From Kapustin-Witten to Extended Bogomolny to Hitchin

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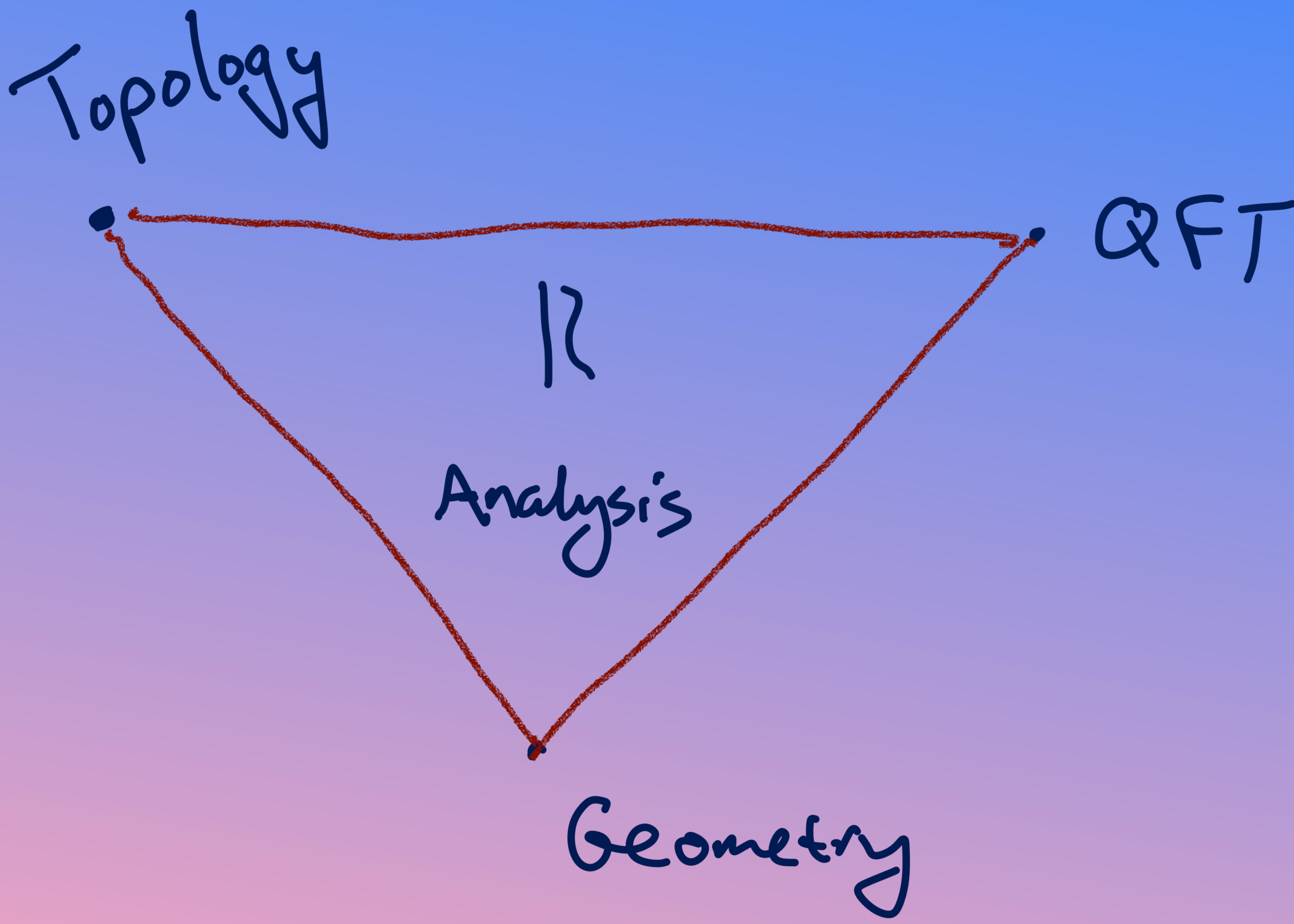
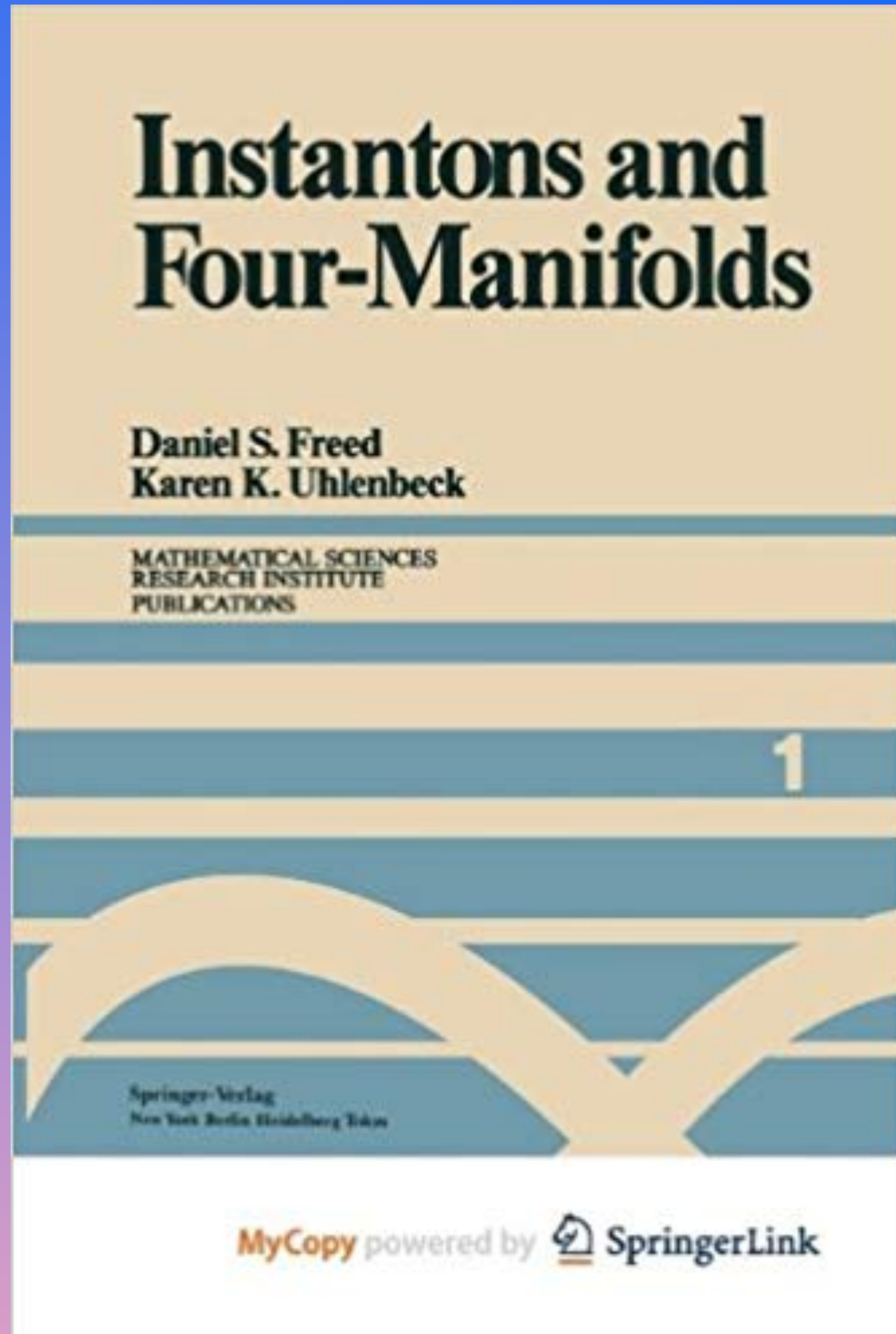
Stanford

Between

Topology and QFT

?


Topology
QFT
Analysis
Geometry


Instantons and Four-Manifolds
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Start with a "new" gauge theory

Kapustin - Witten 2005

(Geometric Langlands)

(X^4, g)

$$\begin{array}{c} E \quad G\text{-bundle} \\ \downarrow \\ X \end{array}$$

(think $G = SU(2)$).

A = connection on E

$\phi \in \Omega^1(X, \operatorname{ad} E)$

$$\left.\begin{aligned} F_A - \phi \wedge \phi + * d_A \phi &= 0 \\ d_A * \phi &= 0 \end{aligned}\right\} \mathrm{KW},$$

c.f. Gagliardo-Uhlenbeck

Specialization of 5D theory (Haydys, Witten)

Embedded in family of equations

$$\left.\begin{aligned} \left(F_A - \phi\wedge\phi + t\, d_A\phi\right)^+ &= 0 \\ \left(F_A - \phi\wedge\phi - t^{-1}\, d_A\phi\right)^- &= 0 \\ d_A * \phi &= 0 \end{aligned}\right\} KW_t$$

Observation:

$$\mathcal{A} = A + i\phi \in \mathcal{G}_{\mathbb{C}}$$

$$F_{\mathcal{A}} = dA + i\, d\phi + (A+i\phi)\wedge(A+i\phi)$$

$$= \left(F_A - \phi\wedge\phi\right) + i\, d_A\phi = 0$$

Include $d_A * \phi = 0$

Thus $F_{\mathcal{U}} = 0$, $d_A * \phi = 0$

$\rightsquigarrow$ sol'ns of KW_t for all t

$KW_t = 0$ only uses half of this.

$F_{\mathcal{U}} = 0$ complex gauge invariant

$d_A * \phi = 0$ real moment map

X^4 closed?

$KW_t(A,\phi) = 0$

$\Rightarrow F_A = 0$

$\rightsquigarrow$ simpler topological theory

(Pf: Weizenböck formula, ...)

Focus instead on

X^4 with boundary W

what are the right boundary conditions?

$$X^4 = W^3_x \times \mathbb{R}^+_y$$

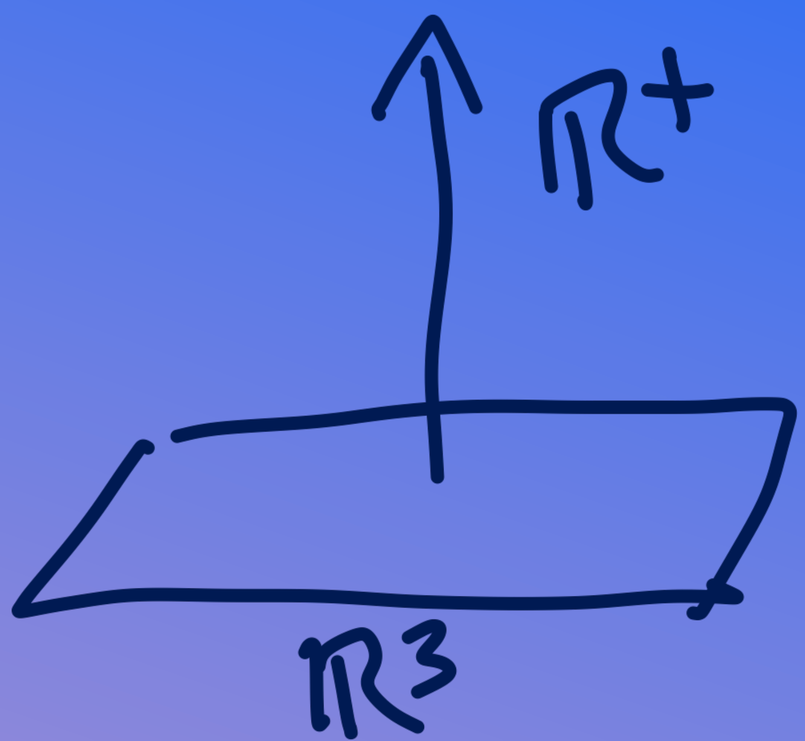
Boundary conditions at $y=0$ and as $y \to \infty$

Demand that solutions become translation-invariant in y as $y \to \infty$

$$A_y = 0, \quad \phi_y = 0 \quad \rightsquigarrow$$

$$\left.\begin{aligned} &F_A - \phi \wedge \phi = 0 \\ &d_A \phi = 0, \quad d_A * \phi = 0 \end{aligned}\right\} \begin{aligned} &F_{\mathcal{A}} = 0 \\ &d_A * \phi = 0 \end{aligned}$$

Note: asymptotics $\rightsquigarrow$ decoupling



$x \in \mathbb{R}^3$, $y \geq 0$

Model: $A_y = 0$, $\phi_y = 0$ and can gauge away A_x too

Remaining eq'n: $d\phi = \phi \wedge \phi$

$\rightsquigarrow \phi = \sum t_a \, dx^a$ where

$$[t_a, t_b] = \varepsilon_{abc} t_c$$

Altogether

Flat $SL(n, \mathbb{C})$ connection (element of Hitchin moduli space).

Nahm condition.

The Nahm boundary condition in general:

Choose principle embedding

$\rho : su(2) \longrightarrow \mathfrak{g}$

i.e. $t_1, t_2, t_3 \in \mathfrak{g}$

$$[t_a, t_b] = \varepsilon_{abc} t_c$$

$\phi_\rho : TW \hookrightarrow \mathrm{ad}\, E|_W$ (isom. when $G = SU_2$)

$\phi_\rho = \sum t_i \, dx^i$

$$A = A_0 + O(y^\varepsilon), \quad \phi = \frac{1}{y} \varphi_\rho + O(y^{-1+\varepsilon})$$

Regularity (M-Witten)

$$A \sim A_0 + x A_1 + \cdots$$

$$\phi \sim \frac{1}{y} \varphi_\rho + \sum \varphi_j \, y^{\lambda_j} \qquad (\lambda_j \to \infty)$$

$A_0 \cong$ Levi-Civita connection

(via intertwining

$\varphi_\rho : TW \hookrightarrow \mathrm{ad}(E)|_W$)

This relies on analysis of linearization:

$$\mathcal{L}_{KW} \sim B_y \partial_y + \sum B_{x_j} \partial_{x_j} + \frac{1}{y} B_0$$

(singular potential)

Also,

$$\mathrm{Ind}(\mathcal{L}_{KW}) =$$

$$-(\dim g)\,\chi(X)$$

$$\left(= 0 \text{ when } X = \mathbb{R}^+ \times W\right).$$

For KW_t, there are 'tilted' Nahm pole conditions

Replace t in KW_t by

$$t = \tan\left(\frac{\pi}{4} - \frac{3}{2}\beta\right) \qquad \left(\begin{matrix} t=1 \\ \beta=0 \end{matrix}\right)$$

$$A_1 - \tan\beta\,\phi_1 = 0$$

x_1 = distinguished direction in boundary

Model ($G = SU(2)$)

$$A_z = \frac{1}{y}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \sin\beta$$

$$A_{\bar{z}} = \frac{1}{y}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \sin\beta$$

$$\phi_z = \frac{1}{y}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cos\beta$$

$$\phi_{\bar{z}} = \frac{1}{y}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cos\beta$$

$$\phi_1 = \frac{i}{2y}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos\beta$$

Note

$$A, \ \phi, \ \phi_1 \quad \text{all} \sim \frac{1}{y}$$

leading parts of A, ϕ

upper/lower triangular

Program:

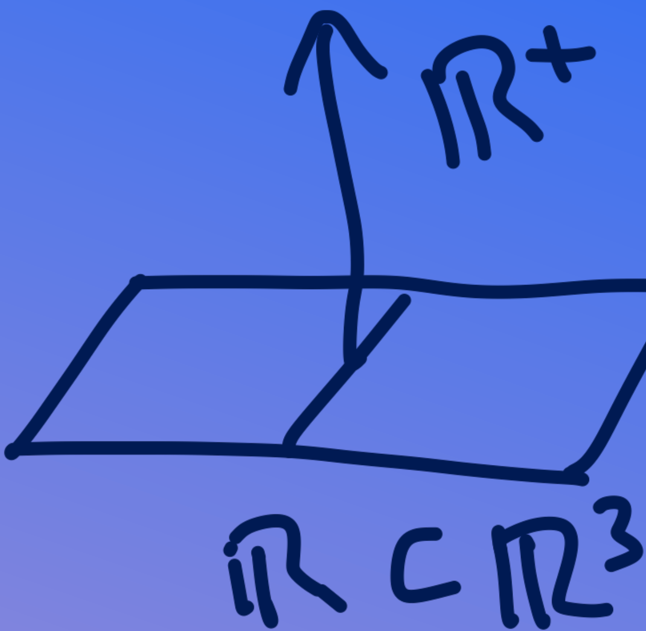
Study sol'ns of $KW_1 = 0$ (or $KW_t = 0$) on $W \times \mathbb{R}^+$ with these boundary conditions.

New invariants of W?

(Gaiotto-Witten)

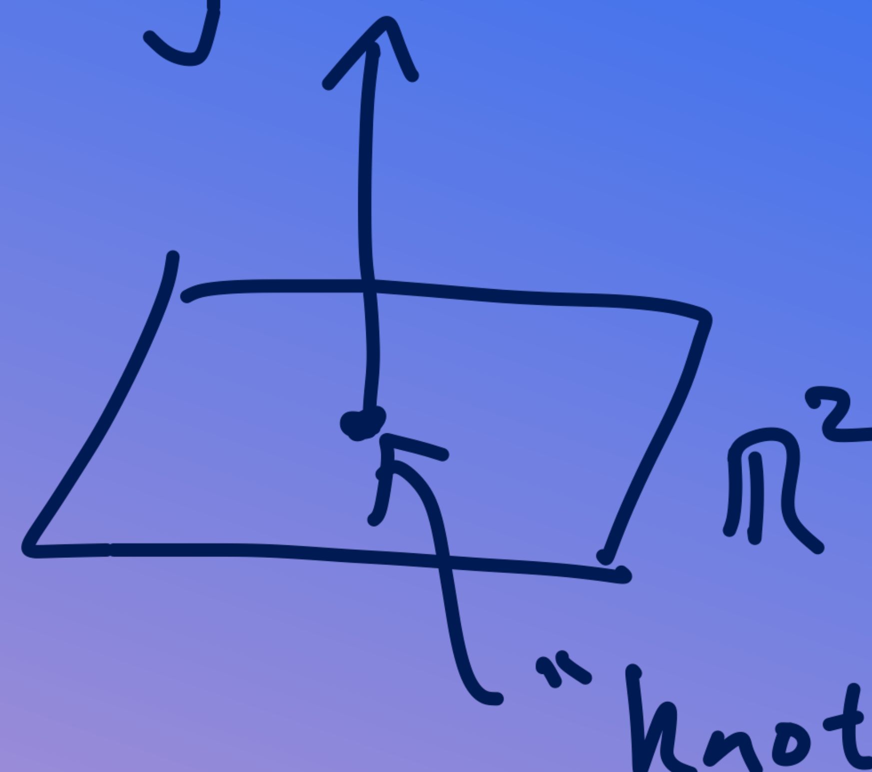
More generally, consider knot $K \subset W$ at $y = 0$

Introduce new boundary conditions with extra singularity at K



model knot

mod out by $\mathbb{R}$



(ρ, θ, ψ)
spherical coords.

$$A = -(n+1)\sin^n\psi \, \frac{F_n(\psi)}{F_{n+1}(\psi)} \, d\theta \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}$$

$$\phi = (\phi_2 - i\phi_3) = \frac{2(n+1)}{\rho} \frac{\sin^n\psi \, e^{in\theta}}{F_{n+1}(\psi)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\phi_1 = \frac{n+1}{\rho} \frac{G_{n+1}(\psi)}{F_{n+1}(\psi)} \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}$$

$$F_\ell(\psi) = (1+\cos\psi)^\ell - (1-\cos\psi)^\ell$$

$$G_\ell(\psi) = \quad " \quad + \quad "$$

For other G
model. knot solution
due to Mikhaylov.

$t \neq 1$?

Similar regularity theory for solutions of $KW(A, \phi) = 0$ with knot singularities.

$$A, \phi \sim \sum \rho^{r_j} s^{\mu_\ell} F_{j\ell}(\theta, x_1)$$

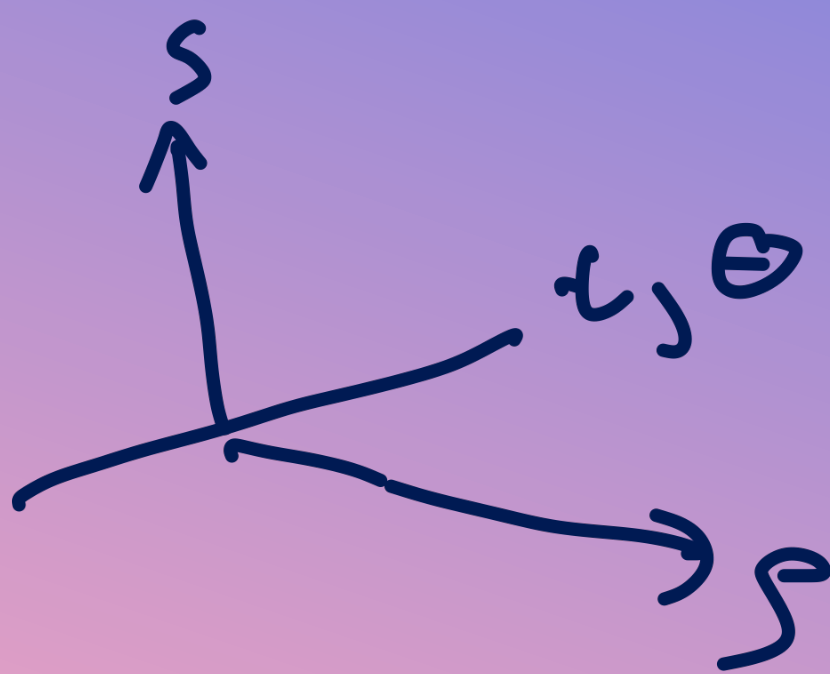
ρ = distance to knot

s = spherical coord

$\sim$ dist. to lower boundary

Based on analysis of

$$\mathcal{L}_{KW} \sim B_\rho \partial_\rho + B_t \partial_t + \frac{1}{\rho}\left(B_s \partial_s + B_\theta \partial_\theta + \frac{1}{s} B_\theta \right)$$



Bigger program (Gaiotto-Witten)

Count of solns $\rightsquigarrow$ coefficients of Jones polynomial of K.

Haydys-Witten equations

$\rightsquigarrow$ categorification

& Khovanov homology

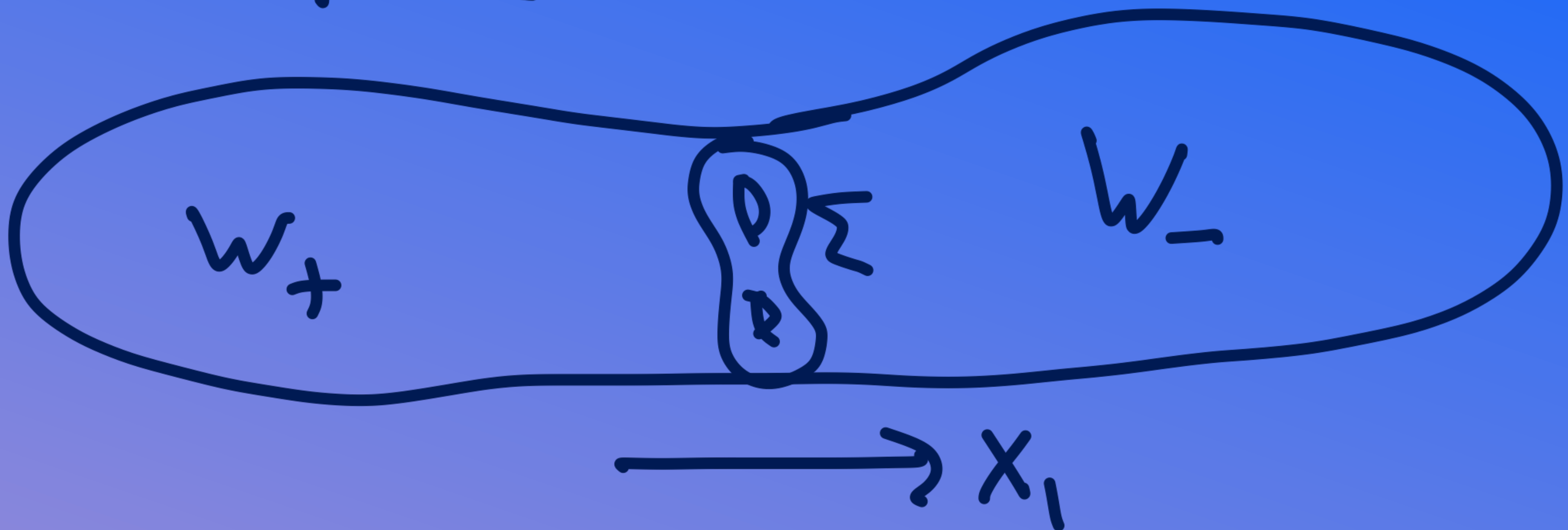
Steps to accomplish all of this

1) Local properties of moduli space (M-Witten, He)
2) Tilted case? $(t \neq 1)$
3) Existence of solutions with or without knots — today
4) Compactness (Taubes).

How to construct solutions?

- dimensionally-reduced cases
- gluing
- ?

$W = W_+ \cup_\Sigma W_-$



elongate along Σ

$\rightsquigarrow$ Sol'ns on $\mathbb{R}_{x_1} \times \Sigma_{x_2, x_3} \times \mathbb{R}^+_y$

Look for x_1-indep sol'ns

This leads to the Extended Bogomolny (EB) equations on $\Sigma \times \mathbb{R}^+$

Fields: $\begin{cases} A = A_\Sigma \\ \phi = \phi_\Sigma \\ \phi_1 \end{cases}$

$$\left.\begin{aligned} F_A - \phi \wedge \phi &= * d_A \phi_1 \\ d_A \phi &= * [\phi, \phi_1] \\ d_A^* \phi &= 0 \end{aligned}\right\} \text{EB}$$

Note: $\left.\begin{array}{l} \phi_1 = 0 \\ y\text{-indep.} \end{array}\right\} \Rightarrow$ Hitchin's eqns

$\phi = 0$ Bogomolny

x-indepentent, $\phi_1 = 0 \Rightarrow$ Nahm

Study KW on

$S^1 \times \Sigma \times \mathbb{R}^+$ (or $S^1 \times \Sigma \times \mathbb{R}$)

Look for S^1-independent

solutions

Solutions on $S^1_{x_1} \times \Sigma \times \mathbb{R}_y$

which are independent of

x_1 and y are

flat $Sl(n, \mathbb{C})$ connections on Σ

i.e. elements of the

Hitchin moduli space.

<u>Theorems</u> (He - M.)

Based on conjectures/predictions of Gaiotto - Witten

1) Solutions with Nahm pole singularity at $y=0$ on Σ (no knots) $\longleftrightarrow$ Hitchin component of flat $Sl(n,\mathbb{R})$ connections

1) (cont.)

In other words

$$\{\text{sdns. to EBE}\}/\mathcal{G} = \mathcal{M}_{EBE}$$

$$\longleftrightarrow \mathcal{M}_{Hit}(Sl_n(\mathbb{R}))$$

diffeomorphism

2) Solutions with knot singularities at $y=0$ on Σ $\longleftrightarrow$

$\{ (\mathcal{E}, \varphi, L) :$ $(\mathcal{E}, \varphi)$ stable Higgs pair (element of Hitchin moduli space) and $L \subset \mathcal{E}$ line subbundle $\}$

knot K = divisor on Σ determined by tangency of $L, \varphi(L), \varphi^2(L)$, etc.

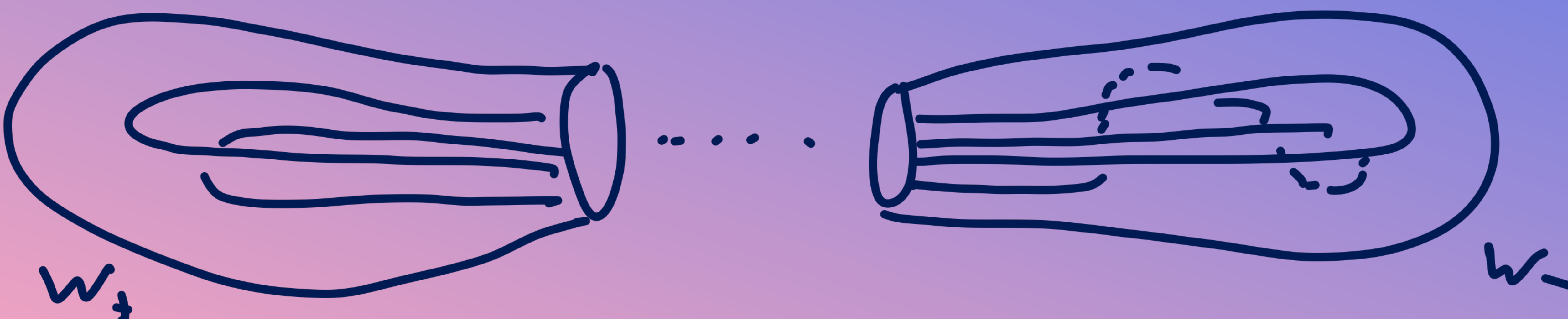
3) All solutions on $S^1 \times \Sigma \times \mathbb{R}^+$ with these bdry conditions are S^1 invariant.

4) Solutions of EBE on $\Sigma \times \mathbb{R}^+$ with tilted Nahm pole at $y = 0$

$\longleftrightarrow$ twisted oper moduli space (inside Hitchin moduli space).

What's missing?

- tilted Nahm boundary conditions
  with knot singularities
  (in progress)
- more general solutions
  $W \neq S^1 \times \Sigma$



We discuss 2) when

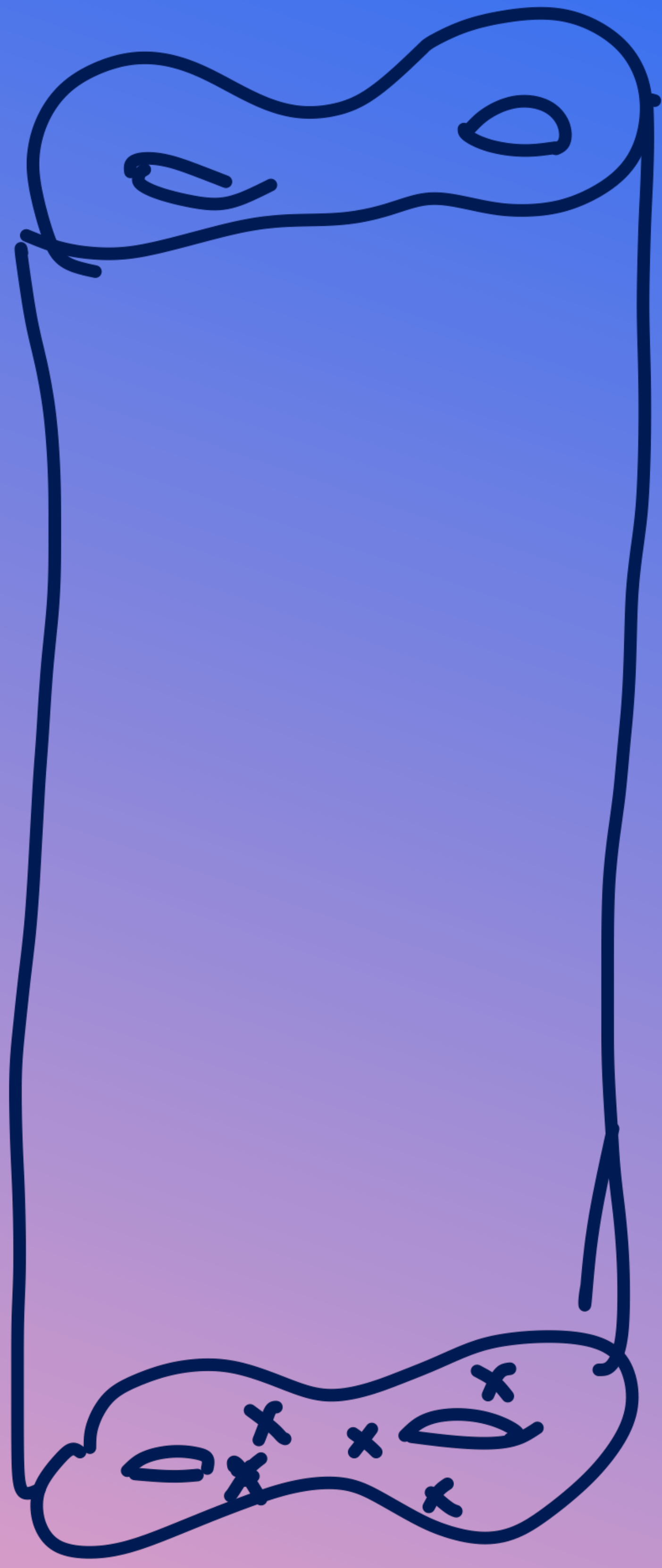
rank $E = 2$

$(\mathcal{E}, \varphi) \in \mathcal{M}_{\text{Hitchin}}$, $\mathcal{L} \subset \mathcal{E}$ line subbundle

$\mathcal{L} \wedge \varphi(\mathcal{L})$ vanishes at

$$\sum n_j p_j$$

The p_j are the knots



$(\mathcal{E}, \varphi, \mathcal{L})$
at $y = \infty$

Soln. of
Extended Bogomolny

knots at $y = 0$

Formalism:

$$\nabla_A = \nabla_2\, dx_2 + \nabla_3\, dx_3 + \nabla_y\, dy$$

$$\phi = \frac{1}{2}\left(\phi_z\, dz + \phi_{\bar{z}}\, d\bar{z}\right)$$

$$\mathcal{D}_1 = (\nabla_2 + i\nabla_3)\, d\bar{z}$$

$$\mathcal{D}_2 = \mathrm{ad}(\phi_2 - i\phi_3) = [\phi_2 - i\phi_3, \cdot\,]$$

$$\mathcal{D}_3 = \nabla_y - i\phi_1$$

EBE $\Longleftrightarrow$

$$[\mathcal{D}_i, \mathcal{D}_j] = 0 \qquad i,j = 1,2,3$$

and

$$\underbrace{\frac{i}{2}\Lambda\left([\mathcal{D}_1, \mathcal{D}_1^{\dagger_H}] + [\mathcal{D}_2, \mathcal{D}_2^{\dagger_H}]\right) + [\mathcal{D}_3, \mathcal{D}_3^{\dagger_H}]}_{\Omega_H} = 0$$

Hermitian Yang-Mills formalism

Donaldson, Uhlenbeck-Yau

$$[\mathcal{D}_i, \mathcal{D}_j] = 0 \quad \mathcal{G}_{\mathbb{C}}\text{-invariant}$$

remaining equation is real moment map

$$H = H_0 e^s \implies$$

$$\Omega_H = \Omega_{H_0} + \gamma(-s) \mathcal{L}_{H_0} s + Q(s)$$

$$\gamma(s) = \left(e^{ad_s} - 1\right) / ad_s$$

$$\mathcal{L}_{H_0} s = \frac{i}{2} \Lambda \left(D_1 D_1^{\dagger_{H_0}} + D_2 D_2^{\dagger_{H_0}} \right.$$

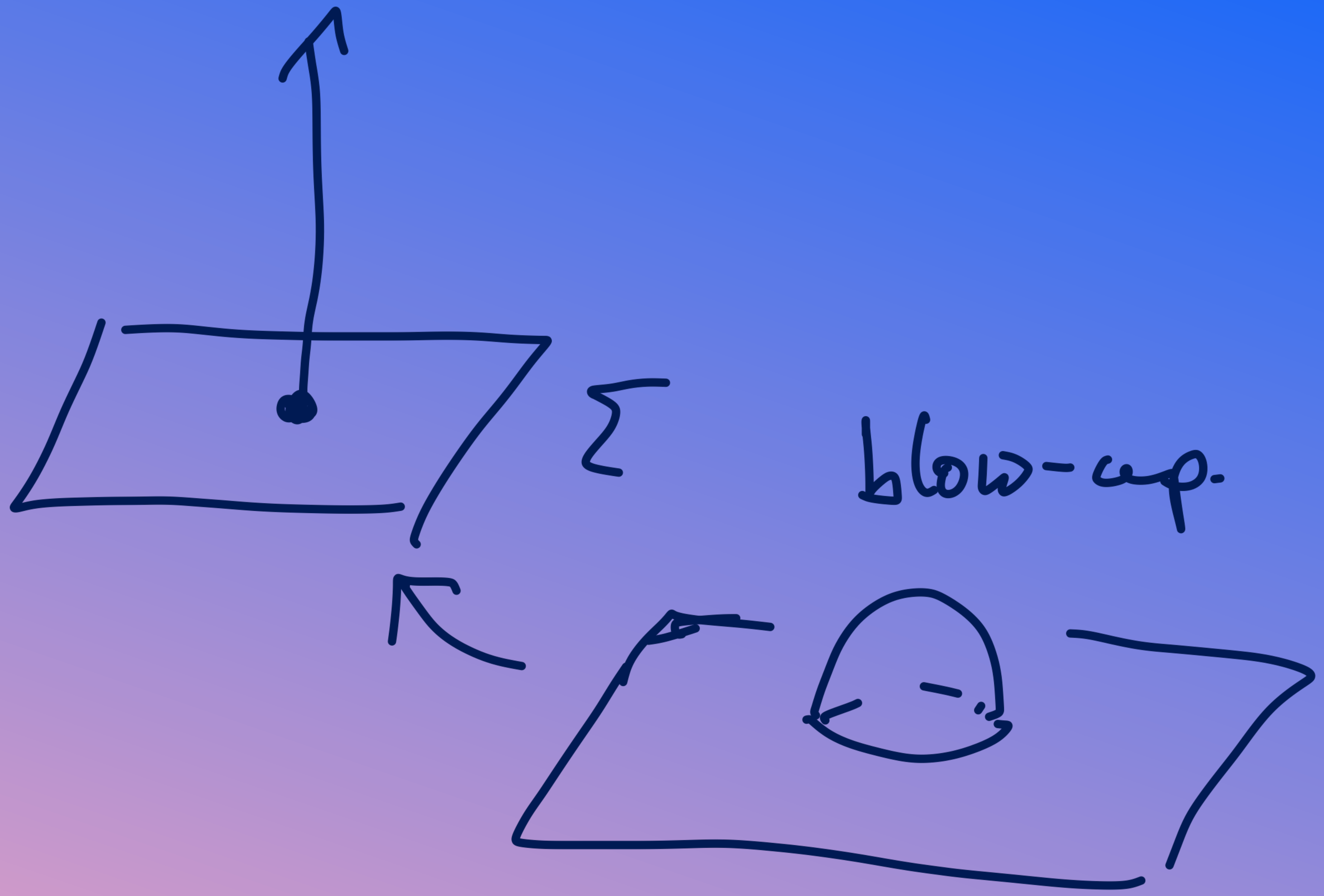
$$\left. + D_3 D_3^{\dagger_{H_0}} \right)$$

(Laplace-type)

- Pick extremely good initial approximate sol'n.

Solve in Taylor series at these two codimension one boundaries

(and at ∞)


Σ
blow-up.

- Then obtain a priori estimates on derivatives of s
- Deform equation

$\Omega^t_H = 0$ (family of equations)

$\Omega^0_H = 0$ has soln. $H = H_0$

$\Omega^1_H = 0$ is what we want.

- $\{t : \exists \text{ soln.}\}$ is

open and closed

implicit function theorem

a priori estimates + Arzela-Ascoli

(all the work is buried here)

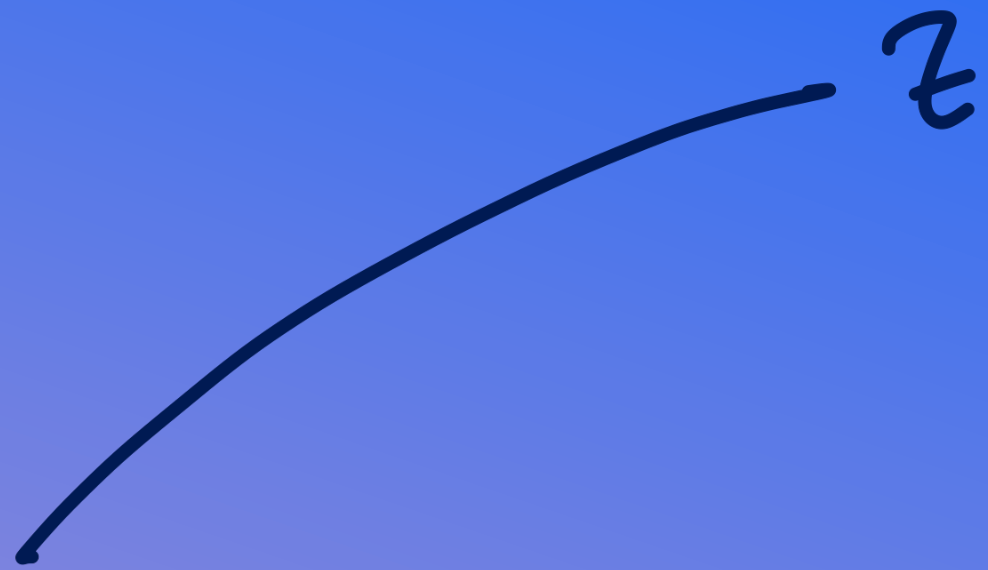
Bach to original problem:

Counting solutions of KW (or just flat $Sl(2,\mathbb{C})$ connections) on X^4 or $W^3 \times \mathbb{R}^+$

Taubes: If (A_j, ϕ_j) is a sequence of solutions, then . . .

Either (A_j, ϕ_j) converges up to unitary gauge

<u>OR</u> a rescaled sequence of the ϕ_j converge to a "$\mathbb{Z}_2$-harmonic spinor"



$L = \mathbb{R}$-line bundle on $X \setminus Z$ with $\mathbb{Z}_2$ holonomy.

ω L-valued 1-form

$$(d + d^*)\,\omega = 0$$

$$|\omega| \leq C \,\mathrm{dist}(\cdot, Z)^{1/2}.$$

Very Overdetermined.

In fact, given any (smooth) Z, (codim 2 in X^n) and $L \to X \setminus Z$, can find ω such that $(d+d^*)\omega = 0$

$$\omega \sim \sum_{j=0}^{\infty} r^{(j-1)/2} \omega_j$$

$r = \mathrm{dist}(\cdot, Z)$

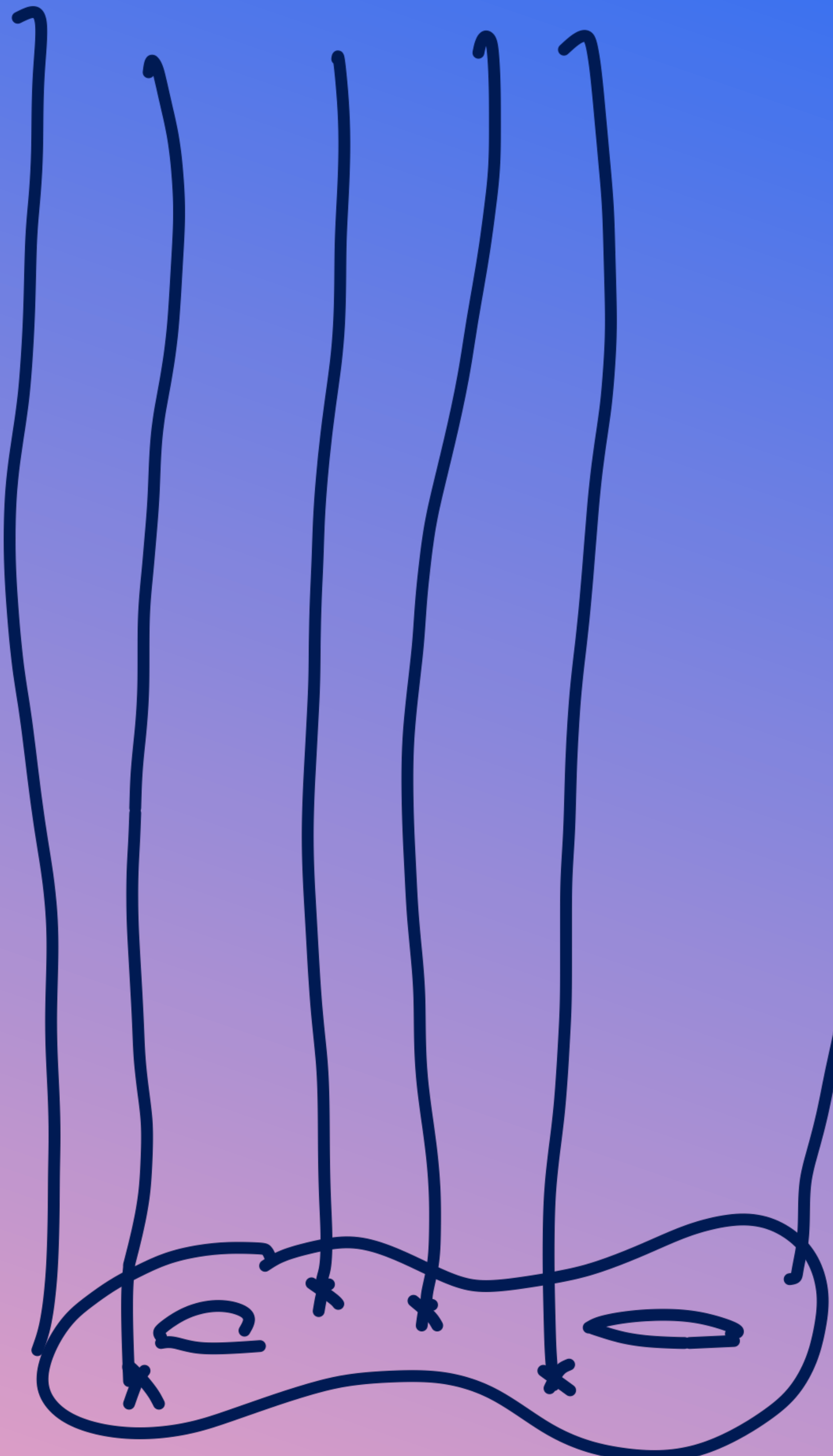
We require $\omega_0 = 0$ (prescribed Cauchy data)

B. Zhang (rectifiability)

Doan-Walpuski (abstract existence theorem)

He-M. (concrete model of this degeneration).

$\times S^1$



diverging sequence in $\mathcal{M}_{\text{Hitchin}}$ (well-understood)

$\rightsquigarrow$ singularities at $p_j \times \mathbb{R}^+$

$(p_1, \dots, p_{4g-4})$ zeroes of holom. quadratic differential.

What is the geometry of the degeneration locus in general X?

These correspond to the virtual KW sol'ns at ∞ (KW_t comes into play.)

Happy
Birthday
Dan !!!