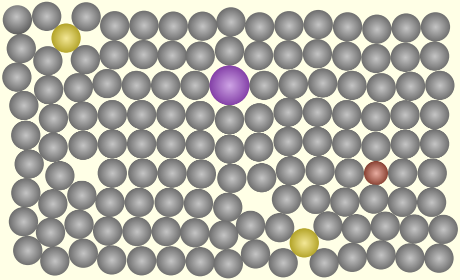
Forms on the space of lattice models



Alexei Kitaev (Caltech)

Aspen 2015

Working group on topological phases

Effective action

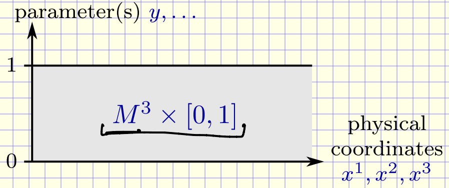
$$-S = \ln Z = \frac{\nu}{4\pi}\int A\wedge dA$$

$$\frac{\partial \ln Z}{\partial y} = \frac{\nu}{2\pi}\int \underbrace{\frac{\partial A}{\partial y}\wedge dA}_{(\tilde F\wedge \tilde F)_{y123}\, dx^1\, dx^2\, dx^3}$$

$$\tilde A_\mu = A_\mu \qquad \tilde A_y = 0$$

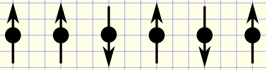
$$\tilde F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\nu}{\partial x^\mu} \qquad \tilde F_{y\nu} = \frac{\partial A_\nu}{\partial y}$$

$$S(1) - S(0) = \frac{\nu}{2\pi}\int_{M^3\times[0,1]} \underbrace{\tilde F\wedge \tilde F}_{\boxed{\Omega^{(n+1)}}}$$



Lattice statistical mechanics models

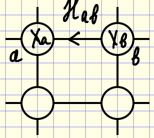
- Ising model



$S_j = \pm 1 \qquad L \equiv 0$

$$Z = \sum_S \prod_j e^{\frac{J}{T} S_j S_{j+1}} = Z(J)$$

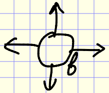
- Tensor models



Edges – f.d. Hilbert spaces
Vertices – tensors

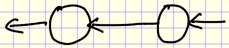
$$\mathcal{H}_{ab} = \mathcal{H}^*_{ba}$$

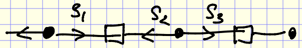
$$X_b \in \mathcal{L}_b = \bigotimes_a \mathcal{H}_{ab}$$



Z = Tensor contraction $(X_1, X_2, \ldots)$

$P \subseteq \prod_b \mathcal{L}_b$ – parameter space



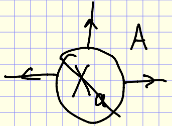


$$\bullet = |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle \in \mathcal{H} \otimes \mathcal{H} \quad , \quad \mathcal{H} = \mathbb{C}^2$$

$$\square = \sum_{s,s'} \langle s, s'| \cdot e^{\frac{J}{T} s s'}$$

$$Z = Z_{Ising}$$

Correlators

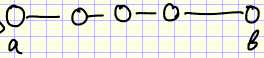


$$Z \to A \cdot \frac{\partial Z}{\partial X_a}$$

$$\langle A \rangle := A \cdot \left(Z^{-1} \frac{\partial Z}{\partial X_a} \right) = \sum_j A^j \left(Z^{-1} \frac{\partial Z}{\partial X_a^j} \right)$$

$$\langle A B \rangle_{ab} = \sum_{j,k} A_j B_k \frac{\partial^2 Z}{\partial X_a^j \, \partial X_b^k} \cdot Z^{-1}$$

$$\langle\langle A, B \rangle\rangle_{a,b} = A \; B \; \frac{\partial^2 \ln Z}{\partial X_a \, \partial X_b} = \langle A B \rangle - \langle A \rangle \langle B \rangle$$



$$\langle\langle A, B, C \rangle\rangle \quad A B C \frac{\partial^3 Z}{\partial X_a \, \partial X_b \, \partial X_c} = \langle A B C \rangle - \langle A B \rangle \langle C \rangle - \langle A C \rangle \langle B \rangle - \langle B C \rangle \langle A \rangle + 2 \langle A \rangle \langle B \rangle \langle C \rangle$$

Main formulas

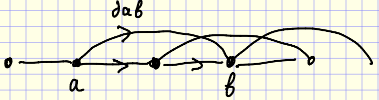
$$\Omega^{(1)}_a = \frac{\partial \ln Z}{\partial X_a}\, dX_a = \sum_j \frac{\partial \ln Z}{\partial x^j_a}\, dx^j_a$$

$$\boxed{\partial \Omega^{(n+1)} = d\Omega^{(n)}}$$

$$\Omega^{(2)}_{ab} = \frac{\partial^2 \ln Z}{\partial X_a \partial X_b}\, dX_a \wedge dX_b = \sum_{j<k} \underbrace{\left(\frac{\partial^2 \ln Z}{\partial x^j_a \partial x^k_b} - \frac{\partial^2 \ln Z}{\partial x^k_a \partial x^j_b} \right)}_{\Omega^{jk}_{ab}} dx^j_a \wedge dx^k_b$$

2-form on P

1-current on the phys. space



$$\boxed{(\partial j)_b = \sum_a j_{ab}}$$

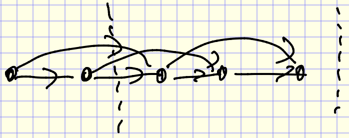
$$\begin{cases} j_{ab} \text{ - decay fast in } |a-b| \\ j_{ba} = -j_{ab} \end{cases}$$

$$\left(\partial\Omega^{(2)}\right)_b = \sum_a \Omega^{(2)}_{ab} = \sum_a \frac{\partial^2 \ln Z}{\partial x^a \partial x^b}\, dx^a \wedge dx^b = d\Big(\underbrace{\frac{\partial \ln Z}{\partial x^b}\, dx^b}_{\Omega^{(1)}_b}\Big)$$

$$\underline{\partial\Omega^{(2)} = d\,\Omega^{(1)}}$$

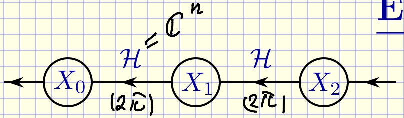
$$j_{ab} = \int_C \Omega_{ab} \qquad\qquad \boxed{\partial j = 0}$$

surface in P



$$J = \sum_{a<0} \sum_{b>0} j_{ab}$$

Example



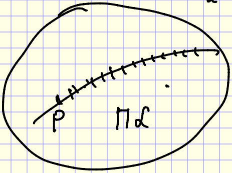
$X_a \in \mathcal{H} \otimes \mathcal{H}^* = \mathcal{L}$

$P \subseteq \prod \mathcal{L}$

$\|$

$\mathbb{C}P^{n-1}$

$X_a = X = |\psi\rangle \otimes \langle\psi|$ $\qquad |\psi\rangle \in \mathbb{C}^n$

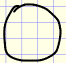


$(\ldots, \overset{=X}{X_1}, \overset{=X}{X_2}, \ldots) \in P$

$Z(\ldots\ldots) = 1$

$$\underbrace{\frac{\partial^2 \ln Z}{\partial X_a \partial X_b} dX_a \wedge dX_b = \langle d\psi_a | d\psi_b \rangle}$$

$$\Omega^{(2)}_{ab} = \langle d\psi | d\psi \rangle = \sum_j d\psi_a^j \wedge d\psi_b^j = -\Omega^{(2)}_{ba} \qquad b = a+1$$



$$\int_{\mathbb{C}P^1} \Omega_{a,a+1} = 2\pi$$

$\Omega^{(n+1)}$ n- current

d-n - form on $\mathbb{R}^d$

$\mathbb{R}^d$ n+1-forms in parameters

d-n - forms on $\mathbb{R}^d$

$\Big\}$ (d+1) forms on $\underline{\underbrace{\mathbb{R}^d \times P}}$

Happy birthday, Dan!