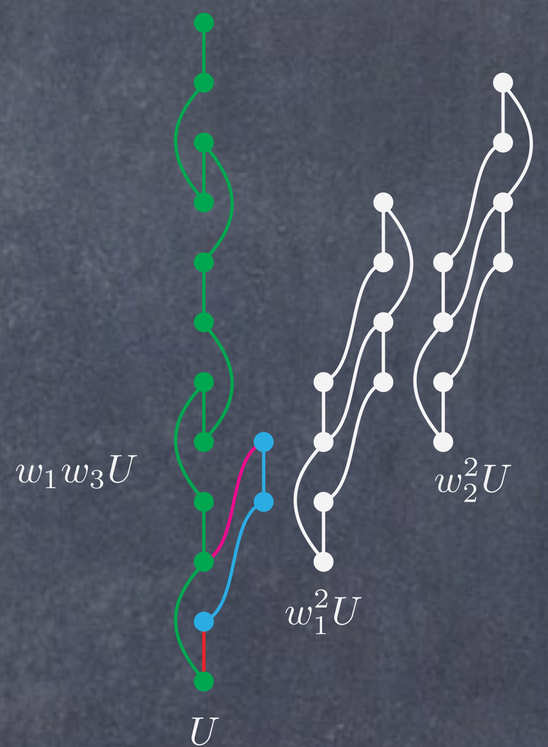


$S^2 \wedge MTO_2$

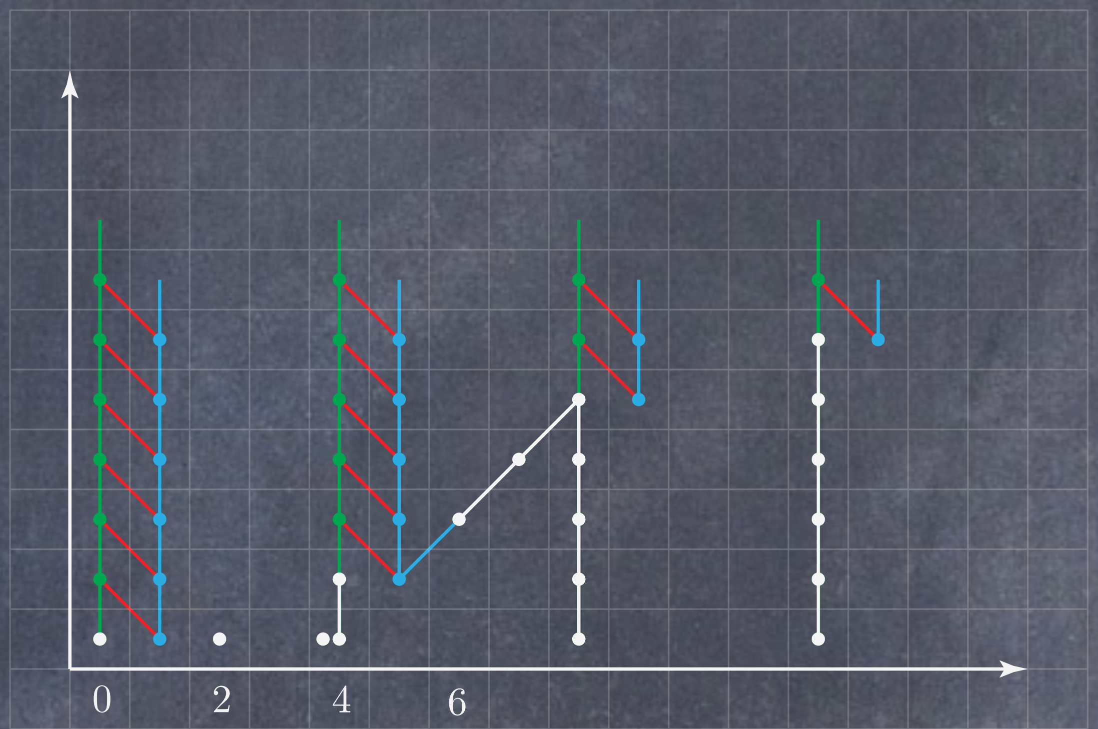


MTG^{+}

$s = 3$



$S^{-3} \wedge MO_3$



Classification of interacting electronic topological insulators in three dimensions

Chong Wang, Andrew C. Potter and T. Senthil
Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
(Dated: November 14, 2013)

The SPT states of neutral bosons with time-reversal symmetry are classified[10,22,23] by $\mathbb{Z}_2^2$, with two fundamental root non-trivial phases. These can both be understood as Mott insulators in topological paramagnet phases. Adding to this the usual $\theta = \pi$ TI captured by band theory we have 3 root states corresponding to a $\mathbb{Z}_2^3$ classification. To establish that there are no other states we need to still consider the other possibility left open for the bulk response: a fermionic monopole.

n	$\ker\Phi \longrightarrow$	$FF_n(\mathrm{Pin}^{\tilde{c}+}) \xrightarrow{\Phi}$	$TP_n(\mathrm{Pin}^{\tilde{c}+}) \longrightarrow$	$\mathrm{coker}\,\Phi$
4	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^3$	$(\mathbb{Z}/2\mathbb{Z})^2$
3	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
2	0	0	0	0
1	0	$\mathbb{Z}$	$\mathbb{Z}$	0
0	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

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2	0	0	0	0
1	0	$\mathbb{Z}$	$\mathbb{Z}$	0
0	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Interacting fermionic topological insulators/superconductors in 3D

Chong Wang and T. Senthil
Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
(Dated: June 1, 2015)

Symmetry class	**Reduction of free fermion states**	**Distinct boson SPT**	**Complete classification**
$U(1)$ only (A)	0	0	0
$U(1) \rtimes \mathbb{Z}_2^T$ with $\mathcal{T}^2 = -1$ (AII)	$\mathbb{Z}_2 \to \mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$
$U(1) \rtimes \mathbb{Z}_2^T$ with $\mathcal{T}^2 = 1$ (AI)	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
$U(1) \times \mathbb{Z}_2^T$ (AIII)	$\mathbb{Z} \to \mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
$U(1) \rtimes (\mathbb{Z}_2^T \times \mathbb{Z}_2^C)$ (CII)	$\mathbb{Z}_2 \to \mathbb{Z}_2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5$
$(U(1) \rtimes \mathbb{Z}_2^T) \times SU(2)$	0	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$
$\mathbb{Z}_2^T$ with $\mathcal{T}^2 = -1$ (DIII)	$\mathbb{Z} \to \mathbb{Z}_{16}$	0	Z_{16} (?)
$SU(2) \times \mathbb{Z}_2^T$ (CI)	$\mathbb{Z} \to \mathbb{Z}_4$	$\mathbb{Z}_2$	$Z_4 \times Z_2$ (?)

TABLE I. Summary of results on classifications of electronic SPT states in three dimensions. The second column gives free fermion states that remain nontrivial after introducing interactions. The third column gives SPT states that are absent in the free fermion picture, but are equivalent to those emerged from bosonic objects such as electron spins and cooper pairs. For symmetries containing a normal $U(1)$ subgroup, we can find the complete classification. In all such examples the complete classifications are simple products of those descending from free fermions and those obtained from bosons. For symmetry class CI, we give suggestive arguments but not a proof that the classification in the last column is complete.

Interacting fermionic topological insulators/superconductors in 3D

Chong Wang and T. Senthil
Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
(Dated: June 1, 2015)

Symmetry class	**Reduction of free fermion states**	**Distinct boson SPT**	**Complete classification**
$U(1)$ only (A)	0	0	0
$U(1) \rtimes \mathbb{Z}_2^T$ with $\mathcal{T}^2 = -1$ (AII)	$\mathbb{Z}_2 \to \mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$
$U(1) \rtimes \mathbb{Z}_2^T$ with $\mathcal{T}^2 = 1$ (AI)	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
$U(1) \times \mathbb{Z}_2^T$ (AIII)	$\mathbb{Z} \to \mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
$U(1) \rtimes (\mathbb{Z}_2^T \times \mathbb{Z}_2^C)$ (CII)	$\mathbb{Z}_2 \to \mathbb{Z}_2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5$
$(U(1) \rtimes \mathbb{Z}_2^T) \times SU(2)$	0	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$
$\mathbb{Z}_2^T$ with $\mathcal{T}^2 = -1$ (DIII)	$\mathbb{Z} \to \mathbb{Z}_{16}$	0	Z_{16} (?)
$SU(2) \times \mathbb{Z}_2^T$ (CI)	$\mathbb{Z} \to \mathbb{Z}_4$	$\mathbb{Z}_2$	$Z_4 \times Z_2$ (?)

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n	$\ker\Phi \longrightarrow$	$FF_n(G^+) \xrightarrow{\Phi}$	$TP_n(G^+) \longrightarrow$	$\operatorname{coker}\Phi$
4	$4\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
3	0	0	0	0
2	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
1	0	0	0	0
0	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

n	$\ker\Phi \longrightarrow$	$FF_n(\mathrm{Pin}^+) \xrightarrow{\Phi}$	$TP_n(\mathrm{Pin}^+) \longrightarrow$	$\operatorname{coker}\Phi$
4	$16\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/16\mathbb{Z}$	0
3	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
2	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
1	0	0	0	0
0	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

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1	0	0	0	0
0	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

don't seem to be in the literature

n	$\ker\Phi$	$\longrightarrow FF_n(\mathrm{Pin}^+)$	$\xrightarrow{\Phi} TP_n(\mathrm{Pin}^+)$	$\longrightarrow \operatorname{coker}\Phi$
4	$16\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/16\mathbb{Z}$	0
3	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
2	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
1	0	0	0	0
0	$2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

Modeling gapped material

Modeling gapped material

(building on work of Kitaev)

(Freed, H-, Kapustin, Kitaev, Moore, Teleman)

$$E = E(I, \phi, k_0) = \mathrm{Map}(MTH, S^2 IZ(1))$$

space of d-dimensional gapped lattice systems with specified internal symmetry. ⟷ 0-space of $S^d E$

Modeling gapped material

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$$E = E(I, \phi, k_0) = \mathrm{Map}(MTH, S^2 IZ(1))$$

space of gapped lattice systems with specified internal symmetry on a space X.

0-space of $S^d E \wedge X$

Modeling gapped material

(from a discussion between Dan and Mike Hermele)

X = 3-space // 180 deg rotation

H = Spin (fermions with no additional symmetry)

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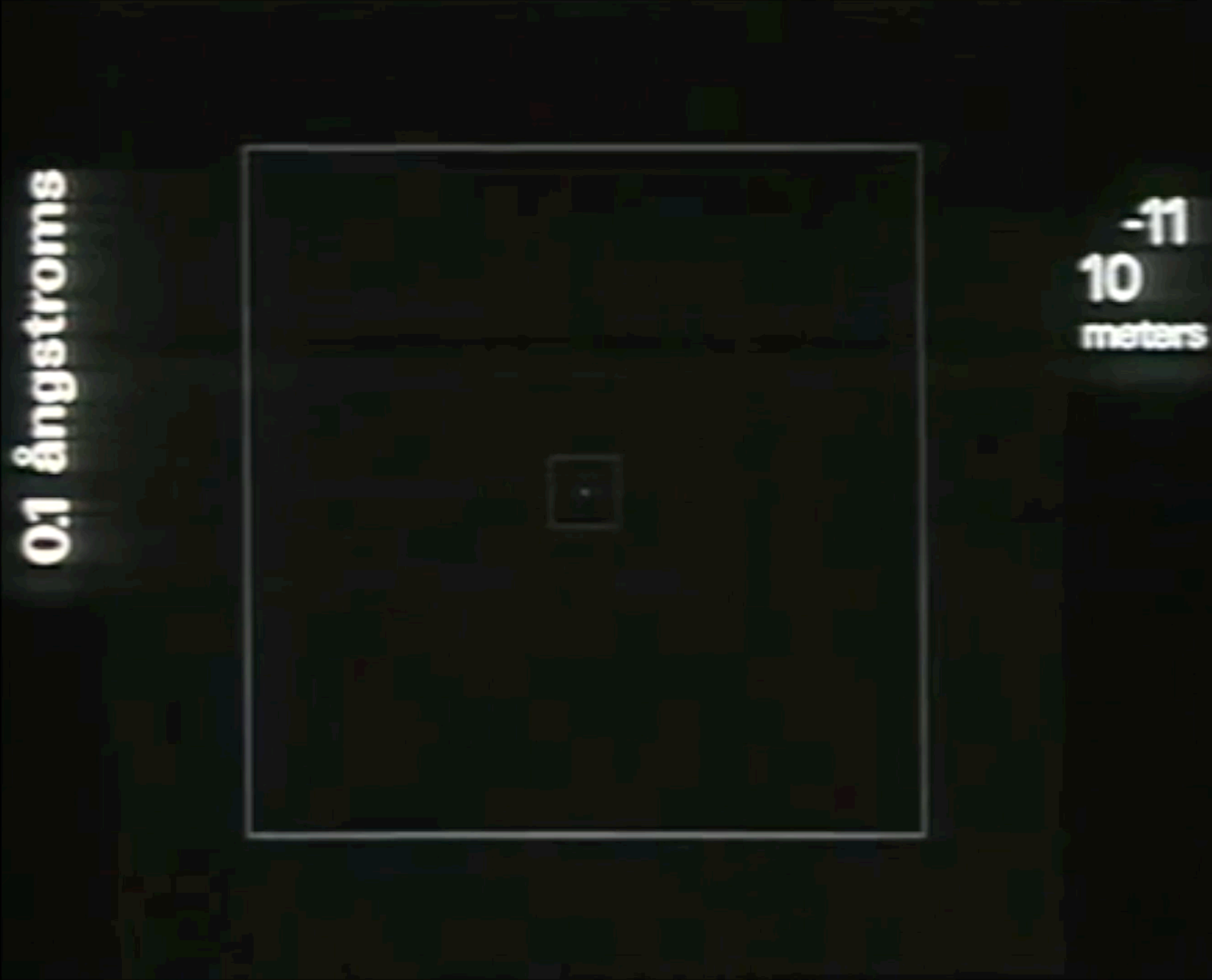
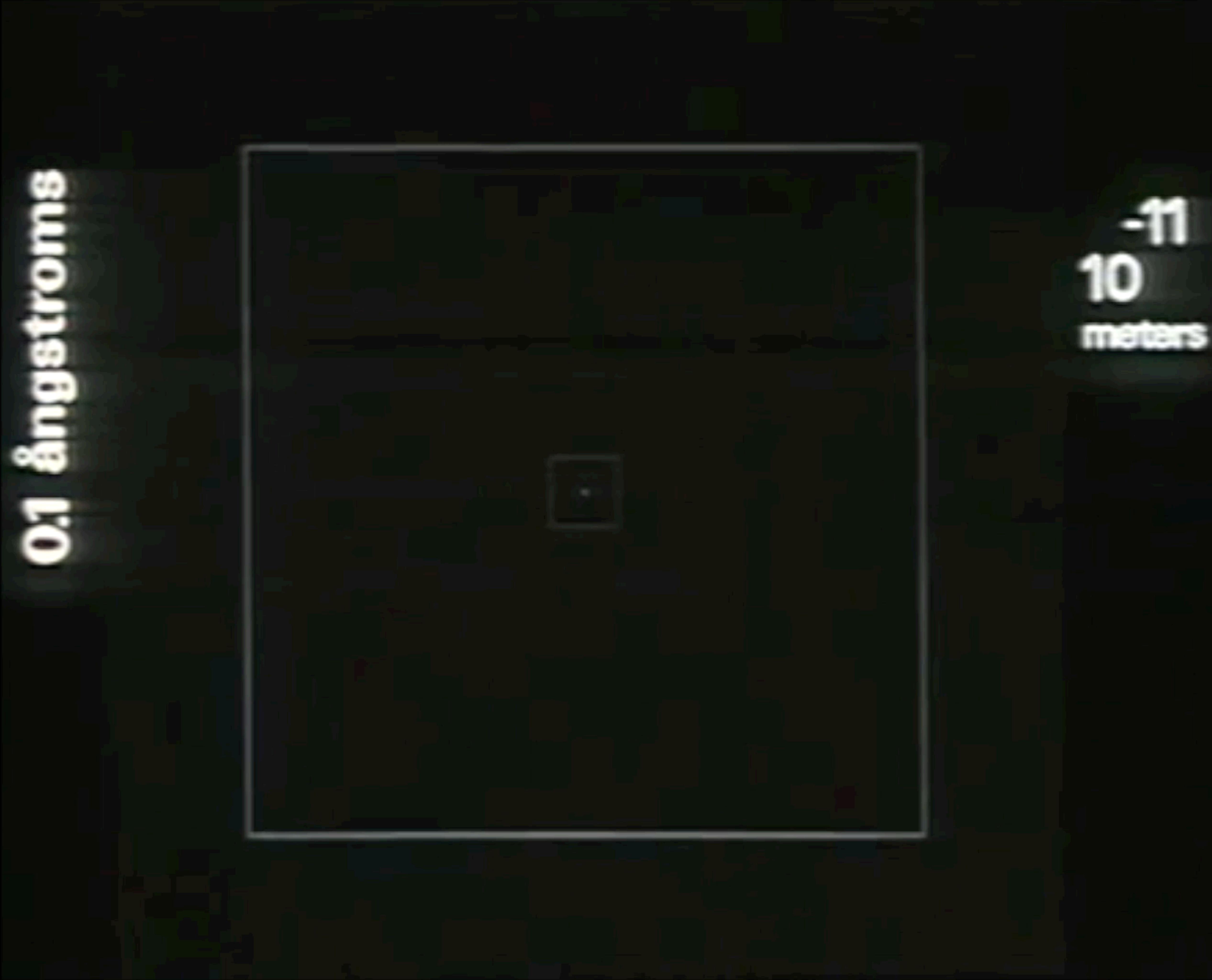
The topological model predicts only one phase.

physically: Kitaev's Majorana chain deforms in 3-space to a tube of $p_x + ip_y$ superconductor material.

Landscape of infinity categories


0.1 ångstroms
10^{-11}
meters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10^{-11}
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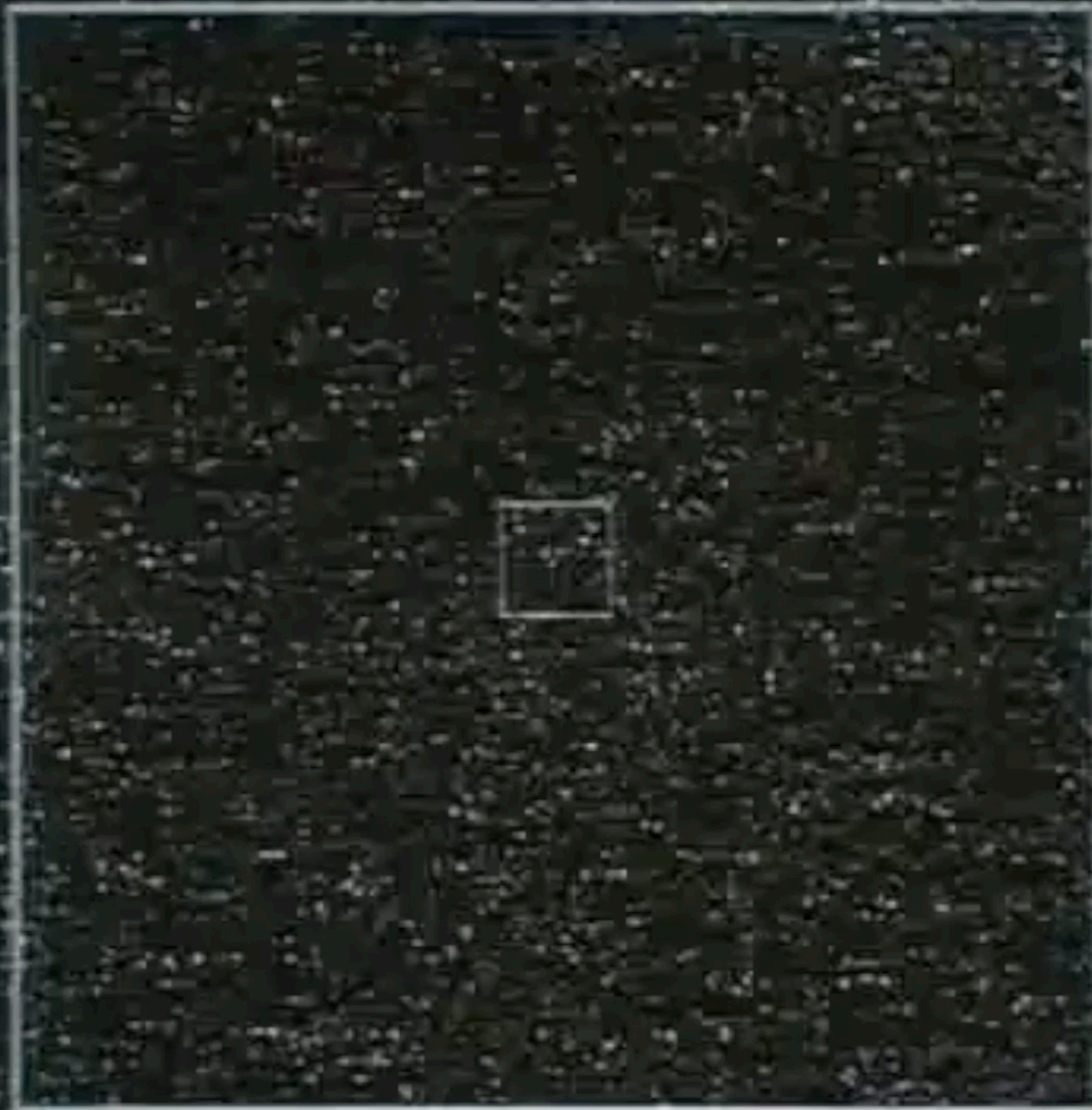
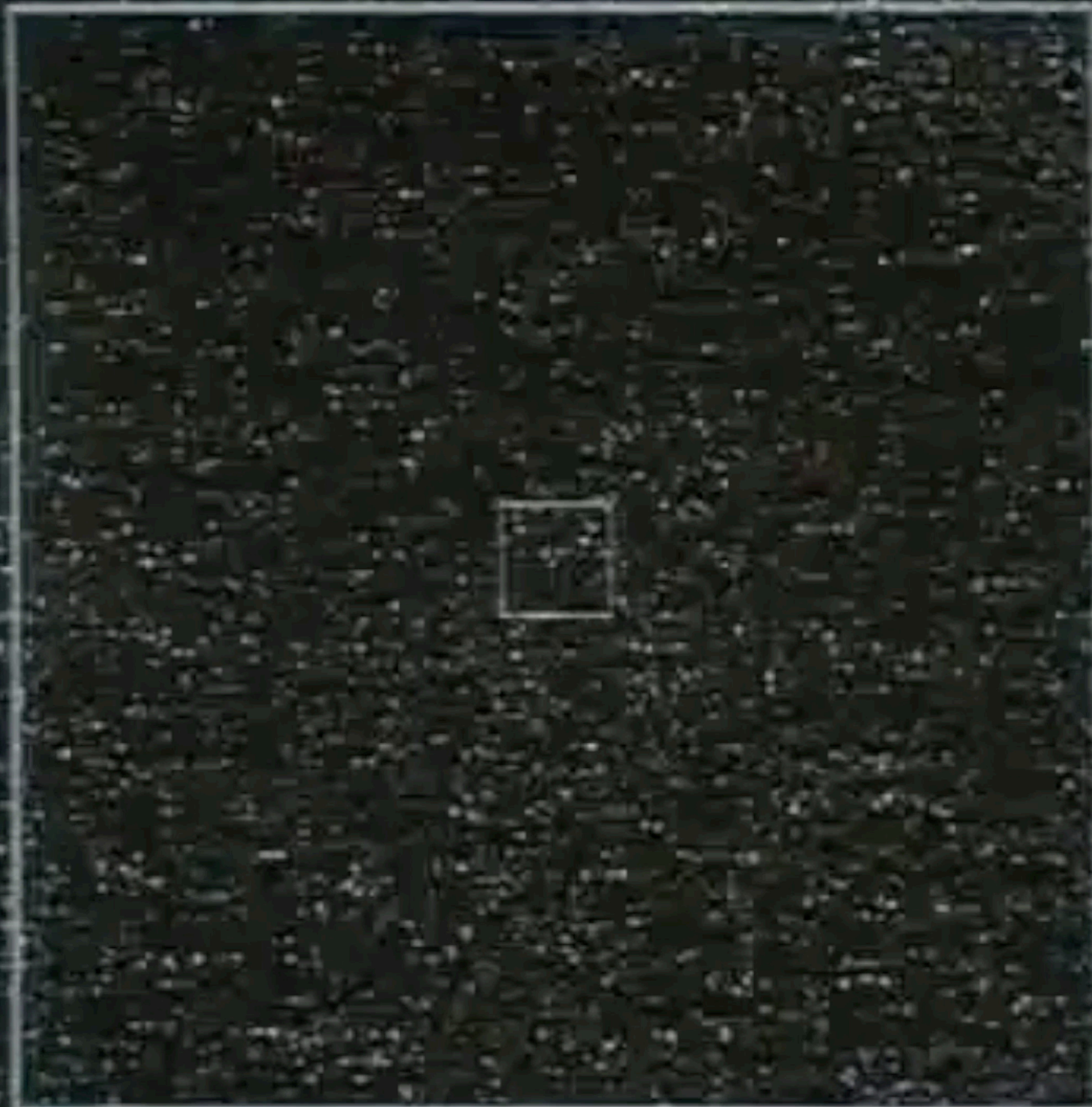
10 ångstroms

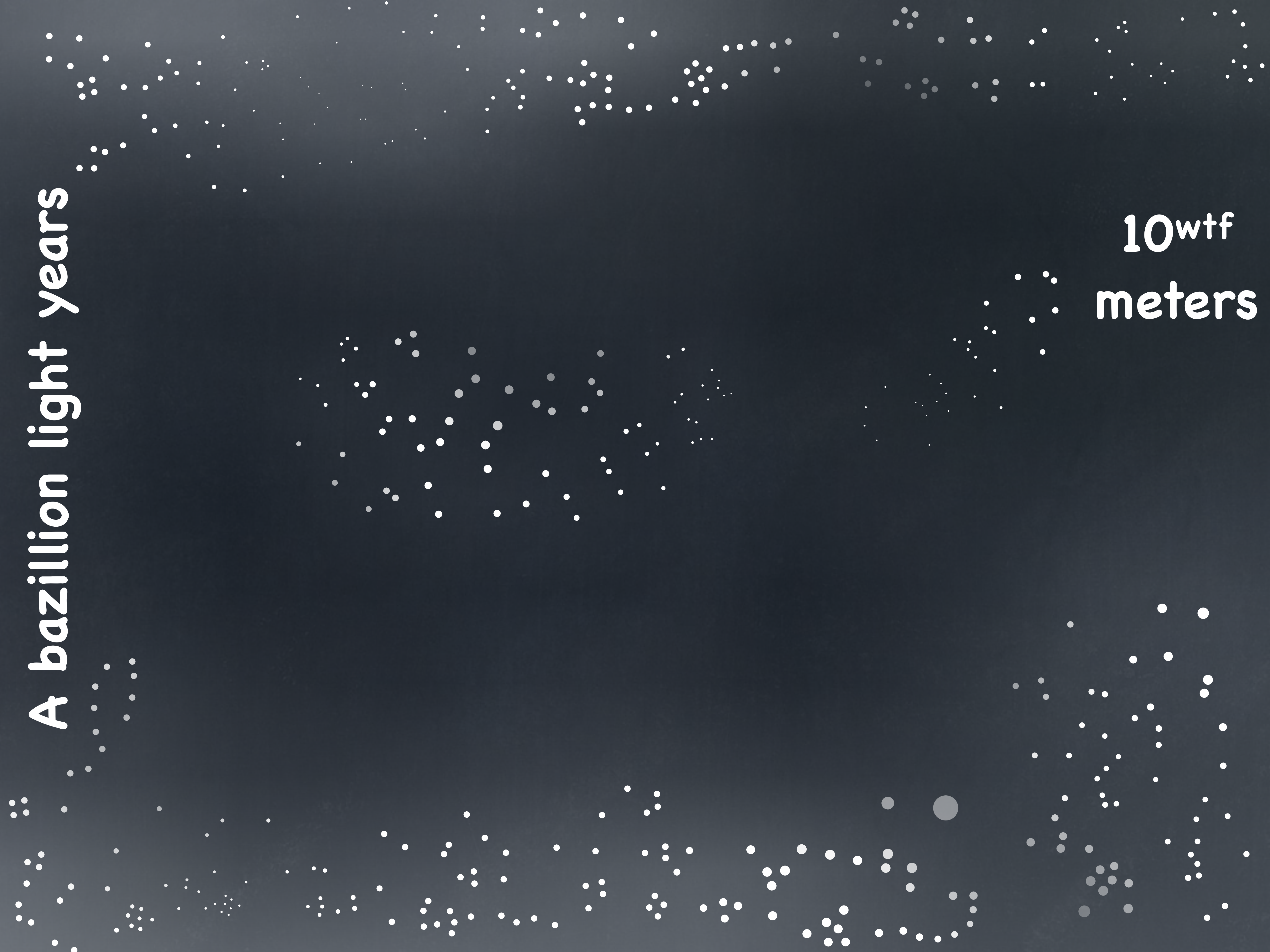


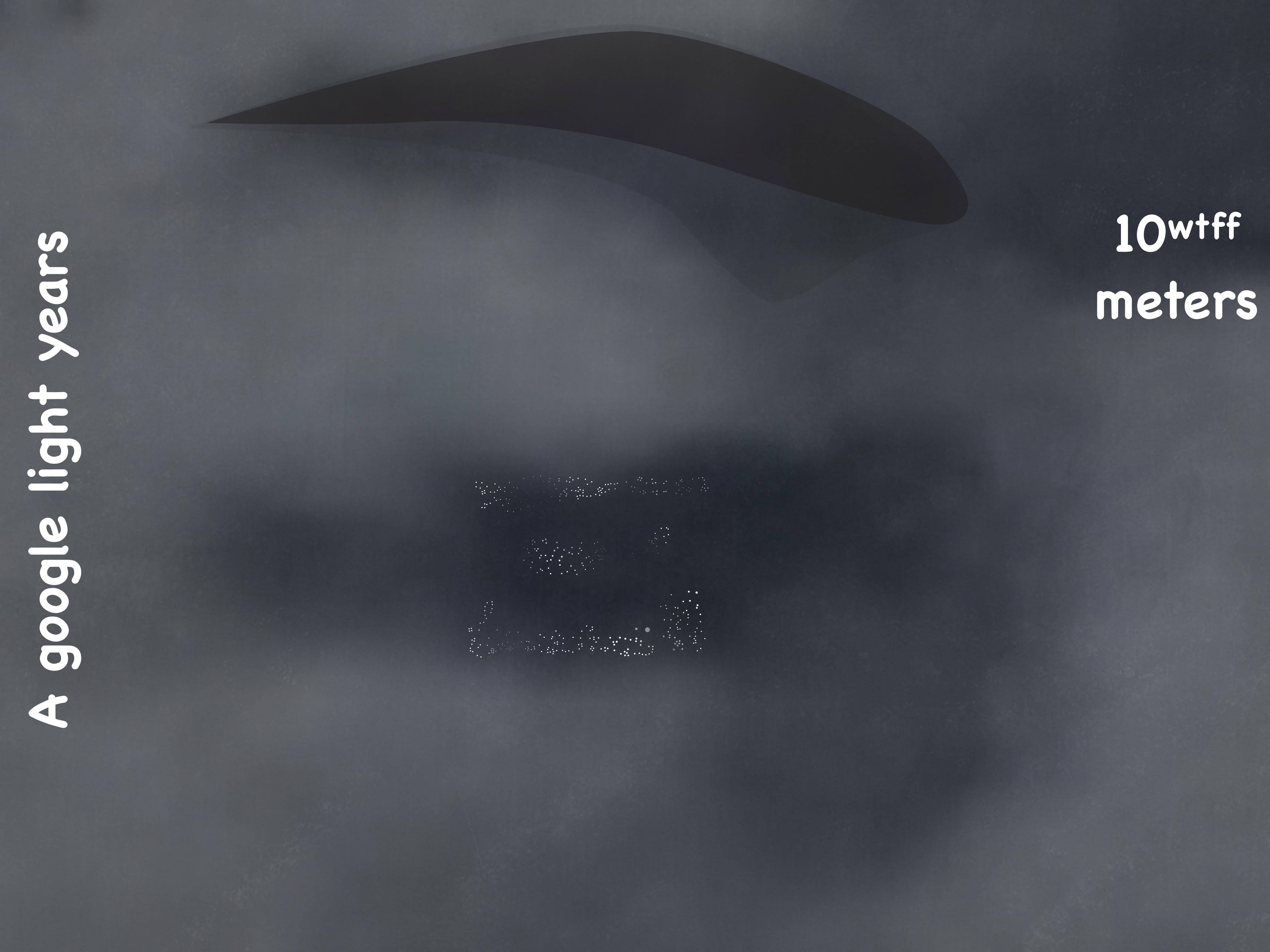
10^{-9} meters



10 ångstroms

10^{-9} meters


A bazillion light years
10^{wtf} meters


A google light years
10wtff
meters


A googleplex-googleplex light years
10^{wtfff}
meters


A googleplex-googleplex light years
10wtfff
meters